A local potential for the Weyl tensor in all dimensions

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Abstract

In all dimensions $n \geq 4$ and arbitrary signature, we demonstrate the existence of a new local potential — a double $(2,3)$-form, $P^{abcde}$ — for the Weyl curvature tensor $C^{abcd}$, and more generally for all tensors $W^{abcd}$ with the symmetry properties of the Weyl tensor. The classical four-dimensional Lanczos potential for a Weyl tensor — a double $(2,1)$-form, $H^{abc}$ — is proven to be a particular case of the new potential: its double dual.

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In $n$ dimensions, the existence of a 1-form potential $A_a$ for the 2-form electromagnetic field $F_{ab}$ enables the electromagnetic field equations to be written as a wave equation (in Lorentz signature) for the potential, which is particularly simple in the differential gauge\textsuperscript{1} $A^a;_a = 0$ \cite{12}:

$$\nabla^2 A^a = F^{ab};_b.$$  

In \textbf{four} dimensions, Lanczos \cite{11} proposed the existence of a double $(2,1)$-form potential $H^{a}_{c} = H^{[ab]}_c$ for the double $(2,2)$-form Weyl tensor $C^{abcd}$ (see e.g. \cite{14} for the definition and properties of $r$-fold forms), and this result was confirmed in \cite{2} (for \textit{any} double

\textsuperscript{1}A semicolon indicates covariant derivative with respect to the canonical connection. As usual, we use round and square brackets to denote symmetrization and antisymmetrization of indices, respectively. Our convention for the Riemann tensor follows from the Ricci identity: $2v_{a;[bc]} = R^d_{abc}v_d$.}
(2,2)-form with the algebraic properties of the Weyl conformal curvature tensor) and equivalently for any symmetric spinor $\phi_{ABCD}$ in \([10]\); see also \([1, 5]\).

It is straightforward to conjecture a direct $n$-dimensional analogue for the Lanczos-Weyl equation (see (13) below) but, unfortunately, it has been shown that such a potential cannot exist, in general, in dimensions $n > 4$ \([6]\). As a consequence, interest in the existence of potentials for the Weyl tensor in dimensions $n > 4$ has diminished.

However there are a number of reasons for a continuing interest in a potential for the Weyl tensor, especially one which is defined in arbitrary dimensions. In particular, one attractive property of any possible potential for the Weyl tensor is that it has units $L^{-1}$, the same as the connection (or the first derivative of the metric). This means that any of its squares (such as its superenergy tensor \([14]\)) would have units $L^{-2}$ which are precisely the units we would expect for quantities related to gravitational ‘energies’, in contrast with the familiar Bel-Robinson superenergy tensor \([3, 14]\) whose units are $L^{-4}$ (Roberts \([13]\) pointed this out for the Lanczos potential). As a matter of fact, another related attractive property of these potentials is that they are at a level similar to the connection. As the potential is an ‘integral’ of the Weyl tensor, which itself is a second derivative of the metric, the potential and the connection are necessarily connected. The advantage is that the potential is a tensorial object. Then, of course, the impossibility of constructing tensors from the metric and its first derivatives manifest itself —as it must— in the lack of uniqueness of the potentials. Thus, already at this stage we realize that any potential must be affected by a gauge freedom.

In this letter we shall show, in arbitrary dimension and signature, that all (2,2)-forms with the algebraic properties of the Weyl tensor, locally have a double (2,3)-form potential $P_{abcd} = P_{[ab][cd]}$. We will further show that, in four dimensions, the double Hodge dual of the new potential, which is a double (2,1)-form, is exactly the classical Lanczos potential. Hence a very natural and very general potential for the Weyl tensor is this new double (2,3)-form potential.

We begin by considering an arbitrary $n$-dimensional pseudo-Riemannian manifold with metric $g_{ab}$ of any signature. We call a Weyl candidate any double (2,2)-form $X_{abcd} = X_{[ab][cd]} = X_{ab[cd]}$ with the algebraic properties of the Weyl conformal curvature tensor:

$$X^a_{\ bca} = 0, \quad X_{a[bc][d]} = 0 \quad (\Rightarrow X_{abcd} = X_{cdab}),$$

so that $X_{abcd}$ is a traceless and symmetric double (2,2)-form. We will now exploit these properties and rearrange as follows,

$$\nabla^2 X_{abcd} = \frac{1}{2} (X_{abcd:e} e + X_{cdab:e} e) = \frac{1}{2} \left(3X_{ab[cd]e} e - 2X_{[abe][c]d} e + 3X_{[ab][ade]} e - 2X_{cd}[a]b e\right)$$

and using the Ricci identity on the second and fourth terms we obtain

$$\nabla^2 X_{abcd} = \frac{1}{2} \left(3X_{ab[cd]e} e - 2X_{[abe][c]d} e + 3X_{[ab][ade]} e - 2X_{cd}[a]b e\right) + 2 \left(R^{e}_{dn[a]n b} X^{n}_{b} e + R^{e}_{cn[a]n b} X^{n}_{n b} e + R_{[n]c n a X^{n}_{b]d} + R_{n[a] n b]c d} - \frac{1}{2} (R_{cden} X^{en}_{ab} + R_{aben} X^{en}_{cd}\right) (2)$$
where \( R_{ab} \) is the Ricci tensor. Using the shorthand notation \( \{ R \otimes X \}_{abcd} \) for all the terms linear in the curvature tensors, that is to say, the second line of (2), we can rewrite by using (1) repeatedly

\[
\nabla^2 X_{abcd} = \frac{1}{2} \left( 3X_{ab[cd;e]} + 4X_{e[ab][c;e]d} + 3X_{cd[ab;e]} + 4X_{e[cd][a;e]b} \right) + \{ R \otimes X \}_{abcd}
\]

\[
= \frac{3}{2} \left( X_{ab[cd;e]} + X^e_{a[bc;e]d} - X^e_{b[ac;e]d} + X^e_{b[ad;e]c} - X^e_{a[bd;e]c} + X_{cd[ab;e]} \right.
\]

\[
+ X^e_{c[da;e]b} - X^e_{d[ca;e]b} + X^e_{d[cb;e]a} - X^e_{c[db;e]a} \right) + \{ R \otimes X \}_{abcd}.
\]

Now define the double (2, 3)-form

\[
P_{abcde} = P_{[ab][cd_e]} = \frac{3}{2} X_{ab[cd;e]}
\]

which will inherit from \( X_{abcd} \) the properties

\[
P_a[bcde] = 0, \quad P_{abc}^b = 0.
\]

Immediate consequences from the first of these are the following useful properties

\[
P_{[bcde]}^e = 0, \quad P_{[abcd]}^e = 0, \quad P_{abcd} = 3P_{[cde]ab} = 3P_{a[cde]b}, \quad P_{a[bc]de} = -P_{a[de]bc}.
\]

Hence (3) can be rewritten as

\[
\nabla^2 X_{abcd} = P_{abcde} + P_{cd[ab]} + 4P_{c[ab][e]} + 4P_{[cd][a;e]} + \{ R \otimes X \}_{abcd}
\]

\[
= P_{abcde} + P_{cd[ab]} + 2P_{[cde][a;e]} + 2P_{[cd][a;e]} + \{ R \otimes X \}_{abcd}.
\]

It is easily confirmed that this expression constructed from \( P_{abcde} \), ignoring the \( \{ R \otimes X \}_{abcd} \) terms, has the necessary Weyl candidate index symmetries.

Now consider any Weyl candidate \( W_{abcd} \). We can always find a Weyl candidate ‘superpotential’ \( X_{abcd} \) locally for \( W_{abcd} \) by appealing to the Cauchy-Kowalewsky theorem which guarantees a local solution of the linear second order equation

\[
\nabla^2 X_{abcd} - \{ R \otimes X \}_{abcd} = W_{abcd}
\]

in a given analytic background space. From the superpotential \( X_{abcd} \) we can then construct the potential \( P_{abcde} \) using (4), and obtain our main result:

**Theorem 1** Any Weyl candidate tensor field \( W_{abcd} \) has a double (2, 3)-form local potential \( P_{abcde} \) with the properties (3) such that

\[
W_{ab}^{cd} = P_{cdi}{^i}^j + P_{ci}{^j}^d - 2P_{[i}{^j}{^k}]^{cd} + 2P_{[{i}{j}{k}]}{^d}{^k}.
\]
Of course the above theorem has a direct application to the Weyl tensor $C^{cd}_{ab}$ of any pseudo-Riemannian manifold.

The number of independent components of the potential can be computed easily by using the properties (5) and the result is $(n+2)n(n-3)(n^2-n+4)/24$ (16 if $n=4$, 70 if $n=5$). This is larger than the number of independent components of a Weyl candidate, which is known to be $(n+2)(n+1)n(n-3)/12$ (that is, 10 if $n=4$, 35 if $n=5$). It is also larger (equal, in the case $n=4$) than the number of independent Ricci rotation coefficients, or of independent components of the connection in a given basis. Although this large number of components may seem unsatisfactory, one must bear in mind that this number can be substantially reduced in any particular case by means of the gauge differential freedom as we show in [7].

The choice of $P_{abcde}$ is not unique. As is usual with potentials — see, e.g., [12] — one can redefine significant parts of $P_{abcde;f}$ without altering the combination in (9) which produces a particular $W_{abcd}$. From known results about $p$-forms, and standard arguments on the independence of the exterior differential versus the divergence, we can easily identify and exploit some of the gauge freedom for the new potential. A detailed discussion on gauge, with explicit formulas for the gauge redefinition of $P_{abcde}$, will be given in [7].

Consider now the special case of $n=4$. It has already been noted that if $n=4$ the Weyl tensor, and more generally all Weyl candidates, have a so-called Lanczos potential [11], [2], [10], [1]. This is a double (2,1)-form $H^{ab}_{[c} = H^{ab}_{[c}$ with the extra properties

$$H_{[abc]} = 0, \quad H_{ab}^b = 0$$

and all Weyl candidates have this type of potentials such that

$$W^{ab}_{cd} = 2H^{ab}_{[c;d]} + 2H^{ad}_{ce} - H^a_{[c} \left( H^{be}_{d]e} + H^{de}_{b[}e \right).$$

What is the relation, if any, between the new potential $P_{abcde}$ and the Lanczos potential?

To answer this question, we first of all remark that a double (2,3)-form is equivalent, via dualization with the Hodge $*$ operator, to a double (2,1)-form in $n=4$. For the formulas and conventions about the Hodge dual operator we refer the reader to [14] — allowing for an extra sign depending on the signature of the space. In particular, for any traceless double (2,2)-form, and denoting the canonical volume element 4-form by $\eta_{abcd} = \eta_{[abcd]}$, we have [14]

$$(\ast W \ast)_{abcd} \equiv \frac{1}{4} \eta_{abef} \eta_{edgh} W^{efgh} \implies (\ast W \ast)_{abcd} = \epsilon W_{abcd}$$

where $\epsilon = \pm 1 = \text{sign}(\text{det}(g_{ab}))$ is a sign depending on the signature ($\epsilon = 1$ in positive-definite metrics, and $\epsilon = -1$ in the Lorentzian case). Similarly, for any double (2,3)-form $P_{abcde}$ we can write [14]

$$(\ast P \ast)_{abc} \equiv \frac{1}{12} \eta_{abef} \eta_{dghce} P^{efdgh} \implies P_{cde}^{ab} = 6\epsilon (\ast P \ast)^{[a}_{cd}e]^{b]}_c, \quad P^{i}_{cab} = \epsilon (\ast P \ast)_{abc}.$$

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Observe that the properties (5) translate for the double dual into, respectively,

\((*P*)_{abc} = 0, \quad (*P*)_{a}{}^{b} = 0\)

which are the Lanczos potential properties (10) exactly. Hence, by taking the double dual of (9), using the previous formulas and after a little bit of algebra we can prove that, in four dimensions

\[ \epsilon W^{ab}_{cd} = 2(*P*)_{abc}^{[cd]} + 2(*P*)_{cd}^{[a;b]} - 2\delta^{[a}_{[c} \left( (*P*)_{b]e}^{]c]d} + (*P*)_{d]}^{]e} \right) \]

which coincides with (11) by identifying

\[ H_{abc} = \epsilon (*P*)_{abc} \cdot \]

For completeness, we give also the inverse formula: 

\[ P_{abcde} = \epsilon (*H*)_{abcde} \cdot \]

Therefore, we have recovered the classical Lanczos potential for the Weyl tensor in four dimensions as the double dual of the new potential \( P_{abcde} \). Note that the familiar differential gauge of the Lanczos potential \( H_{abc} \), (11), (2), (10), (1) becomes \( H_{abc} = \epsilon (*P*)_{abc} \), and so the differential gauge for the new potential resides in \( P_{abc} \), via double dualization.

In dimensions greater than four, a natural generalization of (11) to arbitrary dimension is to keep the double (2,1)-form \( H_{abc} \) with properties (10) and consider the appropriate formula analogous to (11) which keeps the Weyl candidate symmetries. This formula is unique and reads

\[ W^{ab}_{cd} = 2H_{abc}^{[cd]} + 2H_{cd}^{[a;b]} - \frac{4}{n-2} \delta^{[a}_{[c} \left( H_{b]e}^{]c]d} + H_{d]}^{]e} \right) \].

However, as already noted, this Lanczos potential exists exclusively in \( n = 4 \) dimensions. On the other hand, we now know that there is a different counterpart to the Lanczos potential in \( n > 4 \) dimensions; it is the double dual of \( P_{abcde} \), i.e., a double (2, \( n-3 \))-form \( H^{[a}_{c1c2\ldots c_{n-3}] \), and an expression — giving any Weyl candidate in terms of such a potential — can be obtained in any dimension by the same method that led to (12). Clearly such a dimensionally dependent result is much less natural and convenient than our defining formula (9), which is independent of dimension. Therefore, in particular, we have proved that the natural and proper way to consider a potential for a double (2,2)-form is as a double (2,3)-form; this version carries over to all dimensions by means of (7). The traditional double (2,1)-form Lanczos potential \( H^{ab}_{c} \) in \( n = 4 \) dimensions is nothing but the double dual of the general potential \( P_{abc} \).

In electromagnetic theory the local existence of the potential \( A_{a} \) for the field tensor \( F_{ab} \) is a direct consequence of applying Poincare’s Lemma to one of Maxwell’s equations, \( F_{[abc]} = 0 \). However, it is important to note that the local existence of the new potential \( P_{abcde} \) for the Weyl tensor does not require any such ‘field equations’. Our result is actually analogous to the result that any 2-form always has a pair of local potentials, a 1-form \( A_{a} \) and a 3-form \( B_{abc} \) such that

\[ F_{ab} = 2A_{[a;b]} - B_{abc}^{;c} \].


If the 2-form $F_{ab}$ is closed, then $B_{abc}$ can be chosen to be zero, while if it is divergence-free, then $A_a$ can be set to zero. Our result for Weyl candidates is the generalization of this fact but, given the special structure of traceless and symmetric double $(2,2)$-forms, we achieve dealing with just one potential. A detailed discussion will be presented in [7].

Finally we emphasise that all our results are local and depend on the analiticity of the pseudo-Riemannian metric. However, as is usually the case, we expect that these results can be generalized, by using appropriate techniques of existence and uniqueness of solutions to differential equations, to the smooth case and even to spaces of low differentiability; from the point of view of general relativity, we can appeal to stronger theorems when we specialise to spaces with Lorentz signature, and the second order differential equations in the theorems become wave equations.

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