Do Chiral Soliton Models Predict Pentaquarks?\footnote{1}{Contribution to the proceedings of the QCD 2004 Workshop at the University of Minnesota, May 13-16, 2004.}

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Abstract

We reconsider the relationship between the bound state and the $SU(3)$ rigid rotator approaches to strangeness in chiral soliton models. For non-exotic $S = -1$ baryons the bound state approach matches for small $m_K$ onto the rigid rotator approach, and the bound state mode turns into the rotator zero-mode. However, for small $m_K$, there are no $S = +1$ kaon bound states or resonances in the spectrum. This shows that for large $N$ and small $m_K$ the exotic state is an artifact of the rigid rotator approach. An $S = +1$ near-threshold state with the quantum numbers of the $\Theta^+$ pentaquark comes into existence only when sufficiently strong $SU(3)$ breaking is introduced into the chiral lagrangian. Therefore, pentaquarks are not generic predictions of the chiral soliton models.

\section{Introduction}

These lecture notes are largely based on our paper with N. Itzhaki and L. Rastelli\cite{1}.  

\footnote{1}{Contribution to the proceedings of the QCD 2004 Workshop at the University of Minnesota, May 13-16, 2004.}
Recently there has been a flurry of research activity on exotic pentaquark baryons, prompted by reports\cite{2,3,4} of the observation of the $S = +1$ baryon $\Theta^+$ (1540). The original photoproduction experiment\cite{2} was largely motivated by theoretical work\cite{5} in which chiral soliton models were used to predict a rather narrow $I = 0, J^P = \frac{1}{2}^+$ exotic $S = +1$ baryon whose minimal quark content is $uudd\bar{s}$. The method used in\cite{5} to predict the baryon spectrum is the $SU(3)$ collective coordinate quantization of chiral solitons\cite{6,7}. This approach predicts the well-known $8$ and $10$ $SU(3)$ multiplets of baryons, followed by an exotic $\mathbf{10}$ multiplet\cite{8,9,10} whose $S = +1$ member is the $\Theta^+$. The fact that the exotic $\mathbf{10}$ multiplet is found simply by exciting the soliton to the next rotational energy level after the well-known decuplet has led to a widespread belief that the pentaquarks are a robust prediction of chiral soliton models, independent of assumptions about the dynamics. In this talk we argue that this belief is not well-founded. Instead, we will conclude that in soliton models the existence or non-existence of the pentaquarks very much depends on the details of the dynamics, i.e. the structure of the chiral lagrangian. Thus, the soliton models do not produce any miracles that are not obvious from general priciples of QCD. Neither in QCD nor in chiral soliton models is there anything that a priori guarantees the existence of narrow pentaquarks. Indeed, we will show that with the standard set of the Skyrme model parameters, a resonance with the quantum numbers of the $\Theta^+$ does not form. This fact should be kept in mind as some of the more recent searches\cite{11} have failed to confirm the existence of the $\Theta^+$ or other pentaquarks.

Our theoretical discussion follows the basic premise\cite{6} that the semiclassical quantization of chiral solitons corresponds to the $1/N$ expansion for baryons in QCD generalized to a large number of colors $N$. It is therefore important to generalize the discussion of exotic collective coordinate states carried out for $N = 3$ in\cite{5} to large $N$. The allowed multiplets must contain states of hypercharge $N/3$, i.e. of strangeness $S = 0$. In the notation where $SU(3)$ multiplets are labeled by $(p, q)$, the lowest multiplets one finds\cite{8,14,15,16,17,18} are $(1, n)$ with $J = \frac{1}{2}$ and $(3, n - 1)$ with $J = \frac{3}{2}$. These are the large $N$ analogues of the octet and the decuplet. Exactly the same multiplets appear when we construct baryon states out of $N$ quarks. The splittings among them are of order $1/N$, as is usual for soliton rotation excitations.

The large $N$ analogue of the exotic antidecuplet is the representation $(0, n + 2)$ with $J = \frac{1}{2}$, and one finds that its splitting from the lowest multiplets is $O(N^0)$ in the rotator approximation. The fact that the mass splitting is of order one, comparable to the energy of mesonic fluctuations, raises questions\cite{19,16,17} about the validity
of the rigid rotator approach to these states. Instead, a better treatment of these states is provided by the bound state approach\cite{20} where one departs from the rigid rotator ansatz and adopts more general kaon fluctuation profiles; in this approach one describes the $\Theta^+$ as a kaon-skyrmion resonance or bound state of $S = +1$, rather than by a rotator state (a similar suggestion was made independently by Cohen\cite{17}.) In the non-exotic sector, as we take the limit $m_K \to 0$ the bound state description of low-lying baryons smoothly approaches the rigid rotator description, and the bound state wavefunction approaches a zero-mode. However, in contrast with the situation for $S = -1$, for $S = +1$ there is no fluctuation mode that in the $m_K \to 0$ limit approaches the rigid rotator mode. Thus, for large $N$ and small $SU(3)$ breaking, the rigid rotator state with $S = +1$ is an artifact of the rigid rotator approximation (we believe this to be a general statement that does not depend on the details of the chiral lagrangian.)

Next we ask what happens as we increase the $SU(3)$ breaking by varying parameters in the effective lagrangian (such as the kaon mass and the weight of the Wess-Zumino term) and find that a substantial departure from the $SU(3)$-symmetric limit is necessary to stabilize the kaon-skyrmion system. We reach a conclusion that, at least for large $N$, the exotic $S = +1$ state exists only due to the $SU(3)$ breaking and disappears when the breaking is too weak.

## 2 The rigid rotator vs. the bound state approach

Our discussion of chiral solitons is mainly carried out in the context of the Skyrme model, but our conclusions will not be tied to a specific model. The Skyrme approach to baryons begins with the Lagrangian\cite{12}

$$L_{Skyrme} = \frac{f^2}{16} \text{Tr}(\partial^\mu U^\dagger \partial_\mu U) + \frac{1}{32e^2} \text{Tr}([\partial_\mu UU^\dagger, \partial_\nu UU^\dagger]^2) + \text{Tr}(M(U + U^\dagger - 2)),$$

(2.1)

where $U(x^\mu)$ is a matrix in $SU(3)$ and $M$ is proportional to the matrix of quark masses. There is an additional term in the action, called the Wess-Zumino term, whose normalization is proportional to $N$.

Skyrme showed that there are topologically stabilized static solutions of hedgehog form, $U_0 = e^{i r F(r)}$, in which the radial profile function $F(r)$ satisfies the boundary conditions $F(0) = \pi$, $F(\infty) = 0$. The non-strange low-lying excitations of this soliton
are given by rigid rotations of the pion field \( A(t) \in SU(2) \):

\[
U(x, t) = A(t)U_0A^{-1}(t).
\]

(2.2)

For this ansatz the Wess-Zumino term does not contribute. By expanding the Lagrangian about \( U_0 \) and canonically quantizing the rotations, one finds that the Hamiltonian is

\[
H = M_{cl} + \frac{1}{2\Omega} J(J + 1),
\]

(2.3)

where \( J \) is the spin and the c-numbers \( M_{cl} \) and \( \Omega \) are functionals of the soliton profile. For vanishing pion mass, one finds numerically that

\[
M_{cl} \simeq 36.5 \frac{f_\pi}{e}, \quad \Omega \simeq \frac{107}{e^3 f_\pi}.
\]

(2.4)

For \( N = 2n + 1 \), the low-lying quantum numbers are independent of the integer \( n \). The lowest states, with \( I = J = \frac{1}{2} \) and \( I = J = \frac{3}{2} \), are identified with the nucleon and \( \Delta \) particles respectively. Since \( f_\pi \sim \sqrt{N} \), and \( e \sim 1/\sqrt{N} \), the soliton mass is \( \sim N \), while the rotational splittings are \( \sim 1/N \). Adkins, Nappi and Witten\[13\] found that they could fit the \( N \) and \( \Delta \) masses with the parameter values \( e = 5.45, f_\pi = 129 \text{ MeV} \). In comparison, the physical value of \( f_\pi = 186 \text{ MeV} \).

A generalization of this rigid rotator treatment that produces \( SU(3) \) multiplets of baryons is obtained by making the collective coordinate \( A(t) \) an element of \( SU(3) \). Then the WZ term makes a crucial constraint on allowed multiplets\[6, 7, 8, 9\]. The large-\( N \) treatment of this 3-flavor Skyrme model is more subtle than in the 2-flavor case. When \( N = 2n + 1 \) is large, even the lowest lying \((1, n)\) \( SU(3) \) multiplet contains \((n + 1)(n + 3)\) states with strangeness ranging from \( S = 0 \) to \( S = -n - 1 \)[16]. When the strange quark mass is turned on, it introduces a splitting of order \( N \) between the lowest and highest strangeness baryons in the same multiplet. Thus, \( SU(3) \) is badly broken in the large \( N \) limit, no matter how small \( m_s \) is[16]. It is helpful to think in terms of \( SU(2) \times U(1) \) flavor quantum numbers, which do have a smooth large \( N \) limit. In other words, we focus on low strangeness members of these multiplets, whose \( I, J \) quantum numbers have a smooth large \( N \) limit, and identify them with observable baryons.

Since the multiplets contain baryons with up to \( \sim N \) strange quarks, the wave functions of baryon with fixed strangeness deviate only an amount \( \sim 1/N \) into the strange directions of the collective coordinate space. Thus, to describe them, one may
expand the $SU(3)$ rigid rotator treatment around the $SU(2)$ collective coordinate. The small deviations from $SU(2)$ may be assembled into a complex $SU(2)$ doublet $K(t)$. This method of $1/N$ expansion was implemented in [19], and reviewed in [16].

From the point of view of the Skyrme model the ability to expand in small fluctuations is due to the Wess-Zumino term which acts as a large magnetic field of order $N$. The method works for arbitrary kaon mass, and has the correct limit as $m_K \to 0$. To order $O(N^0)$ the Lagrangian has the form

$$L = 4\Phi \dot{K}^\dagger \dot{K} + i \frac{N}{2} (K^\dagger \dot{K} - \dot{K}^\dagger K) - \Gamma K^\dagger K .$$  \hspace{1cm} (2.5)$$

The Hamiltonian may be diagonalized:

$$H = \omega_+ a^\dagger a + \omega_- b^\dagger b + \frac{N}{4\Phi} ,$$  \hspace{1cm} (2.6)

where

$$\omega_\pm = \frac{N}{8\Phi} \left( \sqrt{1 + (m_K/M_0)^2} \pm 1 \right) , \quad M_0^2 = \frac{N^2}{16\Phi \Gamma} .$$  \hspace{1cm} (2.7)$$

The strangeness operator is $S = b^\dagger b - a^\dagger a$. All the non-exotic multiplets contain $a^\dagger$ excitations only. In the $SU(3)$ limit, $\omega_- \to 0$, but $\omega_+ \to \frac{N}{4\Phi} \sim N^0$. Thus, the “exoticness” quantum number mentioned in [18] is simply $E = b^\dagger b$ here, and the splitting between multiplets of different “exoticness” is $\frac{N}{4\Phi}$, in agreement with results found from the rigid rotator [14, 15, 16, 17, 18].

The $O(N^0)$ splittings predicted by the rigid rotator are, however, not exact: this approach does not take into account deformations of the soliton as it rotates in the strange directions [20, 19, 16]. Another approach to strange baryons, which allows for these deformations, and which proves to be quite successful in describing the light hyperons, is the so-called bound state method [20]. The basic strategy is to expand the action to second order in kaon fluctuations about the classical hedgehog soliton. Then one can obtain a linear differential equation for the kaon field, incorporating the effect of the kaon mass, which one can solve exactly. The eigenenergies of the kaon field are then the $O(N^0)$ differences between the masses of the strange baryons and the classical Skyrmion mass. It is convenient to write $U$ in the form $U = \sqrt{U_\pi} U_K \sqrt{U_\pi}$, where $U_\pi = \exp[2i\lambda_j \pi^j/f_\pi]$ and $U_K = \exp[2i\lambda_a K^a/f_\pi]$ with $j$ running from 1 to 3 and $a$ running from 4 to 7. The $\lambda_a$ are the standard $SU(3)$ Gell-Mann matrices. We will collect the $K^a$ into a complex isodoublet $K$:

$$K = \frac{1}{\sqrt{2}} \left( \begin{array}{c} K^4 - iK^5 \\ K^6 - iK^7 \end{array} \right) = \left( \begin{array}{c} K^+ \\ K^0 \end{array} \right) .$$  \hspace{1cm} (2.8)$$
Expanding the Wess-Zumino term to second order in $K$, we obtain

$$L_{WZ} = \frac{iN}{f_\pi^2} B^\mu \left( K^\dag D_\mu K - (D_\mu K)^\dag K \right)$$

where

$$D_\mu K = \partial_\mu K + \frac{1}{2} \left( \sqrt{U^\dag_\pi} \partial_\mu \sqrt{U_\pi} + \sqrt{U_\pi} \partial_\mu \sqrt{U^\dag_\pi} \right) K ,$$

and $B_\mu$ is the baryon number current. Now we decompose the kaon field into a set of partial waves. Because the background soliton field is invariant under combined spatial and isospin rotations $T = I + L$, a good set of quantum numbers is $T, L$ and $T_z$, and so we write the kaon eigenmodes as $K = k(r, t)Y_{TLTz}$. Substituting this expression into $L_{Skyrme} + L_{WZ}$ we obtain an effective Lagrangian for the radial kaon field $k(r, t)$ of the form

$$L = 4\pi \int r^2 dr \left( f(r) k^\dag \dot{k} + i\lambda(r) (k^\dag \dot{k} - \dot{k}^\dag k) - h(r) \frac{d}{dr} k^\dag \frac{d}{dr} k - k^\dag k (m_K^2 + V_{eff}(r)) \right).$$

The formula for the effective potential $V_{eff}(r)$ appears in [21]. The resulting equation of motion for $k$ is

$$-f(r)\ddot{k} + 2i\lambda(r)\dot{k} + \mathcal{O}k = 0,$$

$$\mathcal{O} \equiv \frac{1}{r^2} \partial_r h(r) r^2 \partial_r - m_K^2 - V(r).$$

The eigenvalue equations are

$$(f(r)\omega_n^2 + 2\lambda(r)\omega_n + \mathcal{O})k_n = 0 \quad (S = -1),$$

$$(f(r)\tilde{\omega}_n^2 - 2\lambda(r)\tilde{\omega}_n + \mathcal{O})\tilde{k}_n = 0 \quad (S = +1),$$

with $\omega_n, \tilde{\omega}_n$ positive. Crucially, the sign in front of $\lambda$, which is the contribution of the WZ term, depends on whether the relevant eigenmodes have positive or negative strangeness.

It is possible to examine these equations analytically for $m_K = 0$. Then one finds that the $S = -1$ equation has an exact solution with $\omega = 0$ and $k(r) \sim \sin(F(r)/2)$, which corresponds to the rigid rotator zero-mode [21]. As $m_K$ is turned on, this solution
turns into an actual bound state \cite{20, 21}. On the other hand, the \( S = +1 \) equation
does not have a solution with \( \tilde{\omega} = \frac{N}{4\ell} \) and \( k(r) \sim \sin(F(r)/2) \). This is why the exotic
rigid rotator state is not reproduced by the more precise bound state approach to
strangeness. In section 3 we further check that, for small \( m_K \), there is no resonance
corresponding to the rotator state of energy \( \frac{N}{4\ell} \) in the \( SU(3) \) limit.

The lightest \( S = -1 \) bound state is in the channel \( L=1, T = \frac{1}{2} \), and its mass is
\( M_{cl} + 0.218 \epsilon f_\pi \simeq 1019 \) MeV. This state gives rise to the \( \Lambda(1115) \), \( \Sigma(1190) \), and \( \Sigma(1385) \)
states, where the additional splitting arises from \( SU(2) \) rotator corrections\cite{20, 21}. There is also a \( L = 0, T = \frac{1}{2} \) bound state corresponding to the negative parity hyperon
\( \Lambda(1405) \). The natural appearance of the \( \Lambda(1405) \) is a major success of the bound state
approach\cite{21, 24}.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{phase_shift.png}
\caption{Phase shift as a function of energy in the \( L = 2, T = \frac{3}{2}, S = -1 \) channel. The
energy \( \omega \) is measured in units of \( \epsilon f_\pi \) (with the kaon mass subtracted, so that \( \omega = 0 \) at
threshold), and the phase shift \( \delta \) is measured in radians. Here \( e = 5.45 \) and \( f_\pi = 129 \)
MeV.}
\end{figure}

The same method can be applied also to states above threshold. Such states will
appear as resonances in kaon-nucleon scattering, which we may identify by the stan-
dard procedure of solving the appropriate kaon wave equation and studying the phase
shifts of the corresponding solutions as a function of the kaon energy. In the \( L = 2, T = \frac{3}{2} \) channel there is a resonance at \( M_{cl} + 0.7484 \epsilon f_\pi = 1392 \) MeV (see Figure 1). Upon
the \( SU(2) \) collective coordinate quantization, it gives rise to three states\cite{22} with \( (I, J) \)
given by \( (0, \frac{3}{2}), (1, \frac{3}{2}), (1, \frac{5}{2}) \), with masses 1462 MeV, 1613 MeV, and 1723 MeV respec-
tively (see Table 2). We see that these correspond nicely to the known negative parity resonances \( \Lambda(1520) \) (which is \( D_{03} \) in standard notation), \( \Sigma(1670) \) (which is \( D_{13} \)) and \( \Sigma(1775) \) (which is \( D_{15} \)). As with the bound states, we find that the resonances are somewhat overbound (the overbinding of all states is presumably related to the necessity of adding an overall zero-point energy of kaon fluctuations), but that the mass splittings within this multiplet are accurate to within a few percent. In fact, we find that the ratio

\[
\frac{M(1, \frac{3}{2}) - M(0, \frac{3}{2})}{M(1, \frac{3}{2}) - M(0, \frac{3}{2})} \approx 1.73 \tag{2.14}
\]

while its empirical value is 1.70.

3 Baryons with \( S=+1 \)?

For states with positive strangeness, the eigenvalue equation for the kaon field is the same except for a change of sign in the contribution of the WZ term. This sign change makes the WZ term repulsive for states with \( \bar{s} \) quarks and introduces a splitting between ordinary and exotic baryons\[20\]. In fact, with standard values of the parameters (such as those in the previous section) the repulsion is strong enough to remove all bound states and resonances with \( S = 1 \), including the newly-observed \( \Theta^+ \). It is natural to ask how much we must modify the Skyrme model to accommodate the pentaquark. The simplest modification we can make is to introduce a coefficient \( a \) multiplying the WZ term. Qualitatively, we expect that reducing the WZ term will make the \( S = +1 \) baryons more bound, while the opposite should happen to the ordinary baryons.

The most likely channel in which we might find an exotic has the quantum numbers \( L = 1, T = \frac{1}{2} \), as in this case the effective potential is least repulsive near the origin. For \( f_\pi = 129, 186, \) and 225 MeV, with \( c^3 f_\pi \) fixed, we have studied the effect of lowering the WZ term by hand. Interestingly, in all three cases we have to set \( a \simeq 0.39 \) to have a bound state at threshold. If we raise \( a \) slightly, this bound state moves above the threshold, but does not survive far above threshold; it ceases to be a sharp state for \( a \simeq 0.46 \). We have plotted phase shifts for various values of \( a \) in Figure 2.\[2\] Assuming that the parameters of the chiral lagrangian take values such that the \( \Theta^+ \) exists, we can

\[\text{When the state is above the threshold, we do not find a full \( \pi \) variation of the phase. Furthermore, the variation and slope of the phase shift decrease rapidly as the state moves higher, so it gets too broad to be identifiable. So, the state can only exist as a bound state or a near-threshold state.}\]
then consider the $SU(2)$ collective coordinate quantization of the state, in a manner analogous to the treatment of the $S = -1$ bound states. Here we record our results, assuming that $a = 0.39$ and $f_π = 129$ MeV, and refer the reader to our original paper for details.

The lightest $S = +1$ state we find has $I = 0, J = \frac{1}{2}$ and positive parity, i.e. it is an $S = +1$ counterpart of the Λ. This is the candidate $\Theta^+$ state. Its first $SU(2)$ rotator excitations have $I = 1, J^P = \frac{3}{2}^+$ and $I = 1, J^P = \frac{1}{2}^+$ (a relation of these states to $\Theta^+$ also follows from general large $N$ relations among baryons). The counterparts of these $J^P = \frac{1}{2}^+, \frac{3}{2}^+$ states in the rigid rotator quantization lie in the $27$-plets of $SU(3)$. We find that the $I = 1, J^P = \frac{3}{2}^+$ state is $\sim 148$ MeV heavier than the $\Theta^+$, while the $I = 1, J^P = \frac{1}{2}^+$ state is $\sim 289$ MeV heavier than the $\Theta^+$.

We may further consider $I = 2$ rotator excitations which have $J^P = \frac{3}{2}^+, \frac{5}{2}^+$. Such states are allowed for $N = 3$ (in the quark language the charge +3 state, for example, is given by $uuuu\bar{s}$). The counterparts of these $J^P = \frac{3}{2}^+, \frac{5}{2}^+$ states in the rigid rotator quantization lie in the $35$-plets of $SU(3)$. We find

$$M(2, \frac{5}{2}) - M(0, \frac{1}{2}) \sim 494 \text{ MeV},$$
$$M(2, \frac{3}{2}) - M(0, \frac{1}{2}) \sim 729 \text{ MeV}.$$ (3.15)

Although the specific mass splittings which we have computed depend on the choice of parameters in the chiral lagrangian, it turns out that we may form certain combinations of masses of the exotics which rely only the existence of the $SU(2)$ collective coordinate:

$$2M(1, \frac{3}{2}) + M(1, \frac{1}{2}) - 3M(0, \frac{1}{2}) = 2(M_\Delta - M_N) = 586 \text{ MeV},$$
$$\frac{3}{2}M(2, \frac{5}{2}) + M(2, \frac{3}{2}) - \frac{5}{2}M(0, \frac{1}{2}) = 5(M_\Delta - M_N) = 1465 \text{ MeV},$$
$$M(2, \frac{3}{2}) - M(2, \frac{5}{2}) = \frac{5}{2}(M(1, \frac{1}{2}) - M(1, \frac{3}{2})).$$ (3.16)

where we used $M_\Delta - M_N = \frac{3}{2\Omega}$. These “model-independent” relations have also been derived using a different method.

As another probe of the parameter space of our Skyrme model, we may vary the mass of the kaon and see how this affects the pentaquark. As observed in Section 3, in the limit of infinitesimal kaon mass, there is no resonance in the $S = +1, L = 1, T = \frac{1}{2}$ channel. We find that to obtain a bound state in this channel, we must raise $m_K$ to about 1100 MeV. Plots of the phase shift vs. energy for different values of $m_K$ may be found in [1].

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Figure 2: Phase shifts $\delta$ as a function of energy in the $S = +1$, $L = 1$, $T = \frac{1}{2}$ channel, for various choices of the parameter $a$ (strength of the WZ term). The energy $\omega$ is measured in units of $e f_{\pi}$ ($e = 5.45$, $f_{\pi} = f_K = 129$ MeV) and the phase shift $\delta$ is measured in radians. $\omega = 0$ corresponds to the $K - N$ threshold.

4 Discussion

The main implication of our analysis is that in chiral soliton models there is no “theorem” that exotic pentaquark baryons exist, nor is there a theorem that they do not exist. The situation really depends on the details of the dynamics inherited from the underlying QCD.

The statements above apply to general chiral soliton models containing various lagrangian terms consistent with the symmetries of low-energy hadronic physics. In the bound state approach to the Skyrme model we saw that an $S = +1$ near-threshold state is absent when we use the standard parameters, but comes into existence only at the expense of a large reduction in the Wess-Zumino term. It is doubtful that such a reduction is consistent with QCD. However, one can and should explore other variants of chiral soliton models. For example, in [28] the exotic $S = +1$ resonances were studied in a model containing explicit $K^*$ fields. This model contains a coupling constant which is, roughly speaking, the analogue of the coefficient of the WZ term, $a$, in our approach. The findings of [28] are largely parallel to ours. For a wide range of values of this coupling, the repulsion is too strong, and no $S = +1$ resonances can form. When this coupling is very small, then there exists an $S = +1$ bound state. There is also a narrow intermediate range where this bound state turns into a near-threshold resonance. An important question is whether choosing parameters to lie in this narrow range is consistent with the empirical constrains on the effective lagrangian. If not,
then one may have to conclude that chiral soliton models actually predict the absence of pentaquarks.

Acknowledgments

We are grateful to N. Itzhaki and L. Rastelli for collaboration on a paper reviewed here, and to E. Witten for discussions. This material is based upon work supported by the National Science Foundation Grants No. PHY-0243680 and PHY-0140311. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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