Approximated seventh order calculation of vacuum wave function of 2+1 dimensional SU(2) lattice gauge theory

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Abstract

Using the coupled cluster expansion with the random phase approximation, we calculate the long wavelength vacuum wave function and the vacuum energy of 2+1 dimensional Hamiltonian SU(2) lattice gauge theory (LGT) up to the seventh order. The coefficients $\mu_0$, $\mu_2$ of the vacuum wave function show good scaling behavior and convergence in high order calculations.

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1 Introduction

Lattice gauge theory was developed into a promising first principles approach to the nonperturbative aspects of gauge field systems. Most of our present knowledge about LGT has obtained from numerical simulations. However, in order to gain more physical insight into the theory, it is desirable to develop more analytical methods. Based on Greensite’s proposal [1], we developed the coupled cluster expansions for the Hamiltonian LGT [2, 3, 4]. Although the preliminary research shows that this scheme is reasonable to study the LGT, it suffers from the rapid proliferation and non-independent of clusters produced in the high order expansive calculations.

In Ref.[2], we used the Cayley-Hamilton relations

\[ TrU = TrU^+, \quad U^2 = UTrU - 1 \quad \text{for SU}(2), \]
\[ 2TrU^+ = (TrU)^2 - TrU^2, \quad U^3 = U^2TrU - UTrU^+ + 1 \quad \text{for SU}(3), \]

(1)
(2)

to eliminate redundancies and find the independent cluster bases, where \( U \) is any group element. But those relations are too complicated to be used in the high order expansions. We have tried to use the improved Hamiltonian [5] with the tadpole improvement [6], so that the physical results could be obtained in a relative low order expansion. The research results for 2+1 dimensional U(1) LGT shows that the improvement of convergence is immaterial comparing with the complex of calculation brought about by the improved Hamiltonian [7]. Recently, we introduced the random phase approximation (RPA) into the coupled cluster expansions to circumvent the above problems. the preliminary investigation results were encouraging [8]. In this paper, we use this new method to calculate the vacuum wave function and vacuum energy of 2+1 dimensional SU(2) LGT.

The paper is organized as follows. In Sec. II, we briefly review the scheme of the coupled cluster expansion with RPA. Section III is devoted to the calculation of the vacuum wave function. In Sec. IV, the conclusions and discussions are presented.

2 Formulation and Approximation

The Kogut-Susskind Hamiltonian [9] is

\[ H = \frac{g^2}{2a} \left[ \sum_l E_l^2 - \frac{4}{g^2} \sum_p Tr(U_p) \right], \]

(3)
where the index \( l \) denotes the links between sites, \( a \) is the lattice spacing, and \( U_p \) the plaquette. The vacuum wave function can be written as [1]

\[
\psi_0(U) = e^{R(U)}.
\]  

(4)

where \( R(U) \) consists of various Wilson loops or linked clusters with appropriate symmetries for the state. In the continuum limit (\( a \to 0 \), or equivalently \( \beta = 4/g^2 \to \infty \)), the long wavelength behavior of the vacuum state can be approximated by [2]

\[
\psi(A) = \exp \left( -\mu_0 \int d^2x \text{tr} F^2(x) - \mu_2 \int d^2x \text{tr} [D_i F^2(x)] + \cdots \right).
\]  

(5)

with \( F \) being the field strength tensor and \( D_i \) the covariant derivative. The superrenormalizability of the theory in 2+1 dimensions implies that \( \mu_0 \to \text{const.}/e^2 \) and \( \mu_2 \to \text{const.}/e^6 \), where \( e \) is the invariant charge which is related to the dimensionless coupling constant \( g \) by \( g^2 = e^2 a \).

The eigenvalue equation for the vacuum state of \( H \) is

\[
\sum_l \left( [E_{a,l}, R(U)] + [E_{a,l}, R(U)] [E_{a,l}, R(U)] \right) - \frac{4}{g^4} \sum_p \text{tr} U_p = \frac{2a}{g^2} \epsilon_0,
\]  

(6)

where \( \epsilon_0 \) is the vacuum energy. In the coupled cluster expansion, \( R \) is expanded in terms of a set of linearly independent cluster \( G_{n,i} \) with suitable symmetries

\[
R(U) = \sum_{n=1}^{M} \sum_{i} c_{n,i} G_{n,i},
\]  

(7)

with \( G_{n,i} \) denoting the \( i \)-th cluster of order \( n \), \( c_{n,i} \) being a coefficient to be determined by Eq.(6), and \( M \) the highest order number in the coupled cluster expansion. Substituting it into Eq.(6) and adopting the truncation scheme of Ref. [2, 3, 4], we obtain the truncated eigenvalue equation

\[
\sum_l \left( [E_{l}, \sum_{n=1}^{M} R_n(U)] + \sum_{n+n'\leq M} [E_{l}, R_n(U)][E_{l}, R_{n'}(U)] \right) - \frac{4}{g^4} \sum_p \text{tr} U_p = \frac{2a}{g^2} \epsilon_0.
\]  

(8)

According to the symmetry of the vacuum state, the lowest order term \( R_1 \) is chosen to be composed of just one cluster, which is an elementary plaquette

\[
R_1 = c_{1,1} G_{1,1} = c_{1,1} \sum_p \text{tr} U_p.
\]  

(9)

The term \([E_{l}^q, R_n][E_{l}^q, R_{n'}]\) in Eq.(6) will produce the \((n+n')\)-th order clusters. For example, \([E_{l}^q, R_1][E_{l}^q, R_1]\) generates the second order clusters involving two plaquettes. Obviously, clusters with order \((n+n')\) may contain at most \((n+n')\) Wilson loops. But not all clusters produced in
such a way are independent. They may be related to each other by relation (1). We have to use this relation to identify and get the independent clusters. When clusters have many Wilson loops, relation (1) becomes so complicated that it is impossible to find a set of independent clusters, as pointed in Sec. I. To avoid this difficult, the RPA is applied in the expansions [8]. In RPA, one set of operator in a product of two sets of operators is replaced by its average value [10]. Here, we replace one Wilson loop by its vacuum average value when a cluster produced by \([E_{\alpha}^a, R_n][E_{\alpha}^a, R_{n'}]\) consists of two Wilson loops. It is easy to prove that any cluster produced in the above procedure contains only one Wilson loop. Suppose \(R_n\) and \(R_{n'}\) are linear combinations of clusters which contain only one Wilson loop, then, the new clusters produced by \([E_{\alpha}, R_n][E_{\alpha}, R_{n'}]\) will contain at most two loops, and one loop will be replaced by its vacuum average after applying the RPA. On the other hand, according to Eq. (9) the first order cluster \(G_{1,1}\) has only one Wilson loop. Therefore, all new clusters produced in the calculation consist of only one Wilson loop after using the RPA.

Because the clusters consist of only one Wilson loop, the independent cluster bases for the expansions can be obtained directly. In addition, the number of independent bases of high order expansions is much smaller than that without using the random phase approximation, for example, the number of independent third order clusters is nine in the coupled cluster expansion [2], while it is two after using the RPA (see Sec. III). Therefore, the calculation is simplified very much.

The vacuum average value of a Wilson loop can be determined by the Feynman-Hellman theorem. Let \(G\) be some Wilson loop and \(<G>\) be its vacuum average. Defining \(W = H2a/g^2\), we make the following change [11]:

\[
W \longrightarrow W^G = W + \xi_G G,
\]

where \(\xi_G\) is a variable and will take zero at last. From \(W^G|\psi_0> = w^G_0|\psi_0>\), we get

\[
<G> = \left. \frac{\partial w^G_0}{\partial \xi_G} \right|_{\xi_G=0},
\]

3 Calculation of approximation

We now present the calculation of the expansion. Because

\[
[E_{\alpha}^a, G_{1,1}][E_{\alpha}^a, G_{1,1}] = -4 - 2G_{2,1} + G'_{1} + G'_{2},
\]
there are three new clusters produced in the second order calculation, where the three clusters denoted by \(G_{2,1}, G'_{1}, G'_{2}\) are given in Fig. 1. Two one which consist of two Wilson loops turn to cluster \(G_{1,1}\) times \(<G_{1,1}>\) by RPA. Therefore, there is only one cluster at second order after applying the PRA, that is \(G_{2,1}\), and

\[
R_2 = c_{2,1}G_{2,1}.
\]

Substituting \(R_1\) and \(R_2\) into Eq. (8), we obtain a set of equations about \(c_{1,1}, c_{2,1}\) and \(w_0\) with a parameter \(<G_{1,1}>\). The parameter \(<G_{1,1}>\) can be determined by Eq. (11). Solving those equations, we get the second order approximation of vacuum wave function \(\psi_0(U) \approx e^{R_1(U)+R_2(U)}\).

The long wavelength coefficients up to the second order are

\[
\mu_0 = \left[\frac{c_{1,1}}{2} + 2c_{2,1}\right]g^4,
\]

\[
\mu_2 = -\frac{c_{2,1}}{4}g^8.
\]

The third order clusters are produced by term \([E_{t}^{a}, R_1][E_{t}^{a}, R_2]\), i.e. \(c_{1,1}c_{2,1}[E_{t}^{a}, G_{1,1}][E_{t}^{a}, G_{2,1}]\).

Since

\[
[E_{t}^{a}, G_{1,1}][E_{t}^{a}, G_{2,1}] = -2G_{3,1} - G_{3,2} + G'_{3} + \frac{1}{2}G'_{4} + \frac{3}{2}G'_{5} - 3G_{1,1}
\]

\[
\approx -2G_{3,1} - G_{3,2} + 3 <G_{1,1}> G_{2,1} - 3G_{1,1},
\]

we get

\[
R_3 = c_{3,1}G_{3,1} + c_{3,2}G_{3,2}.
\]

There are only two independent clusters at order 3, while it is nine in the expansion without RPA [2], so the calculation is much simpler. In Eq. (16), when applying RPA to cluster \(G'_{3}, G'_{4}\) and \(G'_{5}\), we replace the small Wilson loop with its vacuum average and let the large one remain unchanged as in Ref. [8]. Thus, only one vacuum average of cluster emerges in the third order calculation. This makes the calculation simple further. On the other hand, the vacuum or exciting states possess definite correlation lengths. Only when the space occupied by glueball is covered with the Wilson loops, the calculation is efficient, so, we replace the small loop with its vacuum average and preserve the large loop. Submitting \(R_1, R_2\) and \(R_3\) into Eq. 8, and solving the equation, we obtain the third order approximation of the vacuum wave function.

Higher order calculation can be carried out similarly. We have done the calculation up to the seventh order. the number of independent clusters are 1, 1, 2, 6, 14, 44, and 109 at order 1, 2, 3, 4, 5, 6, and 7 respectively. The results for \(\mu_0\) and \(\mu_2\) from the fourth order to the seventh order are presented in Fig. 2. In Fig. 3, the vacuum energy \(w_0\) against \(\beta\) are plotted.
4 Results and discussions

From Fig. 2, we see that the curves of $\mu_0$ (or $\mu_2$) show good scaling behavior and convergent trend in weak coupling region $\beta = 5.5$ to $9.5$. The sixth and seventh order values of $\mu_0$ are coincident in the scaling region, which shows that the approach is effective and rapidly convergent. From the seventh order results, we obtain

$$\mu_0 = 2.3,$$
$$\mu_2 = -0.3.$$  \hspace{1cm} (18) \hspace{1cm} (19)

A Monte Carlo measurement was gave by Arisue [12]

$$\mu_0 = (0.91 \pm 0.02),$$
$$\mu_2 = -(0.19 \pm 0.05).$$  \hspace{1cm} (20)

The value of $\mu_0$ in Eq. (18) is larger than Arisue’s value. It is perhaps that there is some systematic error introduced by the random phase approximation in our procedure. But it is also probable that the Monte Carlo results of Eq. (20) was not enough accurate. Because the values in Eq. (20) can be reproduced in the low order (the third order) expansion without RPA, while in higher order (the fourth order) calculation the values became large [2].

In Fig. 3, the curve of the sixth and seventh order results are almost coincident, which proves the expansion is able to converge rapidly again. We also give the third order result of the vacuum energy calculated without RPA for comparing. The third order values of the vacuum energy with RPA are lower than that without RPA. Such a case was also true in the case of SU(3) LGT [8].

We think it means that the vacuum state obtained by using RPA is better than that without using RPA, and it may explain the rapid convergence of the approach.

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References


Figure Captions

Fig.1  The linked clusters used in the expansions up to order 3.

Fig.2  $\mu_0$ and $\mu_2$ as a function of $\beta = 4/g^2$. The four curves represents the results from the fourth order to the seventh order expansion with RPA respectively.

Fig.3  The vacuum energy vs $\beta$. We also give the third order vacuum energy calculated without using RPA.
Fig. 1  The linked clusters used in the expansions up to order 3.