Sufficient integral criteria for instability of the free charged surface of an ideal liquid

Nikolay M. Zubarev and Olga V. Zubareva

Institute of Electrophysics, Ural Branch, Russian Academy of Sciences,
106 Amundsen Street, 620016 Ekaterinburg, Russia

Applying the method of integral estimates to the analysis of three-wave processes we derive the sufficient criteria for the hard loss of stability of the charged plane surface of liquids with different physical properties. The influence of higher-order wave interactions on the instability dynamics is also discussed.

As we know [1, 2, 3, 4], the dispersion relation for electrocapillary waves on a charged liquid surface has the following form:

\[ \omega^2 = gk + \frac{\alpha}{\rho} k^3 - \frac{P}{4\pi\rho} k^2 \]  

(1)

where \( k = |k| \) is the wave number, \( \omega \) is the frequency, \( g \) is the acceleration of gravity, \( \alpha \) is the surface tension, \( \rho \) is the mass density, and \( P \) is an external control parameter, depending on liquid nature. So, \( P = E^2 \) for an ideal perfectly conducting liquid (liquid metal in applications), where \( E \) is the external electric field strength. For an ideal dielectric liquid the parameter \( P \) equals the expression \( E^2 (\varepsilon - 1)^2 / (\varepsilon^2 + \varepsilon) \), where \( \varepsilon \) is the permittivity. For liquid helium and liquid hydrogen it holds \( P = E_+^2 + E_-^2 \), where \( E_+ \) and \( E_- \) designate the electric field strength above and below the fluid surface, respectively. One can see from Eq. (1) that if the control parameter \( P \) exceeds the critical value,

\[ P_c = 8\pi\sqrt{g\alpha\rho}, \]

then \( \omega^2 < 0 \) and, as a consequence, an aperiodic instability develops. Thus, the condition \( P > P_c \) is the criterion for the surface instability with respect to infinitesimal perturbations of the surface shape and the velocity field.

As shown in Refs. [5, 6, 7], nonlinear interactions between three standing waves, which form a hexagonal structure, can lead to the hard loss of stability of a charged liquid surface.

*Electronic address: nick@ami.uran.ru
Then, even if the value of the control parameter is subcritical, i.e., $P < P_c$, a fairly large-amplitude perturbation can remove the system from equilibrium, resulting either in the formation of a perturbed stationary surface profile or in infinite growth of the amplitudes of surface perturbations in accordance with character of higher-order wave processes. An important task of physical significance is therefore to obtain criteria for instability of the plane surface with respect to perturbations with finite amplitudes, namely, to determine the initial conditions (surface configuration and distribution of the velocity field) which lead to the development of an instability.

In the present Letter we construct and discuss such criteria for the case of a conducting liquid in an external electric field (all our results may be extended to other fluids) by using the method of majoring equations, which was applied previously to the nonlinear Schrödinger equation (see, for example, Refs. [8, 9]), the nonlinear Klein-Gordon equation [10, 11], different modifications of the Boussinesq equation [12], and so on. This method allows us to derive a number of sufficient integral criteria for the loss of stability of the plane charged liquid surface, with most of them corresponding to the subcritical values of the control parameter $P$, when the surface is stable in the linear approximation and the instability onset is caused by nonlinear processes.

So, let us consider the irrotational motion of a perfectly conducting ideal liquid of infinite depth with a free surface $z = \eta(x, y, t)$ in an external electric field $E$ directed along the $z$ axis. The velocity potential $\Phi$ and the potential of the electric field $\varphi$ obey the Laplace equations

$$\nabla^2 \Phi = 0, \quad \nabla^2 \varphi = 0$$

with the conditions at infinity:

$$\Phi \to 0, \quad z \to -\infty,$$

$$\varphi \to -Ez, \quad z \to \infty.$$

The conditions on the free surface are

$$\frac{\partial \Phi}{\partial t} + \frac{(\nabla \Phi)^2}{2} = \frac{(\nabla \varphi)^2 - E^2}{8\pi \rho} + \frac{\alpha}{\rho} \frac{\nabla \cdot \nabla \eta}{\sqrt{1 + (\nabla \eta)^2}} - g \eta, \quad z = \eta,$$

$$\frac{\partial \eta}{\partial t} = \frac{\partial \Phi}{\partial z} - \nabla \eta \cdot \nabla \Phi, \quad z = \eta,$$
and, since the surface of a conducting liquid is equipotential,

\[ \varphi = 0, \quad z = \eta. \]

The functions \( \eta(x, y, t) \) and \( \phi(x, y, t) = \Phi|_{z=\eta} \) are canonically conjugate \[13\], so that the equations of motion take the Hamiltonian form,

\[ \frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta}, \quad \frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}, \]

where the Hamiltonian coincides with the total energy of the system:

\[ H = H_{\text{kin}} + H_{\text{pot}}, \]

\[ H_{\text{kin}} = \int_{z \leq \eta} \frac{(\nabla \Phi)^2}{2} d^3r, \]

\[ H_{\text{pot}} = -\int_{z \geq \eta} \frac{(\nabla \varphi)^2}{8\pi\rho} d^3r + \int \left[ \frac{g\eta^2}{2} + \frac{\alpha}{\rho} \left( \sqrt{1 + (\nabla \eta)^2} \right) \right] d^2r. \]

It is possible to express \( H \) explicitly in terms of the canonical variables. Rewriting the Hamiltonian in the form of a surface integral with the help of Green’s formulas and expanding the integrand in powers series of \( \psi \) and \( \eta \), we obtain

\[ H_{\text{kin}} = \int \frac{\psi^2}{2} \left( \hat{T}_+ \hat{k}^{\hat{T}_+^{-1}} \psi - \nabla_\perp \eta \cdot \hat{T}_+ \nabla_\perp \hat{T}_+^{-1} \psi \right) d^2r, \]

\[ H_{\text{pot}} = -\int \frac{E^2 \eta}{8\pi\rho} \left( \hat{T}_- \hat{k}^{\hat{T}_-^{-1}} \eta + \nabla_\perp \eta \cdot \hat{T}_- \nabla_\perp \hat{T}_-^{-1} \eta \right) d^2r \]

\[ + \int \left[ \frac{g\eta^2}{2} + \frac{\alpha}{\rho} \left( \sqrt{1 + (\nabla_\perp \eta)^2} \right) \right] d^2r. \]

Here \( \hat{k} \) is the two-dimensional integral operator with a difference kernel whose Fourier transform is equal to the absolute value of the wave vector \( (\hat{k}e^{i\mathbf{kr}} = |\mathbf{k}|e^{i\mathbf{kr}}) \), and the nonlinear operators \( \hat{T}_\pm \) are defined by the expressions

\[ \hat{T}_\pm = \sum_{n=0}^{\infty} \frac{(\pm \eta)^n \hat{k}^n}{n!}. \]

For values of the control parameter \( \epsilon = (P - P_c)/P_c \) close to threshold \( (|\epsilon| \ll 1) \), as is clear from the dispersion relation (1), the surface perturbations with \( k \approx k_0 = \sqrt{g\rho/\alpha} \) have the
maximum linear growth rates. Then the nonlinear dynamics of the surface perturbations can be effectively studied with the help of the amplitude equation approach. Assuming the surface-slope angles to be small, $|\nabla_\perp \eta|$, we introduce the slowly varying amplitudes $A_j$, $j = 1, 2, 3$, by means of the substitutions

$$
\eta(r,t) = \frac{2}{3k_0} \sum_{j=1}^{3} A_j(x_j, y_j, t) \exp(i k_j r) + \text{c.c.,}
$$

$$
\psi(r,t) = \frac{2}{3k_0^2} \sum_{j=1}^{3} \frac{\partial A_j(x_j, y_j, t)}{\partial t} \exp(i k_j r) + \text{c.c.},
$$

where the wave vectors $k_j$ with $|k_j| = k_0$ make angles of $2\pi/3$ with each other, and the variables $x_j$, $y_j$ form the orthogonal coordinate systems with $x_j$ axis directed along the wave vectors $k_j$. This representation for $\eta$ and $\psi$ corresponds to a hexagonal structure of the surface perturbations, which is preferred at the initial stages in the development of the instability.

Substituting the relations for $\eta$ and $\psi$ into Eqs. (2), (5) and (6), in the leading order we get the following expression for the averaged Hamiltonian:

$$
H = \int \left[ \sum_{j=1}^{3} \left( |A_j|^2 + |\hat{L}_j A_j|^2 - \epsilon |A_j|^2 \right) - A_1 A_2 A_3 - A_1^* A_2^* A_3^* \right] d^2 r,
$$

(7)

where we convert to dimensionless variables,

$$
r \rightarrow r/(\sqrt{2}k_0), \quad t \rightarrow t/\sqrt{2gk_0}, \quad H \rightarrow 8gH/(9k_0^2),
$$

and introduce the linear differential operators

$$
\hat{L}_j = \frac{\partial}{\partial x_j} - \frac{i}{2} \frac{\partial^2}{\partial y_j^2}, \quad j = 1, 2, 3.
$$

The equations for the complex amplitudes $A_j$ corresponding to this Hamiltonian are

$$
\frac{\partial^2 A_j}{\partial t^2} = \epsilon A_j + \hat{L}_j^2 A_j + \frac{A_1^* A_2^* A_3^*}{A_j^*}, \quad j = 1, 2, 3.
$$

(8)

It should be noted that the multiplier $p = (\epsilon - 1)/(\epsilon + 1)$ and the multiplier $s = (E_+^2 - E_-^2)/P_c$ appear before the nonlinear terms in the right-hand sides of Eqs. (8) for dielectric liquid and for liquid helium, respectively (the equations reduce to the form (8) by scaling the amplitudes $A_j \rightarrow A_j/p$ or $A_j \rightarrow A_j/s$).
Let us find the sufficient criteria for unlimited growth of the amplitudes $A_j$ in a finite time, i.e., the criteria of the blow-up for Eqs. (8), which describe the nonlinear interactions between three standing waves. Consider the time evolution of the following positive quantity:

$$X = \sum_{j=1}^{3} X_j, \quad X_j(t) = \int |A_j|^2 d^2 r.$$ 

Differentiating $X$ twice with respect to $t$ and then making use of Eqs. (8), we get

$$X_{tt} = \int \left[ 2 \sum_{j=1}^{3} \left( |A_j|^2 - |\hat{L}_j A_j|^2 + \epsilon |A_j|^2 \right) + 3A_1A_2A_3 + 3A_1^*A_2^*A_3^* \right] d^2 r.$$ 

Excluding the cubic terms from the integrand with the help of the expression (7), we come to the relation

$$X_{tt} + 3H = -\epsilon X + \sum_{j=1}^{3} \int \left[ 5|A_j|^2 + |\hat{L}_j A_j|^2 \right] d^2 r. \quad (9)$$

It follows from the integral Hölder inequality for the functions $|A_j|$ and $|A_j|^2$ that

$$4X_j \int |A_j|^2 r^2 \geq X_j^2.$$ 

On the other hand, the algebraic Cauchy inequality yields

$$\left( \sum_{j=1}^{3} X_j \right) \cdot \left( \sum_{j=1}^{3} \frac{X_j^2}{X_j} \right) \geq \left( \sum_{j=1}^{3} X_j \right)^2.$$ 

As a consequence, we can estimate the second term in the right-hand side of Eq. (9):

$$\sum_{j=1}^{3} \int |A_j|^2 d^2 r \geq \frac{X_t^2}{4X}.$$ 

In addition, taking into account that

$$\int |\hat{L}_j A_j|^2 d^2 r \geq 0,$$

we obtain the following second order differential inequality:

$$X_{tt} + 3H \geq \frac{5}{4} \frac{X_t^2}{X} - \epsilon X. \quad (10)$$

Solving the inequality one may find the sufficient conditions under which the integral value $X$ becomes infinite in a finite time. It should be noted that the similar majoring inequalities were derived in Refs. [8–12] as a result of investigation of the blow-up in different well-known nonlinear partial differential equations.
Under the substitution, \( Y = X^{-1/4} \), inequality (10) takes the Newtonian form

\[
Y_{tt} \leq -\frac{\partial P(Y)}{\partial Y}, \quad P(Y) = -\frac{1}{8} \left( \epsilon Y^2 + HY^6 \right),
\]

where \( Y \) can be considered as a coordinate of some particle and the function \( P(Y) \) plays the role of the potential. Let the particle velocity \( Y_t \) be negative (in this case \( X_t > 0 \)). Then, multiplying the inequality (11) by \( Y_t \) and integrating it over time, we get

\[
U_t(t) \geq 0, \quad U(t) = Y_t^2/2 + P(Y),
\]

i.e., the particle energy \( U \) increases with time. It is clear that if the condition \( U_t = 0 \) holds, which corresponds to the equality sign in the expression (11), and the particle does not encounter a potential barrier, then it reaches the origin, i.e., the point \( Y = 0 \), and, consequently, the positive-definite quantity \( X \) becomes infinite. The collapse takes place

(a) if \( \epsilon < 0 \) and \( H > 0 \), provided that \( Y(t_0) < |\epsilon|^{3/4}/(3)^{1/2} \) and \( 12U(t_0) \leq |\epsilon|^{3/4}/(3H)^{1/2} \);

(b) if \( \epsilon < 0 \) and \( H > 0 \), provided that \( 12U(t_0) > |\epsilon|^{3/4}/(3H)^{1/2} \);

(c) if \( \epsilon < 0 \) and \( H \leq 0 \);

(d) if \( \epsilon \geq 0 \), provided that \( U(t_0) > 0 \),

where \( t = t_0 \) corresponds to the initial moment. The collapse time \( t_c \) at which the amplitudes \( A_j \) go to infinity can be estimated from above as follows,

\[
t_c \leq t_0 + \int_0^{Y(t_0)} \frac{dY}{\sqrt{2U(t_0) - 2P(Y)}}.
\]

Note that the condition \( Y_t(0) < 0 \) is optional for the cases (a) and (c). Since \( U_t(t) \leq 0 \) for \( Y_t > 0 \), in these two cases the particle always reaches the point \( Y = 0 \) after the reflection from the potential barrier.

It is important that the conditions (a)–(d) may serve as the sufficient criteria for the instability of the plane surface of a conducting liquid in the near-critical electric field with respect to finite amplitude perturbations. This distinguishes our criteria from the simplest criterion for linear instability, \( P > P_c \), which corresponds to infinitesimal perturbations. Actually, the conditions (a), (b) and (c) are relative to the case of the subcritical electric
field strength, $P < P_c$, when the surface is stable in the linear approximation. It is obvious that we deal with hard excitation of the electohydrodynamic instability.

Thus, we have shown that if the conditions (a)–(d) are valid, then the equations (8) describe the unlimited growth of the amplitudes $A_j$. However, the applicability of Eqs. (8), i.e., the possibility of limiting the treatment to three-wave processes, assumes that the perturbation amplitudes are small ($|A_j|$ are the order of magnitude of the control parameter $\epsilon$). The question arises as to whether the neglected higher-order nonlinearities stabilize the instability, or, on the contrary, higher-order wave processes promote an explosive growth of the amplitudes. Notice that both the experimental data [14] and the results of the numerical calculations [15] indicate that the higher-order nonlinearities have the destabilizing influence.

Let us show that, in particular, the four-wave interaction does not saturate the explosive instability. Consider the simplest case when the perturbation amplitudes do not depend on the spatial variables $x$ and $y$, and the surface configuration is given by

$$\eta(r, t) = 3 \sum_{j=1}^{3} (\xi_j + 2k_0\xi_j^2) + \frac{k_0}{2}(5\sqrt{3} + 6)(\xi_1\xi_2^* + \xi_2\xi_3^* + \xi_3\xi_1^*) + \text{c.c.},$$

$$\xi_j(r, t) = a_j(t) \exp(i k_j r),$$

where the nonlinear interactions between fundamental and combination harmonics, $k_0 \leftrightarrow 2k_0$ and $k_0 \leftrightarrow \sqrt{3}k_0$, are taken into account. Substituting these expressions into (6), we obtain the fourth-order correction for the potential energy (4):

$$H^{(4)}_{\text{pot}} = -g k_0^2 \sum_{j=1}^{3} \int \left[ \frac{11}{4}|a_j|^4 + \frac{1}{2}(25\sqrt{3} + 13)|a_j|^{-2} \prod_{j=1}^{3}|a_j|^2 \right] d^2 r.$$  

This functional is the negative-definite quantity, $H^{(4)}_{\text{pot}} \leq 0$. Consequently, we can assume that the potential energy $H_{\text{pot}}$ decreases indefinitely as the perturbations grow. Then the kinetic energy $H_{\text{kin}}$, which is the positive-definite quantity, increases infinitely (see Eqs. (2) and (3)). Thus, the higher-order nonlinearities do not retard the explosive growth of the amplitudes in the model (8), and the integral criteria (a)–(d) can be considered as the sufficient criteria for the unlimited growth of the perturbations of a conducting liquid surface in an applied electric field.

As for a dielectric liquid in the near-critical electric field, we have

$$H^{(4)}_{\text{pot}} = -g k_0^2 \sum_{j=1}^{3} \int \left[ \left( 4p^2 - \frac{5}{4} \right)|a_j|^4 + \frac{1}{2} \left( 15 - 12\sqrt{3} + p^2(37\sqrt{3} - 2) \right)|a_j|^{-2} \prod_{j=1}^{3}|a_j|^2 \right] d^2 r.$$
It can readily be seen that $H^{(4)}_{\text{pot}} \leq 0$ for arbitrary $a_j$ if $\varepsilon \geq \varepsilon_{2D}$ holds, where $\varepsilon_{2D} \approx 3.53$ (at this value of the permittivity the hard instability regime changes to the soft one in 2D geometry \cite{16}). Then the relations (a)-(d) represent the criteria for the blow-up-type dynamics of the surface perturbations. If $1 < \varepsilon \leq \varepsilon_{3D}$ holds, where $\varepsilon_{3D} \approx 2.05$ (the functional $H^{(4)}_{\text{pot}}$ with $|a_1| = |a_2| = |a_3|$ changes the sign precisely at this value of the permittivity $\varepsilon$ \cite{6}), then $H^{(4)}_{\text{pot}} \geq 0$ and, consequently, the four-wave processes can stabilize the instability, resulting in the appearance of the stationary hexagonal structures. In this case the conditions (a)-(c) are the criteria for the hard excitation of the stationary wave patterns on the free surface of an ideal dielectric liquid.

For liquid helium (or hydrogen) with a charged surface it holds

$$H^{(4)}_{\text{pot}} = -g k_0^2 \sum_{j=1}^{3} \int \left[ \left( 4s^2 - \frac{5}{4} \right) |a_j|^4 + \frac{1}{2} \left( 16\sqrt{3} - 29 + s^2(9\sqrt{3} + 42) \right) |a_j|^2 \prod_{j=1}^{3} |a_j|^2 \right] d^2r.$$  

One can find that $H^{(4)}_{\text{pot}} \leq 0$ for $|s| \geq |s_{2D}|$, where $s_{2D} \approx 0.56$ (at this value of the parameter $s$ the functional $H^{(4)}_{\text{pot}}$ changes its sign in 2D case \cite{4}), and $H^{(4)}_{\text{pot}} \geq 0$ for $|s| \leq |s_{3D}|$, where $s_{3D} \approx 0.33$.

The authors are grateful to E.A. Kuznetsov for stimulating discussions, and also to A.M. Iskoldsky and N.B. Volkov for their interest in this work. The work was supported by the Russian Fund for Fundamental Research (Project No. 00-02-17428) and, partly, by the INTAS Fund (Project No. 99-1068).

\begin{thebibliography}{99}
\bibitem{1} L. Tonsk, Phys. Rev. 48 (1935) 562.
\bibitem{2} Ya.I. Frenkel, Zh. Teh. Fiz. 6 (1936) 347.
\bibitem{5} A. Gailitis, Magnitnaya Gidrodinamika 1 (1969) 68.
\bibitem{7} H. Ikezi, Phys. Rev. Lett. 42 (1979) 1628.
\end{thebibliography}