The Doppler effect from a uniformly moving mirror

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ABSTRACT

The formula for the frequency shift of a plane-polarized light wave reflected from a uniformly moving mirror is derived directly from the constant light speed postulate and the basic principles of wave optics. Unlike the original derivation by Einstein, our derivation employs only the notion of the wave nature of light, and it does not refer to its electromagnetic character. As such, it lies within the scope of a first course in special relativity.

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A plane-polarized light wave reflected from a uniformly moving mirror will suffer a frequency change. This special case of the Doppler effect was predicted and described by Einstein in his famous 1905 paper on special relativity [1]. Einstein employed the standard Lorentz-transformation procedure on the electromagnetic characteristics of the wave to show that the value of this frequency shift depends on the angle of incidence of the wave and the velocity of the moving mirror. The purpose of the present paper is to derive the formula for this Doppler shift directly from the constant light speed postulate and the basic principles of wave optics without referring to the electromagnetic character of the light.
Figure 1 shows a plane-polarized light wave incident on a vertical plane mirror. The mirror is moving at a constant velocity $v$ to the right. We denote by $\alpha$ the angle of incidence, and by $\beta$ the angle of reflection of the light. We let $A$ and $B$ to be the points of two successive wavefronts of the incident light a distance $AB = \lambda_0$ apart, where $\lambda_0$ equals the wavelength of the incident light. The wavefront at point $B$ will be reflected by the mirror at time $t_0$. On the other hand, the wavefront at $A$ will have to travel the path $AA' = c(t - t_0)$ before being reflected by the mirror. The reflection will occur off the point $A'$ at time $t$, when the mirror is located a distance $v(t - t_0)$ to the right from its position at time $t_0$. For this same amount of time $(t - t_0)$ required for the wavefront at $A$ to cover the path $AA'$, the wavefront reflected at $B$ will traverse the path $BB' = c(t - t_0)$.

We have taken into account that the speed of the reflected wavefront will equal $c$ as a consequence of the constant light speed postulate.

Observe from Fig. 1 that the paths of the reflected wavefronts will not coincide due to the shift of the wavefronts caused by the motion of the mirror [2]. Also observe that the distance $A'B'$ is the shortest (orthogonal) distance between the planes of the reflected wavefronts located at $A'$ and $B'$, and it equals the wavelength $\lambda$ of the reflected light. From Fig. 1 we have

$$AA' = AB + BA', \quad (1)$$

and

$$A'B' = A'B + BB'. \quad (2)$$

From the triangles $A'BC$ and $A'A''B$, we have

$$BA' = \frac{v(t - t_0)}{\cos \alpha}, \quad (3)$$

and

$$A'B = BA' \cos(\alpha + \beta) = v(t - t_0) \frac{\cos(\alpha + \beta)}{\cos \alpha}. \quad (4)$$
Taking into account Eqs. (3) and (4) and the discussion associated with Fig. 1, Eqs. (1) and (2) can be rewritten as

\[ c(t-t_0) = \lambda_0 + \frac{v(t-t_0)}{\cos \alpha}, \quad (5) \]

and

\[ \lambda = v(t-t_0) \frac{\cos(\alpha + \beta)}{\cos \alpha} + c(t-t_0). \quad (6) \]

We eliminate the term \((t-t_0)\) from Eqs. (5) and (6) to get

\[ \frac{\lambda}{\lambda_0} = \frac{c + v \cos(\alpha + \beta)}{c - \frac{v}{\cos \alpha}}. \quad (7) \]

By using the law of reflection of light from a uniformly moving vertical mirror [1,2,3]

\[ \cos \beta = \frac{-2 \frac{v}{c} + \left(1 + \frac{v^2}{c^2}\right) \cos \alpha}{1 - 2 \frac{v}{c} \cos \alpha + \frac{v^2}{c^2}} \quad (8) \]

and some algebra, Eq. (7) can be transformed into

\[ \frac{\lambda}{\lambda_0} = \frac{1 - \frac{v^2}{c^2}}{1 - 2 \frac{v}{c} \cos \alpha + \frac{v^2}{c^2}}. \quad (9) \]

Hence, the formula that connects the frequencies \(f_0\) and \(f\) of the incident and the reflected light, respectively, states
\[
f = f_0 \frac{1 - 2 \frac{v}{c} \cos \alpha + \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}},
\]

and it coincides with Einstein’s formula obtained with Lorentz transformations [1].

The formula for the Doppler effect from a uniformly moving mirror is usually not considered in the standard textbooks on special relativity, although its origin belongs to the foundations of the theory. We hope that the method of derivation in this paper will bring this important issue into the undergraduate classroom and will enthuse the student due to its simplicity to arrive at the same Doppler formula.

REFERENCES


Fig. 1. The derivation of the formula for the Doppler shift from a uniformly moving vertical mirror. The mirror is moving at a constant velocity $v$ to the right. The planes of the incident and the reflected wavefronts are perpendicular to their paths.