Pion scattering on the lattice with chirally improved fermions

Christof Gattringer, Dieter Hierl, and Rainer Pullirsch
(for the BGR [Bern-Graz-Regensburg] Collaboration)

Institut f"ur Theoretische Physik, Universit"at Regensburg, 93040 Regensburg, Germany.

We report preliminary results for the $I = 2$ pion scattering length $a_0$, calculated with chirally improved fermions. After chiral extrapolation our results are in good agreement with both, experimental and theoretical predictions.

1. Introduction

Since the pion is the goldstone boson of the spontaneous breaking of chiral symmetry, observables related to the pion give insight into the chiral dynamics of QCD. For example the pion scattering length is a parameter entering the effective chiral lagrangian of QCD and a precise determination of this observable in an ab-initio lattice calculation is desirable. On the lattice scattering data of stable particles can be calculated using L"uscher’s formula \[1\] which relates the scattering length to the energy $W$ of the scattering state. The crucial technical challenge is a very precise determination of $W$ and advanced strategies for error reduction have to be used. For chirally sensitive quantities, such as the pion scattering length, it is important to be able to work with small pion masses. Using fermions based on the Ginsparg-Wilson equation it is now possible to reach pion masses much lower than what was accessible with more traditional formulations. This implies that the chiral extrapolation of the results to the physical pion mass becomes more reliable.

2. Setting of our calculation

Our calculation is based on the chirally improved (CI) Dirac operator \[2\]. CI fermions are a systematic approximation of a solution of the Ginsparg-Wilson equation and have been tested successfully \[3\] in spectroscopy down to pion masses of $m_\pi = 265$ MeV.

We use gauge configurations generated with the L"uscher-Weisz action. Our quark masses are in the range of $am = 0.015$ to $am = 0.2$. The scale was determined in \[4\] using the Sommer parameter. The parameters of the three ensembles we used are listed in Table 1.

3. L"uscher’s formula

If two stable particles are confined in a box of size $L$, the energy $W$ of the corresponding scattering state can be expanded \[1\] in a power series in $L^{-1}$. The coefficients of this series are related to the elastic scattering amplitude,

\[
W - 2m_\pi = -\frac{4\pi a_0}{m_\pi L^3} \left[1 + c_1 \frac{a_0}{L} + c_2 \left(\frac{a_0}{L}\right)^2 \right] \ldots \ , \ (1)
\]

where the omitted terms are $O(L^{-6})$ and $c_1$, $c_2$ are numerical constants that are known analytically ($c_1 = -2.837297, c_2 = 6.375183$). Thus, for calculating the $s$-wave scattering length $a_0$, one has to compute $m_\pi$ and $W$ on the lattice and to solve the cubic equation \[1\] for $a_0$. The challenge

<table>
<thead>
<tr>
<th>ensemble</th>
<th>$\beta$</th>
<th>$a$(fm)</th>
<th>$L$(fm)</th>
<th>size</th>
<th>conf.</th>
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<tr>
<td>A</td>
<td>7.90</td>
<td>0.15</td>
<td>2.4</td>
<td>$16^3 \times 32$</td>
<td>97</td>
</tr>
<tr>
<td>B</td>
<td>7.90</td>
<td>0.15</td>
<td>1.8</td>
<td>$12^3 \times 24$</td>
<td>100</td>
</tr>
<tr>
<td>C</td>
<td>8.35</td>
<td>0.10</td>
<td>1.6</td>
<td>$16^3 \times 32$</td>
<td>97</td>
</tr>
</tbody>
</table>

Table 1. Parameters and statistics of our quenched ensembles.
is to obtain the difference $W - 2m_\pi$ with high accuracy since the solution of the cubic equation is very sensitive to this difference.

4. Two pion scattering state energy

We calculate the pion mass $m_\pi$ from a 2-parameter fit of the pseudoscalar correlator ($T$ denotes the temporal extent of the lattice)

$$\langle C_2(t) \rangle = \langle \hat{P}(t) P^\dagger(0) \rangle \sim A \cosh (m_\pi [T/2 - t]),$$

(2)

where $P^\dagger$ generates $I = 1$ pseudoscalar mesons and $\hat{P}$ denotes the zero-momentum Fourier transform of the corresponding annihilator.

The energy $W$ of the scattering states can be extracted from

$$\langle C_4(t) \rangle = \langle \hat{P}(t) \hat{P}(t) P^\dagger(0) P^\dagger(0) \rangle.$$  

(3)

In both, the 2- and the 4-point correlators, we use Jacobi smeared quark sources and sinks. We also experimented with wall-type sources but found no improvement of the signal.

While the pion mass can be extracted quite accurately from (2), a direct determination of the scattering state energy $W$ from a fit to (3) is unsatisfactory. A successful method [5] to reduce the statistical noise is to consider the ratio

$$\langle C_4(t) \rangle / \langle C_2(t)^2 \rangle \sim \frac{A + B \cosh (W [T/2 - t])}{\cosh (m_\pi [T/2 - t])^2},$$

(4)

where fluctuations of the correlators on single configurations cancel. We first determine $m_\pi$ from (2) and subsequently use this value in the 3-parameter fit (4) which determines $W$. Our fits are fully correlated and errors were determined with the jackknife method.

5. Chiral extrapolation

For the chiral extrapolation of our data we use chiral perturbation theory. In one-loop chiral perturbation theory one finds [6]

$$a_0 = \frac{-m_\pi}{16\pi f_0^2} \left[ 1 - \frac{m_\pi^2}{f_0^2} \left( G_\Lambda - \frac{7 \ln(m_\pi^2)}{32\pi^2} + \ldots \right) \right].$$

(5)

$G_\Lambda$ is a (scale dependent) combination of LEC’s and $f_0$ is the pion decay constant in the chiral limit. However, we do not fit our data directly to the form (5), but consider two combinations of $a_0$ with $m_\pi$. To remove the leading $m_\pi$-dependence of the scattering length we study the ratio

$$F_1(m_\pi^2) = \frac{a_0}{m_\pi}.$$  

(6)

We also consider the product

$$F_2(m_\pi^2) = a_0 m_\pi,$$  

(7)

which is often used by experiments. We perform the chiral extrapolation for both $F_1$ and $F_2$ using the accordingly modified right hand side of (5). We furthermore experimented with adding higher order terms from chiral perturbation theory but did not observe a large effect.

6. Results

In Fig. 1 we present our results for the ratio $F_1$. The open symbols are our data and the filled symbols are their extrapolation to the physical pion mass.
mass. The asterisk represents the experimental value. When comparing the two ensembles B (triangles) and C (circles) which are similar in size ($L = 1.8$ fm and $L = 1.6$ fm) but have a different lattice spacing ($a = 0.15$ fm and $a = 0.1$ fm) we find, at least for larger pion masses, good agreement of the data points. This indicates that discretization errors are small, similar to what was already found in quenched hadron spectroscopy with the chirally improved Dirac operator [3]. A comparison with the data obtained on our larger volume (ensemble A, $L = 2.4$ fm, $a = 0.15$ fm) shows, however, that at the smaller volume with $L = 1.8$ fm we still observe a finite size effect.

The same picture also holds for the analysis of $F_2$. Again the two ensembles with small physical volume but different lattice spacing agree reasonably well, indicating that scaling violations are small. The larger volume gives lower values for $F_2$, showing that at $L = 1.8$ fm finite size effects cannot be neglected.

Because of the good scaling and the finite size effect we expect the results from ensemble A ($16^3 \times 32, a = 0.15$ fm, $L = 2.4$ fm) to be our best estimate. For this ensemble we list in Table 2 our results for $a_0 m_\pi$ obtained from the extrapolation of both $F_1(m_\pi^2)$ and $F_2(m_\pi^2)$ to the physical pion mass. We compare our final results to recent experimental data [7] and to a two-loop result from chiral perturbation theory [8]. Given the fact that our calculation is quenched, the agreement of our results with the experimental number is surprisingly good.

<table>
<thead>
<tr>
<th></th>
<th>extrap.</th>
<th>our data</th>
<th>experiment</th>
<th>$\chi$-PT</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1(m_\pi^2)$</td>
<td>-0.0453(35)</td>
<td>-0.0454(34)</td>
<td>-0.0444(10)</td>
<td></td>
</tr>
<tr>
<td>$F_2(m_\pi^2)$</td>
<td>-0.0425(97)</td>
<td>-0.0454(34)</td>
<td>-0.0444(10)</td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Our final results for $a_0 m_\pi$ compared to experimental data and to 2-loop chiral perturbation theory.

Acknowledgement

We thank Jimmy Juge for important comments and discussions. Our computations were done on the Hitachi SR8000 at LRZ in Munich and at the RZ Regensburg.

REFERENCES