Heavy meson chiral perturbation theory in finite volume

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We present the first step towards the estimation of finite volume effects in heavy-light meson systems using heavy meson chiral perturbation theory. We demonstrate that these effects can be amplified in both light-quark and heavy-quark mass extrapolations (interpolations) in lattice calculations. As an explicit example, we perform a one-loop calculation for the neutral $B$ meson mixing system and show that finite volume effects, which can be comparable with currently quoted errors, are not negligible in both quenched and partially quenched QCD.

1. INTRODUCTION

Advances in lattice QCD have lead to the prospect of numerical calculations for hadronic matrix elements with high precision, which will have significant impact on the progress of particle physics in the near future. One of the areas where lattice QCD plays an essential rôle is the search for physics beyond the Standard Model via over-constraining the $b-d$ unitarity triangle in the Cabibbo-Kobayashi-Maskawa (CKM) matrix. This requires the control of systematic errors at the percentage level for certain hadronic matrix elements in heavy-light meson mixing and decays.

Here we present the first step towards the estimation of finite-volume effects in heavy-light meson systems\textsuperscript{[1]} in the framework of heavy meson chiral perturbation theory (HM\textsubscript{\chi}PT)\textsuperscript{[2] [3] [4] [5] [6]}, with first order $1/M_P$ and chiral corrections, assuming the mass hierarchy

$$M_{GP} \ll \Lambda_{\chi} \ll M_P,$$

where $M_{GP}$ is the mass of any Goldstone particle, $M_P$ is the mass of the heavy-light meson, and $\Lambda_{\chi}$ is the chiral symmetry breaking scale. Under this assumption, we discard corrections of the size $M_{GP}/M_P$. Concerning the finite volume, we work with the condition that

$$M_{GP}L \gg 1,$$

where $L$ is the spatial extent of the cubic box. The temporal direction of the lattice is taken to be infinite.

The main task of this work is to study the volume effects due to the presence of the scales

$$\Delta = M_{P^\ast} - M_P,$$

and

$$\delta_s = M_{P_s} - M_P,$$

where $P^\ast$ and $P$ are the heavy-light vector and pseudoscalar mesons containing a $u$ or $d$ anti-quark\textsuperscript{2}, and $P_s$ is the heavy-light pseudoscalar meson with an $s$ anti-quark. The scale $\Delta$ appears due to the breaking of heavy quark spin symmetry that is of $O(1/M_P)$ and $\delta_s$ comes from light flavour $SU(3)$ breaking in the heavy-light meson masses\textsuperscript{3}. These scales are comparable to the pion mass in the real world and in lattice simulations. Therefore it is important to understand how they combine with the infra-red scales $1/L$ and $M_{GP}$ in finite volume.

2. HEAVY-LIGHT MESONS IN FINITE VOLUME

Finite volume effects in heavy-light meson systems are dominated by the Goldstone particles.

\textsuperscript{2}We work in the isospin limit in this paper.

\textsuperscript{3}Under the assumption of Eq. 1, $\Delta$ is independent of the light quark mass, and $\delta_s$ does not contain any $1/M_P$ corrections, to the order we are working.
which couple to the heavy meson and can “wrap around the world”. In HMχPT, the heavy-light pseudoscalar P meson can scatter into the vector $P^*$ meson by emitting a pion. In the limit the heavy quark mass is infinite, where the heavy-quark spin symmetry is exact, both P and $P^*$ mesons are on-shell static sources and there is a velocity superselection rule. This can be seen from their propagators

$$\frac{i}{2(v \cdot k + i\epsilon)} \sim \frac{-i(g_{\mu\nu} - v_\mu v_\nu)}{2(v \cdot k + i\epsilon)},$$

which are proportional to $\theta(t)\delta^{(3)}(\vec{x})$ in position space. In this situation, finite volume effects can only be of the form $\exp(-|\vec{r}|M_{GP}L)/(|\vec{r}|M_{GP}L)$, which is the position-space propagator for a Goldstone particle wrapping around the world $n_i$ times in the spatial direction $i$.

Away from the strict heavy-quark limit, there is a mass difference, $\Delta \sim 1/M_P$, between $P^*$ and P mesons. This mass shift, in the presence of the velocity superselection rule, renders the propagator of the $P^*$ meson into the form

$$\frac{-i(g_{\mu\nu} - v_\mu v_\nu)}{2(v \cdot k - \Delta + i\epsilon)},$$

and brings it off-shell with the virtuality $\Delta$. The time uncertainty conjugate to this virtuality, $\delta t \sim 1/\Delta$, restricts the period during which the Goldstone particles can propagate to wrap around the world, hence alters volume effects. Therefore in the class of diagrams involving the $P-P^*-\pi$ coupling in HMχPT, volume effects decrease with increasing $\Delta$ and can only depend on $M_{GP}/\Delta$.

The above physical picture can be observed in a momentum-space calculation by considering a typical sum in one-loop HMχPT,

$$J(M_{GP}, \Delta) = \frac{1}{L^3} \sum_k \int \frac{dk_0}{2\pi} \frac{1}{(k^2 - M_{GP}^2 + i\epsilon)(v \cdot k - \Delta + i\epsilon)},$$

where the spatial momentum $\vec{k}$ is quantised in finite volume as $2\pi\vec{i}/L$, with $\vec{i}$ being a three-dimensional integer vector. In the infinite-volume limit, the sum $J$ becomes an integral with $\sum_{\vec{i}}/L^3$ replaced by $\int d^3k/(2\pi)^3$. Using the Poisson summation formula, it can be shown that finite volume effects in this sum are (with $n = |\vec{n}|$)

$$J_{FV}(M_{GP}, \Delta) = \sum_{\vec{n} \neq \vec{0}} \left( \frac{1}{8\pi n L} \right) e^{-nM_{GP}L}A,$$

in the asymptotic limit $M_{GP}L \gg 1$, where

$$A = e^{(z^2)[1 - \text{Erf}(z)]} + \mathcal{O}\left( \left( \frac{1}{nM_{GP}L} \right) \right),$$

with

$$z = \left( \frac{\Delta}{M_{GP}} \right) \sqrt{\frac{nM_{GP}L}{2}}.$$

The quantity $A$ is the alteration of finite volume effects due to the presence of a non-zero $\Delta$. It multiplies the factor $\exp(-nM_{GP}L)$, which results from the Goldstone particles wrapping around the world. Notice that it is possible to analytically compute the higher order corrections of $A$ in powers of $1/(nM_{GP}L)$ to achieve any desired numerical precision. The function $J_{FV}$ is plotted in Fig. 1 at $L = 2.5$ fm. This plot shows

Figure 1. $J_{FV}(M_{GP}, \Delta)$ at $L = 2.5$ fm, plotted as a function of $M_{GP}$. The Goldstone mass $M_{GP} = 0.197$ GeV corresponds to $M_{GP}L = 2.5$, and $M_{GP} = 0.32$ GeV corresponds to $M_{GP}L = 4$.

that $J_{FV}$ is strongly dependent on $\Delta$, hence $M_P$, and it decreases as $\Delta$ increases.

### 3. Calculation for $B^0 - \bar{B}^0$ Mixing

We have performed a one-loop calculation for the $B_{(s)}^0 - \bar{B}_{(s)}^0$ mixing system, including the
$B_{B_s}$ parameters and decay constants\(^4\) in full, quenched and $N_f = 2 + 1$ partially-quenched QCD. Especially, we investigate volume effects in the SU(3) breaking ratios:

$$\xi_f = \frac{f_{B_s}}{f_B} \text{ and } \xi_B = \frac{B_{B_s}}{B_B} \quad (11)$$

which are important inputs for the global CKM fit. Furthermore, we define

$$(\xi_f)_{\text{FV}} \text{ and } (\xi_B)_{\text{FV}} \quad (12)$$

to be the contributions from volume effects. They are estimated by calculating the volume-dependent one-loop corrections with respect to the lowest-order values of $f_{B_s}$ ($B_{B_s}$) and $f_B$ ($B_B$), then taking the difference between the results. We find that volume effects are more salient in $\xi_B$ than $\xi_f$. For existing and future lattice calculations, $(\xi_B)_{\text{FV}}$ are typically $\sim 5\%$ for quenched, $\sim 4\%$ for partially quenched and $\sim 2\%$ for full QCD, in the parameter space where lattice simulations are carried out, and they can be amplified in both light-quark and heavy-quark mass extrapolations (interpolations). An example in quenched QCD is shown in Fig. 4. From these plots, it is clear that finite volume effects have strong dependence on both light-quark and heavy-quark masses and can exceed the currently quoted errors on $\xi_B$.

REFERENCES


\(^4\)Notice that the results presented here can be used to analyse the box-diagram contribution to the matrix elements of $D^0 - \bar{D}^0$ mixing as well.

\[\text{Figure 2. } (\xi_B)_{\text{FV}} \text{ in QQCD plotted against } M_B, \text{ with } L = 1.6 \text{ fm, strange-quark mass set to its physical value and choices of the couplings } g \text{ (the } P-P^* - \pi \text{ coupling) and } \gamma \text{ (the } P-P^* - \eta' \text{ coupling). The pion mass } M_\pi = 0.35 \text{ GeV corresponds to } M_\pi L = 2.8, \text{ and } M_\pi = 0.5 \text{ GeV corresponds to } M_\pi L = 4 \text{ in this plot. We set } \alpha = 0 \text{ and } M_0 = 700 \text{ MeV. Notice that at } \Delta \sim 50 \text{ MeV and 150 MeV at physical } M_B \text{ and } M_B \text{ respectively.}\]