IMPROVING EXTRACTIONS OF $|V_{cb}|$ AND THE $B$ QUARK MASS FROM SEMILEPTONIC INCLUSIVE $B$ DECAY

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Recent advances in improving extractions of $|V_{cb}|$ and $m_b$ from spectra of semileptonic inclusive $B$ decay are reported. Results of a general moment analysis of the lepton energy spectrum and the hadronic invariant mass spectrum are summarized. The calculation of the general $O(\alpha_s)$ structure functions for semileptonic $B$ decay is reported, which has allowed the calculation of the $O(\alpha_s \Lambda_{QCD}/m_b)$ terms for the hadronic invariant mass moments to be carried out. Recent theoretical advances and improvements in experimental data has allowed extractions of the CKM matrix element $|V_{cb}|$ to improve to the 2% level.

1. Introduction

Inclusive semileptonic $B \rightarrow X_c \ell \bar{\nu}$ decay offers the opportunity to measure the CKM parameter $|V_{cb}|$ and the bottom quark mass from experiment.\[1] Experimental studies of moments of the differential decay spectra of $B \rightarrow X_c \ell \bar{\nu}$, combined with a measurement of the total inclusive decay rate, allow the extraction of these parameters to occur with high precision.

The idea is that sufficiently inclusive observables of this decay can be calculated without model dependence from QCD using an operator product expansion (OPE). The OPE demonstrates that in the $m_b \gg \Lambda_{QCD}$ limit inclusive $B$ decay rates are equal to inclusive $b$ quark decay rates. Corrections to this rate are suppressed by powers of the ratio $\Lambda_{QCD}/m_b$ and $\alpha_s$. The terms in the $\Lambda_{QCD}/m_b$ expansion are parameterized using heavy quark effective theory (HQET) with two parameters introduced for corrections at $O(\Lambda_{QCD}^2/m_b^2)$, typically labeled $\lambda_{1,2}$, while six new parameters enter the expansion at $O(\Lambda_{QCD}^3/m_b^3)$, typically labeled $\rho_{1,2}$ and $\tau_{1,2,3,4}$. To extract $|V_{cb}|$ from inclusive decay spectra with high precision one needs to acce-
rately know the b quark mass and the largest terms in the OPE that the 
$B \to X_c \ell \bar{\nu}$ semileptonic decay rate depends upon.

Experimental results have been reported by the BABAR, BELLE, CDF,
CLEO and DELPHI collaborations measuring various B meson decay spec-
tra and moments\textsuperscript{10,11,12,13,14,15,16}. A number of authors have per-
formed global fits to this data\textsuperscript{17,18,19,20}, the most recent of which\textsuperscript{21} finds
$|V_{cb}| = (41.3 \pm 0.6 \pm 0.1 \tau_B) \times 10^{-3}$ and $m_{b}^{1S} = 4.68 \pm 0.04$ GeV. Determin-
ing $m_b$ and $|V_{cb}|$ to this level of precision, and prospects of further improve-
ment, require that the a high degree of confidence be obtained that the
OPE fits the data and that the extraction of the unknown matrix elements
of the leading order operators in the OPE be as accurate as possible.

Recent work aimed at these goals has followed the approach of calcu-
lating general observables in order to maximize the amount of informa-
tion obtained from experimental improvements in measured $B \to X_c \ell \bar{\nu}$
spectra\textsuperscript{22,23}. The calculation of the $O(\alpha_s)$ structure functions for $B \to X_c \ell \bar{\nu}$ has also been completed\textsuperscript{23} recently to reduce the theoretical uncer-
tainty in hadronic invariant mass moments in determining these parameters.

The purpose of this paper is to review recent work in this area; focusing
in particular on basic features of the general moments analysis and the
calculation of the $O(\alpha_s)$ structure functions.

\section{General Moment Approach}

A $B \to X_c \ell \bar{\nu}$ observable calculated by an OPE is a double expansion in
terms of the strong coupling $\alpha_s(m_b)$ and the ratio $\Lambda_{\text{QCD}}/m_b$. In obtaining
predictions for observables the triple differential decay spectrum is decom-
posed in terms of the invariant mass of the W boson $\hat{y} = q^2/m_b^2$, where $q^\mu$ is the momentum of the lepton pair, the $c$ quark jet invariant mass
$\hat{z} = (m_b v - q)^2/m_b^2$, and the charged lepton energy $\hat{E}_\ell = E_\ell/m_b$.

In extracting a b quark mass parameter, the pole mass is related to
the short distance 1S mass\textsuperscript{24,25} to avoid the pole mass renormalon\textsuperscript{26,27}
ambiguity that leads to unnecessarily badly behaved perturbation series.
We choose to express the b quark pole mass perturbatively in terms of
the 1S mass and use the fact that $\frac{m_{\Upsilon}}{2} - m_{b}^{1S} \sim \Lambda_{\text{QCD}}$ to expand in the
parameter $\Lambda_{1S} \equiv \frac{m_{\Upsilon}}{2} - m_{b}^{1S}$.

Following this approach we obtain predictions for the lepton energy
spectrum and the hadronic invariant mass spectrum of $B \to X_c \ell \bar{\nu}$. Observ-
ables are usually calculated with experimentally required cuts to reduce
backgrounds, such as a cut on the charged lepton energy, so that in general
a prediction for an nth moment of an observable \( (O^n) \) is an expansion of the following form

\[
\frac{1}{\Gamma_0} \int_{\hat{E}_\ell^{\min}} O^n \left[ \hat{y}, \hat{z}, \hat{E}_\ell \right] \frac{d\Gamma}{d\hat{y} d\hat{z} d\hat{E}_\ell} d\hat{y} d\hat{z} d\hat{E}_\ell = f_0 \left[ n, \hat{E}_\ell^{\min} \right] + f_1 \left[ n, \hat{E}_\ell^{\min} \right] \frac{\Lambda_{1S}}{m_T} + \sum_{i=1}^{2} f_{i+1} \left[ n, \hat{E}_\ell^{\min} \right] \frac{\lambda_i}{m_T^2} + O \left( \alpha_s, \frac{\Lambda_{QCD}^3}{m_T^3} \right),
\]

where

\[
\Gamma_0 = \frac{G_F^2 |V_{cb}|^2 (m_T)}{192\pi^3}.
\]

The motivation of the general moment approach is to exploit the degree of experimental and theoretical control that exists over \( n \) and \( \hat{E}_\ell^{\min} \) by examining directly the dependence of the various coefficient functions \( f_i \) on these terms. Using this control, one can uncover moments that are well suited to measure the HQET matrix elements and test our understanding of B decay.

### 3. Lepton Energy Spectrum

In the lepton energy spectrum, the ratio of lepton energy moments was considered using this approach:\[22\]

\[
R[n, E_{\ell 1}, m, E_{\ell 2}] = \frac{\int_{E_{\ell 1}}^{m \nu/2} E_{\ell 1}^n \frac{d\Gamma}{dE_{\ell 1}}}{\int_{E_{\ell 2}}^{m \nu/2} E_{\ell 2}^m \frac{d\Gamma}{dE_{\ell 2}}}. \tag{3}
\]

The parameter space of \( n, m, E_{\ell 1}, \) and \( E_{\ell 2} \) was then examined. Moments were found with suppressed third order contributions in the nonperturbative expansion that improve the theoretical error in the extractions of \( m_b \), thru \( \Lambda_{1S} \), and \( \lambda_1 \). Sets of moments of this type were found and have been experimentally examined as shown in Fig. 1.

Another possible source of theoretical error in extractions of \( |V_{cb}| \) and \( m_b \) is a quark-hadron duality violation effect that is not incorporated in the OPE analysis. Some authors advocate that duality violating effects could be as large as order \( \Lambda_{QCD}/m_b \)[28] while other authors argue that duality violations are small. As duality violation is difficult to reliably quantify theoretically, it is important to quantify duality violation as directly as possible from experiment.

In global fits, duality violations effects could appear as a poor \( \chi^2 \), and no evidence has been found in recent fits for large duality violations[24]. These
indirect tests of duality violation are limited by our knowledge of $m_b$ and $\lambda_1$ and recent fits have improved our knowledge of to $\lambda_1 = -0.24 \pm 0.06$ GeV$^2$ and $m_b^{1S} = 4.68 \pm 0.04$ GeV.

Another way to quantify the possible size of duality violating terms is to use a duality testing moment uncovered by examining the general lepton energy moment. This complementary technique is limited to a lesser degree by the error on $\lambda_1$ and $m_b$: as a duality testing moment is defined to be a moment with the leading order unknown terms in the OPE suppressed. Moments of this type can be predicted to percent level accuracy and the following table illustrates the impressive level of agreement that these moments exhibit with experimental data.

<table>
<thead>
<tr>
<th></th>
<th>$D_3$</th>
<th>$D_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theoretical Prediction</td>
<td>$0.5200 \pm 0.0014$</td>
<td>$0.6053 \pm 0.0018$</td>
</tr>
<tr>
<td>Experimental Results</td>
<td>$0.5193 \pm 0.0008$</td>
<td>$0.6036 \pm 0.0006$</td>
</tr>
</tbody>
</table>
4. Hadronic Invariant Mass Spectrum

Calculating a general hadronic invariant mass moment presents unique challenges. Proceeding as in the lepton energy case one can attempt to calculate

\[
S[n, E_{\ell_1}, m, E_{\ell_2}] = \int_{E_{\ell_1}} s_H^n d\hat{y} d\hat{z} d\hat{E}_\ell d\hat{E}_\ell',
\]

where

\[
s_H = m_B^2 - m_B m_b (1 - \hat{z} + \hat{y}) + m_b^2 \hat{y}.
\]

However in order to perform the integrations required for the analysis one has to expand the general moment of \( s_H \) in the following manner,

\[
s_H^n = \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{\Gamma(n+1)}{\Gamma(n+1-k)\Gamma(k)} C^k_l \hat{y}^l \hat{z}^{n-k} m_b^{2n}.
\]

The coefficient functions \( C^k_l \) are \( O(\Lambda^k_{\text{QCD}}/m_b^k) \). Expanding up to \( O(\Lambda^3_{\text{QCD}}/m_b^3) \) in the nonperturbative expansion one finds,

\[
s_H^n = \hat{z}^n m_b^{(2n)} \left[ C^0_0 \frac{n}{\hat{z}} + C^1_0 + \hat{y} C^1_1 \right] + \frac{n}{4! \hat{z}^2} \left( C^2_0 + \hat{y} C^2_1 + \hat{y}^2 C^2_2 \right) + \frac{n}{2^4 \hat{z}^3} \left( C^3_0 + \hat{y} C^3_1 + \hat{y}^2 C^3_2 + \hat{y}^3 C^3_3 \right)
\]

where the \( C^k_l \) are functions of \( n \) and the nonperturbative matrix elements. For integer moments this expression has no \( 1/\hat{z} \) dependence, however, for non-integer moments one obtains contributions of order \( \hat{z}^{-k} \) where \( k \geq n \) is the ceiling of the fractional moment power \( n \). As the lower limit of \( z \) is \( \rho = m_c^2/m_b^2 \) this corresponds to a \( m_b \Lambda_{\text{QCD}}/m_b^2 \) expansion entering into the calculations of fractional moments. Formally this expansion is well behaved in the SV limit where \( m_b \sim m_c \gg m_b - m_c \gg \Lambda_{\text{QCD}} \). The precise manner to reliably estimate this uncertainty is currently under study.

In the hadronic invariant mass spectrum duality testing moments can also be found and the predictions of these moments should be compared to experimental measurements. The hadronic invariant mass spectrum also offers the opportunity to measure the b quark mass with minimal error due to \( \lambda_1 \): as moments have been found that have a strong dependence on the b quark mass, while having a suppressed dependence on \( \lambda_1 \).
5. $\mathcal{O}(\alpha_s)$ Hadronic Structure Functions

In determining $|V_{cb}|$ and $m_b$, the perturbative corrections to the hadronic invariant mass spectrum and the lepton energy spectrum are currently calculated to $\mathcal{O}(\alpha_s^2 \beta_0)$. The $\mathcal{O}(\alpha_s)$ spectra have been known for some time and the $\mathcal{O}(\alpha_s^2 \beta_0)$ corrections are obtained from the $\mathcal{O}(\alpha_s)$ result via a dispersion integral. As observables calculated in the OPE are a double expansion in the parameters $\alpha_s$ and ratio $\Lambda_{\text{QCD}}/m_b$, the largest cross term of these expansions is the $\mathcal{O}(\alpha_s \Lambda_{\text{QCD}}/m_b)$ terms. Calculations of the $\mathcal{O}(\alpha_s \Lambda_{\text{QCD}}/m_b)$ terms in the lepton energy spectrum were obtained from the known $\mathcal{O}(\alpha_s)$ spectra but the lepton energy cut dependence of the $\mathcal{O}(\alpha_s \Lambda_{\text{QCD}}/m_b)$ terms was unknown for hadronic invariant mass observables until recently.

This lack of knowledge of the lepton energy cut dependence of the $\mathcal{O}(\alpha_s \Lambda_{\text{QCD}}/m_b)$ terms was the largest theoretical uncertainty in hadronic invariant mass observables. In order to determine this dependence, the $\mathcal{O}(\alpha_s)$ corrections to the structure functions for a massive final state had to be determined; which required a systematic calculation of cuts across all intermediate state contributions, while keeping the final state mass scale, to the diagrams shown in Fig. 2. This challenging calculation has recently been completed and the $\mathcal{O}(\alpha_s)$ corrections to the structure functions for a massive final state are now known. As a check of this calculation, the massless limit of all regular terms of the $\mathcal{O}(\alpha_s)$ structure functions was taken and found to be in agreement with the $\mathcal{O}(\alpha_s)$ contributions to the structure functions for a massless final state, which have been known for some time.

It deserves to be emphasized that the $\mathcal{O}(\alpha_s)$ corrections to all inclusive $B \to X_c \ell \bar{\nu}$ observables, with arbitrary cuts on kinematic variables, can now be determined in a systematic fashion. In particular, these general results were used to determine the lepton energy cut dependence of the $\mathcal{O}(\alpha_s \Lambda_{\text{QCD}}/m_b)$ terms for the hadronic invariant mass spectrum, improving the agreement of these theoretical expressions with the data and removing the largest theoretical uncertainty in these observables.

6. Conclusions

The determination of $|V_{cb}|$ with a theoretical uncertainty below the 2% level is a significant theoretical achievement. The work reported on in this note represents a part of the efforts of a large number of theorists over the past decade and a half in developing and applying the OPE techniques required in this extraction. Improvements in extracting other CKM matrix elements,
in particular $|V_{ub}|$, to this level of precision, will allow the consistency of the CKM description of CP violation to be precisely tested in the near future.

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