Baryonic sources using irreducible representations of the double-covered octahedral group

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Irreducible representations (IRs) of the double-covered octahedral group are used to construct lattice source and sink operators for three-quark baryons. The goal is to achieve a good coupling to higher spin states as well as ground states. Complete sets of local and nonlocal straight-link operators are explicitly shown for isospin 1/2 and 3/2 baryons. The orthogonality relations of the IR operators are confirmed in a quenched lattice simulation.

1. INTRODUCTION

The determination of the hadronic spectrum is a crucial goal of lattice QCD. Lattice simulations should answer such open questions as the nature of Roper resonance, masses and structures of higher spin states, and explore the existence of pentaquarks, hybrids, and other exotic states. This paper focuses on the construction of three-quark operators with zero and nonzero orbital angular momentum. The method is applicable to any multi-quark hadron. Half-integer spin objects on the lattice can be characterized by IRs of the spinorial rotation group known as the double-covered octahedral group \cite{1,2}. The group has three IRs: $G_1$, $G_2$, and $H$ with respective dimensions 2, 2, and 4. For states transforming according to $G_1$ possible spins are $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, for $G_2$ spins are $\frac{3}{2}, \frac{5}{2}, \ldots$, and for $H$ spins are $\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \ldots$.

2. LOCAL OPERATORS

Assuming that up and down quark masses are degenerate, a color-neutral nucleon source operator formed from three quarks is

$$N_{s_1 s_2 s_3}^{\rho_1 \rho_2 \rho_3} = 2^{-\frac{3}{4}} \varepsilon_{abc} (s_1^a \sigma_{s_2}^{\rho_1 b} - s_1^a \sigma_{s_2}^{\rho_2 b}) s_3^c, \quad (1)$$

where all quark fields are located at $(x,t)$. Color indices are labeled by $a, b,$ and $c$, and spinor indices are written in terms of two-component $\rho$-spin and two-component $s$-spin ($\rho$-spin denotes upper/lower component of Dirac spinor and $s$-spin denotes up/down ordinary spin). The $\rho$-spin is the eigenvalue of intrinsic parity: $+/−$ for upper/lower components. Thus the product $\rho_1 \rho_2 \rho_3$ is the overall intrinsic parity of an operator.

The states of three spin $1/2$ objects span a space of three IRs: a two-dimensional mixed antisymmetric IR ($MA$); a two-dimensional mixed symmetric IR ($MS$); and a four-dimensional totally symmetric IR ($S$). The $MA$ and $MS$ are $G_1$ IRs and $S$ is $H$ IR. Three $\rho$-spins can be categorized into the same combinations. Ten direct products of $\rho$-spin and $s$-spin that span the positive-parity IRs of the group and yield the $MA$ Young tableaux of Dirac indices are listed in Table 1. Ten negative-parity IR operators are obtained by reversing $\rho_1, \rho_2$ and $\rho_3$. These are three embeddings of $G_1$. Though there exists no group element that transforms an operator from $(G_1, i)$.
to \((G_1,j \neq i)\), in lattice QCD simulations different embeddings should overlap because they share the same quantum numbers. The optimal linear combination of operators from different embeddings is found by the variational technique for ground and excited states \([3,4]\).

Table 1

\[
\begin{array}{cc}
\text{Table 1} & \text{Table 2} \\
I = I_z = \pm \frac{1}{2}, \text{ local, positive-parity operators.} & \text{local, positive-parity operators. Only } \rho_1 \rho_2 \rho_3 \text{ indices are listed. Those indices in curly brackets represent the sum over all cyclic permutations of indices.} \\
\hline
\Psi_{1}^{[I_3, \pm]} & \Xi_{\sigma_{1,2,3}}^{[\rho_1 \rho_2 \rho_3]} \text{ notation} \\
\hline
\Psi_{3/2,3/2}^+ & \Psi_{3/2,3/2}^+ \equiv \mathcal{N}_{1,2,3}^{[\rho_1 \rho_2 \rho_3]} \\
\Psi_{3/2,1/2}^+ & \Psi_{3/2,1/2}^+ \equiv \mathcal{N}_{1,2,3}^{[\rho_1 \rho_2 \rho_3]} \\
\Psi_{3/2,-1/2}^+ & \Psi_{3/2,-1/2}^+ \equiv \mathcal{N}_{1,2,3}^{[\rho_1 \rho_2 \rho_3]} \\
\Psi_{3/2,-3/2}^+ & \Psi_{3/2,-3/2}^+ \equiv \mathcal{N}_{1,2,3}^{[\rho_1 \rho_2 \rho_3]} \\
\Psi_{1/2,1/2}^+ & \Psi_{1/2,1/2}^+ \equiv \mathcal{N}_{1,2,3}^{[\rho_1 \rho_2 \rho_3]} \\
\Psi_{1/2,-1/2}^+ & \Psi_{1/2,-1/2}^+ \equiv \mathcal{N}_{1,2,3}^{[\rho_1 \rho_2 \rho_3]} \\
\hline
\end{array}
\]

Ten positive-parity operators for \(\Delta^{++}\) (two \(H\) IRs and one \(G_1\) IR) are given in Table 2. Ten negative-parity IR operators are obtained by reversing \(\rho_1, \rho_2\) and \(\rho_3\).

Table 2 can be used for other baryon channels that have \(M\)A flavor content, \(\rho_1, \rho_2, \rho_3\) indices are suppressed in \(\overline{O}_i^{(n)}\). There are six possible link directions for \(\overline{O}_i^{(n)}\) that transform amongst themselves according to the octahedral group. They are reduced into three IRs: \(A_1, T_1, \text{ and } E\) with dimensions 1, 3, and 2, respectively.

The \(A_1\)-link operator is the sum of all six \(\overline{O}_i\). Because its spatial distribution is cubically symmetric, IR operators for \(A_1\)-link can be obtained directly from Table 1 or Table 2. We choose \(T_1\) and \(E\)-link operators to be discretized versions of a spherical basis, following [5,6]:

\[
\begin{align*}
D_0 \bar{B} & \equiv (-i/2\alpha)(\overline{O}_x - \overline{O}_z); \\
D_\pm \bar{B} & \equiv (\pm i/2\sqrt{2}\alpha)(\overline{O}_x - \overline{O}_z)\pm i(\overline{O}_y - \overline{O}_w); \\
E_0 \bar{B} & \equiv (1/\sqrt{6}\alpha^2)[2(\overline{O}_x + \overline{O}_z) - (\overline{O}_y + \overline{O}_w) - (\overline{O}_y + \overline{O}_w)]; \\
E_2 \bar{B} & \equiv (1/\sqrt{2}\alpha^2)(\overline{O}_x + \overline{O}_z) - (\overline{O}_y + \overline{O}_w),
\end{align*}
\]

where \(D_m(E_m)\) acting on the third quark of baryon source \(\bar{B}\) are \(T_1\)-\((E)\) link operators. The \(T_1\)-\((E)\) link operators have odd(even) spatial parity. The decomposition of vectorial IR \(\otimes\) spinorial IR is,

\[
\begin{align*}
T_1 \otimes H & \equiv H \otimes H \oplus G_1 \otimes G_2, \\
T_1 \otimes G_1 & \equiv H \otimes G_1, \\
E \otimes H & \equiv H \oplus G_1 \oplus G_2, \\
E \otimes G_1 & \equiv H.
\end{align*}
\]

3. STRAIGHT-Link OPERATORS

Here, we consider the simplest nonlocal operator—the third quark displaced along a straight path from the other two,

\[
\overline{O}_i^{(n)} \equiv \epsilon_{abc} \eta_{A_{1,2,3}}(x) \eta_{A_{2,3}}(x)[f(U) \eta_{A_{1,3}}(x + na \hat{e}_i)]^c,
\]

where \(n\) is an integer specifying the number of sites displaced, and \(f(U)\) is a product of gauge links connecting the third quark to point \(x\). Dirac
Tables 3 and 4 show all IR operators formed from straight-link constructions according to Eq. (2).

The local operators $\Psi_{S, S'}^\pm$ are selected from Table 1 or Table 2. The $D_m$ straight-link operators are analogous to the discretized spherical harmonics $Y_{1m}$, hence we list the $J_z$ values in Table 3 for operators having Clebsch-Gordan coefficients. Straight-link operators include $G_2$ operators, which couple to spin states with $J \geq 5/2$.

To verify that the operators in Table 1 satisfy the appropriate orthogonality relations, we have performed a test using 287 quenched, $16^3 \times 64$
lattices with the Wilson fermion action. We have confirmed that the correlation functions vanish within one jackknife error for all timeslices when the source and sink operators belong to different IRs ($G_1, G_2, H$), have different parities, or belong to different rows of the same IR.

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