Gauge-covariant S-matrices for field theory and strings

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Abstract

S-matrices can be written Lorentz covariantly in terms of free field strengths for vector states, allowing arbitrary gauge choices. In string theory the vertex operators can be chosen so this gauge invariance is automatic. As examples we give four-vector (super)string tree amplitudes in this form, and find the field theory actions that give the first three orders in the slope.

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1 Introduction

1.1 Gauges

An interesting feature of four-point amplitudes with four external gauge fields in both $D = 10$ superstrings and maximally supersymmetric gauge theories in $D \leq 10$ (and by supersymmetry, arbitrary external massless states) is that the kinematic factors are identical at the tree and one-loop level [1]. Because lower-point amplitudes vanish in these theories, the one-loop four-point amplitude consists of one-particle-irreducible graphs in the field theory case, and is thus expressed directly in terms of field strengths as a contribution to the effective action in a background-field gauge calculation, as the “non-field-strength” contributions (from non-spin couplings) exactly cancel [2]. On the other hand, tree graphs are never expressed in terms of field strengths, so the identity of these kinematic factors seems somewhat mysterious.

In general field theories, the fact that S-matrices always have external propagators amputated means that the generating functional for the S-matrix (as opposed to that for Green functions) can always be expressed in terms of fields rather than sources [3]. (Consider, e.g., the external vector states for the tree amplitude of an electron in an external electromagnetic field.) In fact, the external line factors of Feynman graphs are (asymptotic) fields, and satisfy their (free) wave equations and (linear) gauge conditions. However, the gauge conditions imposed on external states generally do not match those applied to internal ones, neither for propagators in loops (“quantum gauge”) nor in attached trees (“background gauge”): Usually the latter two gauges are some variation of the Fermi-Feynman gauge, while the external states satisfy a Landau gauge, further restricted to some type of unitary gauge (lightcone or Coulomb) by the residual gauge invariance. An exception is when external polarizations are summed over in a cross section, a procedure that is often more cumbersome because cross sections involve double sums (i.e., over both amplitudes and their complex conjugates).

The consistency of this procedure follows from the fact that in general three independent gauges can be chosen in the calculation of an S-matrix element from Feynman diagrams, corresponding to three steps in the procedure: (1) First calculate the effective action, using the background field method. The gauge for the “quantum” fields, which appear inside the loops, is fixed but the background fields are not gauge-fixed. The resulting effective action, which depends only on the background fields, is thus gauge invariant, not merely BRST invariant (and in fact is not a functional of the ghosts). (2) Calculate the generating functional for the S-matrix from “tree”
graphs of the effective action, treating the full effective action as “classical”, fixing
the gauge for the (background) fields of the effective action. The result can always be
expressed as a functional of linearized, on-shell field strengths only, in a Lorentz and
gauge covariant way. (3) Calculate a specific S-matrix element, choosing a (linear)
unitary gauge condition for the external gauge fields, or expressing the external field
strengths directly in terms of polarizations.

It is the second step that will be the focus of this paper. We will also examine its
analog in string and superstring theory. In that case, with the usual first-quantized
methods, the effective action does not appear, so the procedure reduces to two steps:
(1) Calculate the S-matrix in terms of field strengths by using gauge-covariant vertex
operators \[4\]. (2) Same as step 3 of the field theory case. The main difference in the
string case is that gauge invariance at the next-to-last step is automatic (although
there is still some work to rearrange the result in terms of field strengths). The
advantages of having the third gauge invariance are similar to those of the other gauge
invariances, since the result (a) can be applied to different gauges (e.g., lightcone or
Coulomb), depending on the application, (b) is generally simpler, since various terms
of various derivatives of gauge fields can be combined into field strengths, (c) is
more unique, simplifying comparison of different contributions, and (d) is manifestly
Lorentz covariant.

Some of these advantages can also be obtained by instead using a twistor formal-
ism (“spinor helicity” \[5\], “spacecone” \[6\], etc.), but that approach does not generalize
conveniently to higher dimensions. In fact, the two methods are somewhat related in
\(D = 4\). As an example, consider the “maximally helicity violating” tree amplitudes
of Yang-Mills theory \[7\]: In the usual twistor notation, these are written as
\[
A = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}, \quad \langle kl \rangle = \lambda_\alpha^k \lambda_\alpha^l
\]
for an \(n\)-point amplitude with \(i\) and \(j\) labeling the lines whose helicites differ from
the rest. The twistors themselves are “square roots” of the momenta,
\[
p_{\alpha \dot{\alpha}} = \lambda_\alpha \bar{\lambda}^{\dot{\alpha}}
\]
so no residue of gauge invariance is visible, but manifestation of Lorentz invariance is
possible because in \(D = 4\) the little group is just \(U(1)\), as represented by helicity. On
the other hand, a twistor can also be interpreted as the square root of (the selfdual \(f\)
or anti-selfdual \(\bar{f}\) part of) an antisymmetric tensor: In an appropriate normalization
for external lines,
\[
f_{\alpha \beta} = \lambda_\alpha \lambda_\beta, \quad \bar{f}_{\dot{\alpha} \dot{\beta}} = \bar{\lambda}_{\dot{\alpha}} \bar{\lambda}^{\dot{\beta}}
\]
as follows from Maxwell’s equations. Thus the result can easily be expressed in terms of field strengths and the usual (helicity-independent) momentum invariants by completing the denominator of the amplitude to the square of its absolute value (thus making the usual pole structure obvious): In $2 \times 2$ matrix notation,

$$A = \frac{\text{tr}(f_if_j)\text{tr}(p_j\bar{f}_{j+1}...\bar{f}_n\bar{f}_1...\bar{f}_{i-1}p_i^*)\text{tr}(p_i\bar{f}_{i+1}...\bar{f}_j\bar{f}_1...\bar{f}_{j-1}p_j^*)}{p_1 \cdot p_2...p_n \cdot p_1}$$

In string theory, the gauge-boson vertex operator $A(X) \cdot \partial X$, expanded in plane waves as $A(X) = \epsilon \epsilon^{ik\cdot X}$, is not gauge covariant, and requires the gauge condition $\partial \cdot A = 0$ for worldsheet conformal invariance. In a previous paper \[4\] we derived gauge-covariant vertex operators for (super)strings and used them to calculate the three-vector vertex: The result was the cubic term from the gauge-unfixed $F^2$ Yang-Mills action (and in the bosonic string, also an $F^3$ term). In this paper, we will use this gauge-covariant vertex operator to compute the gauge-invariant tree amplitude between 4 gauge bosons. In particular, to our knowledge a complete, explicit expression for this amplitude (i.e., not simply as a functional derivative of some generating functional) in bosonic string theory has not appeared previously in the literature. Then we will reproduce the same amplitudes at order $\alpha'$ and $\alpha'^2$ from the appropriate $F^2$, $F^3$ (for the bosonic string), and $F^4$ terms in a field theory action.

### 1.2 Results

For the bosonic string we find the amplitude (see subsection 2.1):

$$\left(K_0 + \alpha'K_1 + \alpha'^2stuK_2\right) \alpha'^2 \frac{\Gamma(-\alpha's)\Gamma(-\alpha't)}{\Gamma(1-\alpha's-\alpha't)} \hspace{1cm} (1)$$

where we have factored out the usual coupling constants and momentum conservation $\delta$-function, as well as Chan-Paton factors for cyclic ordering. The kinematic factors are

$$K_0 = \left(4\tilde{F}_{\mu
u}^1\tilde{F}_{\nu\rho}^2\tilde{F}_{\rho\sigma}^3\tilde{F}_{\sigma\mu}^4 - \tilde{F}_{\nu\mu}^2\tilde{F}_{\nu\rho}^3\tilde{F}_{\rho\sigma}^4\tilde{F}_{\sigma\mu}^3\tilde{F}_{\sigma\mu}^4\right) + 2 \text{ permutations}$$

$$\equiv \epsilon^{\mu\nu\rho\sigma\alpha\beta\gamma\delta} F_{\mu\nu}^1 F_{\rho\sigma}^2 F_{\alpha\beta}^3 F_{\gamma\delta}^4, \hspace{1cm} (2)$$

$$K_1 = \left[4(\tilde{F}_{\mu\nu}^1\tilde{F}_{\nu\mu}^4)(k_1^1 - k_4^1)\tilde{F}_{\nu\rho}^2 \tilde{F}_{\rho\sigma}^3 k_4^4 + 8\tilde{F}_{\nu\mu}^1 \tilde{F}_{\nu\rho}^2 \tilde{F}_{\rho\sigma}^3 k_4^1 k_4^4 \tilde{F}_{\gamma\delta}^1 \tilde{F}_{\gamma\delta}^4 \right] + 3 \text{ permutations} \hspace{1cm} (3)$$

$$K_2 = -2 \left(\frac{\tilde{F}_{\mu\nu}^1 \tilde{F}_{\nu\mu}^4 \tilde{F}_{\rho\sigma}^3 \tilde{F}_{\rho\sigma}^4}{t(1+\alpha't)} + \frac{\tilde{F}_{\nu\mu}^1 \tilde{F}_{\nu\rho}^2 \tilde{F}_{\rho\sigma}^3 \tilde{F}_{\sigma\mu}^4}{s(1+\alpha's)} + \frac{\tilde{F}_{\nu\mu}^1 \tilde{F}_{\nu\rho}^2 \tilde{F}_{\rho\sigma}^3 \tilde{F}_{\sigma\mu}^4}{u(1+\alpha'u)} \right) \hspace{1cm} (4)$$
Here, the permutations in \( K_0 \) are the order 1342 and 1423 which replace the cyclic order 1234 and the 3 permutations in \( K_1 \) are the replacing of 1234 by 2341, 3412 and 4123. We also have the definitions

\[
\tilde{F}^i_{\mu\nu} = k^i_{[\mu} \epsilon^j_{\nu]} = k^i_{\mu} \epsilon^j_{\nu} - k^i_{\nu} \epsilon^j_{\mu}
\]

and

\[
s = -(k^1 + k^2)^2, \quad t = -(k^1 + k^4)^2, \quad u = -(k^1 + k^3)^2.
\]

Because \( \Box \tilde{F} = 0 \) is gauge invariant, we can always set any \((k^i)^2 = 0\) once all external line factors have been written in terms of \( \tilde{F} \)'s. The \( K_2 \) term in (1) can be regarded as the contribution of tachyon poles in the \( s \) and \( t \) channels (the apparent \( u \) pole is canceled by the \( \Gamma \)'s), and will be absent in the corresponding superstring amplitude, while the \( K_1 \) term corresponds to the contribution from an \( F^3 \) term in the field theory action, and hence is also absent in the presence of supersymmetry. (These amplitudes agree with earlier results obtained in the Landau gauge [8].)

Expanding this amplitude in orders of 1, \( \alpha' \) and \( \alpha'^2 \), it follows from the classical gauge theory action (see subsection 2.2):

\[
S = \frac{1}{2\pi i} \int d^D x \left[ -\frac{1}{4} Tr(F^{\mu\nu} F_{\mu\nu}) - \frac{2i\alpha'}{3} Tr(F_{\mu\nu} F_{\alpha\beta} F_{\gamma\delta}) \right. \\
\left. - \frac{\alpha'^2}{4!} t^{\mu\nu\rho\sigma\alpha\beta\gamma\delta} Tr(F_{\mu\nu} F_{\rho\sigma} F_{\alpha\beta} F_{\gamma\delta}) + \frac{\alpha'^2}{2} Tr(F_{\mu\nu} F_{\nu\mu} F_{\rho\sigma} F_{\sigma\rho} - F_{\mu\nu} F_{\rho\sigma} F_{\nu\mu} F_{\sigma\rho}) \right].
\]

(6)

In the Neveu-Schwarz sector of the superstring, the 4-point tree amplitude is

\[
K_0 \alpha'^2 \frac{\Gamma(-\alpha's) \Gamma(-\alpha't)}{\Gamma(1 - \alpha's - \alpha't)}
\]

(7)

with the same \( K_0 \) as defined in the bosonic case. Then, the low energy limit in \( O(\alpha'^0) \), \( O(\alpha') \) and \( O(\alpha'^2) \) follows from the classical action (see section 3):

\[
S = \frac{1}{2\pi i} \int d^D x \left[ -\frac{1}{4} Tr(F^{\mu\nu} F_{\mu\nu}) - \frac{\alpha'^2}{4!} t^{\mu\nu\rho\sigma\alpha\beta\gamma\delta} Tr(F_{\mu\nu} F_{\rho\sigma} F_{\alpha\beta} F_{\gamma\delta}) \right].
\]

(8)

(These actions agree with those obtained from non-gauge-covariant amplitudes [9].)

2 Bosonic string

2.1 Four-point amplitudes

For the bosonic string, as in the previous paper, using the BRST operator

\[
Q = \oint \frac{1}{2\pi i} dz (-\frac{1}{4\alpha'} c \partial X \cdot \partial X + bc \partial c)
\]

(9)
and the integrated vertex operator for gauge bosons
\[ \oint W = \oint A \cdot \partial X = \oint \epsilon \cdot \partial X e^{ik \cdot X}, \]  
we found an unintegrated BRST invariant vertex operator without gauge fixing
\[ V = cA \cdot \partial X - \alpha'((\partial c) \partial \cdot A = c\epsilon \cdot \partial X e^{ik \cdot X} - \alpha'(\partial c)ik \cdot \epsilon e^{ik \cdot X}. \]

Using the integrated vertex operator \( \oint W \) (10) and the gauge invariant vertex operator \( V \) defined in (11), the gauge invariant N-point amplitude for gauge bosons can be constructed in the bosonic string. Specifically, The 4-point amplitude is:
\[ A_4 = \frac{g^2}{2\alpha'}(V(y_1) \int dy_2 W(y_2) V(y_3) V(y_4)) \]

In the upper-half complex plane, the \( X \) propagator is \(-2\alpha'\ln|z' - z|\eta^{\mu\nu}\) and
\[ \langle c(y_1)c(y_2)c(y_3) \rangle = y_{12}y_{13}y_{23} \]
\[ \langle \partial_{y_1}c(y_1)c(y_2)c(y_3) \rangle = \partial_{y_1}(y_{12}y_{13}y_{23}), \cdots . \]

Conventionally, set \( y_1 = 0, y_3 = 1, y_4 \to \infty \) and integrate \( y_2 \) from 0 to 1.

To write the 4-point amplitude in a gauge covariant form, the \( gauge-invariant \) equation of motion of the free vector is necessary:
\[ \partial^\mu F_{\mu
u} = 0 \quad or \quad k^2 \epsilon^\mu - k^\mu (k \cdot \epsilon) = 0 \]

Notice
\[ \int_0^1 dy y^a(1 - y)^b = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} \]
with
\[ \Gamma(a) = \int_0^\infty dt t^{a-1}e^{-t}, \quad \Gamma(a + 1) = a\Gamma(a). \]

For \( y_4 \to \infty \) and \( k^1 + k^2 + k^3 + k^4 = 0 \), the factor appearing in \( A_4 \)
\[ |y_{14}|^{2\alpha'k^1 \cdot k^4}|y_{24}|^{2\alpha'k^2 \cdot k^4}|y_{34}|^{2\alpha'k^3 \cdot k^4} \to |y_4|^{-2\alpha'k^4 \cdot k^4}. \]

Using the equation of motion (14), this factor is just 1 if the amplitude is written in a gauge-covariant form. Finally, the amplitude between 4 gauge bosons is given by eq. (1).

Taking the expansion in \( \alpha' \)
\[ \frac{\Gamma(-\alpha's)\Gamma(-\alpha't)}{\Gamma(1 - \alpha's - \alpha't)} = \frac{1}{\alpha'^2 st} - \frac{\pi^2}{6} + O(\alpha'), \]
Figure 1: The $s$- and $t$-channel diagrams for 4 gauge bosons coupled by $F^2$ vertices in $S_1$.

the lower orders till $O(\alpha'^2)$ of the amplitude $\mathcal{A}_4$ are

$$K_0 + \alpha' K_1 + \alpha'^2 \left(-\frac{\pi^2}{6} K_0 + u K'_2\right), \quad (19)$$

with

$$K'_2 = -2 \left(\frac{\hat{\sigma}_1^1 \hat{\sigma}_1^2 \hat{\sigma}_1^3 \hat{\sigma}_1^4}{s} + \frac{\hat{\sigma}_1^1 \hat{\sigma}_1^2 \hat{\sigma}_1^3 \hat{\sigma}_1^4}{t} + \frac{\hat{\sigma}_1^1 \hat{\sigma}_1^2 \hat{\sigma}_1^3 \hat{\sigma}_1^4}{u}\right) \quad (20)$$

2.2 Action in effective Yang-Mills theory

Clearly, the amplitude in $O(\alpha'^0)$ in (19) corresponds to the 4-point amplitude from 3 Feynman diagrams in Yang-Mills theory

$$S_1 = \frac{1}{g_Y^2 M} \int d^D x \left[-\frac{1}{4} Tr(F^\mu\nu F^\mu\nu)\right], \quad (21)$$

as shown in Fig. 1.

For the bosonic case, as mentioned in the previous paper, there is a cubic interaction for gauge bosons:

$$S_2 = \frac{2 i \alpha'}{3 g_Y^2 M} Tr(F^\mu \nu F^\omega \nu F^\mu \omega) \quad (22)$$

Thus, in the field theory to $O(\alpha')$ there are 5 Feynman diagrams for the 4-point amplitude, as shown in Fig. 2.

The summation of amplitudes from these 5 diagrams is

$$-2 \alpha'(K'_1 + 3 \text{ permutations}) \quad (23)$$

in which $K'_1$ is

$$\frac{p_a}{p^2}(\hat{\sigma}_1^a \hat{\sigma}_1^b F^1_{ab} F^1_{br} - \hat{\sigma}_1^a \hat{\sigma}_1^b F^4_{ab} F^4_{br})[2\epsilon^2_r(\epsilon^3 \cdot k^2) - 2\epsilon^3_r(\epsilon^2 \cdot k^3) + (k^3 - k^2)\epsilon^2_r(\epsilon^2 \cdot \epsilon^3)] + \hat{\sigma}_1^a \hat{\sigma}_1^b F^4_{ab}(\epsilon^3 \epsilon_a^4 - \epsilon^3 \epsilon_b^4) \quad (24)$$
Figure 2: The $s$- and $t$-channel diagrams for 4 gauge bosons coupled by one $F^2$ vertex in $S_1$ and one $F^3$ vertex in $S_2$.

and $p = -k^1 - k^4$. The 3 permutations are the replacing of 1234 by 2341, 3412 and 4123 in $K'_1$. To rewrite it in gauge covariant form, apply the gauge transformation

$$\epsilon^i_{\mu} \rightarrow \epsilon^i_{\mu} - k^i_{\mu} \epsilon^i_{k^i+1} = -\frac{k^i_{\nu+1} F^i_{\mu \nu}}{k^i \cdot k^i+1}, \quad (25)$$

where $i + 1 \rightarrow 1$ for $i = 4$. Then, by using the gauge-invariant equation of motion of the free vector $(14)$ and the Bianchi identity

$$k_{[\mu} \overset{\circ}{F}_{\nu \sigma]} = 0 \quad (26)$$

the amplitude $(23)$ is exactly the same as $O(\alpha')$ in the amplitude $A_4$ in $(19)$. It agrees with the existence of $F^3$ terms in the Lagrangian density as predicted by the three-point amplitude in the previous paper.

It is known that the superstring predicts a higher-derivative gauge interaction $F^4$. In the bosonic case, $O(\alpha'^2)$ in $(1)$ will give similar interactions.

Because there is a cubic interaction $S_2$ in $(22)$, two Feynman diagrams, as shown in Fig. 3, will give directly a gauge covariant amplitude in $O(\alpha'^2)$

$$\alpha'^2 \left( \frac{s - u}{t} F^4_{\mu \nu} F^3_{\nu \rho} F^3_{\rho \delta} + \frac{t - u}{s} F^2_{\mu \nu} F^3_{\nu \rho} F^3_{\rho \sigma} \right) \quad (27)$$

But this is not equal to the $O(\alpha'^2)$ part of the string amplitude $(19)$. The difference between them represents higher-derivative interactions, i.e., the $F^4$ interactions.
in the effective theory. The difference is composed of two parts. One is

\[ B_1 = -\frac{\pi^2}{6} \alpha'^2 K_0, \]  

and the other is

\[ B_2 = \alpha'^2 \left( \hat{F}^1_{\mu\nu} \hat{F}^4_{\nu\rho} \hat{F}^3_{\rho\sigma} \hat{F}^2_{\sigma\mu} + \hat{F}^1_{\mu\nu} \hat{F}^2_{\nu\rho} \hat{F}^3_{\rho\sigma} \hat{F}^4_{\sigma\mu} - 2 \hat{F}^1_{\mu\nu} \hat{F}^3_{\nu\rho} \hat{F}^4_{\sigma\mu} \hat{F}^2_{\rho\sigma} \right). \]

To convert the amplitude to the Lagrangian density, replace \( \hat{F}_{\mu\nu} \) by \( -i F_{\mu\nu} \) and include a factor of 1/4 for the cyclic identity (as well as the usual overall factor \( 1/g_{YM}^2 \)). So from \( B_1 \) we obtain

\[ -\frac{\pi^2 \alpha'^2}{4g_{YM}^2} t^{\mu\nu\rho\sigma\alpha\beta\gamma\delta} Tr(F_{\mu\nu} F_{\rho\sigma} F_{\alpha\beta} F_{\gamma\delta}), \]

which we will see is same as the one from the superstring, while using the same method, from \( B_2 \) we obtain

\[ \frac{\alpha'^2}{2g_{YM}^2} Tr(F_{\mu\nu} F_{\nu\rho} F_{\rho\sigma} - F_{\mu\nu} F_{\rho\sigma} F_{\nu\rho} F_{\sigma\mu}), \]

which is absent in the superstring case. The low energy limit (19) of amplitude \( \mathcal{A}_4 \) in (1) then corresponds to the effective action given in eq. (6).

3 Superstring

In the case of the Neveu-Schwarz sector of the Ramond-Neveu-Schwarz formulation of the superstring, the language of the “Big Picture” [10] will be used. Define

\[ Z = (z, \theta), \quad X^\mu(Z) = x^\mu(z) + i\theta \psi(z), \quad C = c + \theta \gamma, \quad D_\theta = \partial_\theta + \theta \partial_z \]
where $c$ and $\gamma$ are the anticommuting and commuting superconformal ghosts. As mentioned in our previous paper, the integrated vertex operator is

$$
\oint W = \oint A(X) \cdot D_\theta X = \oint \epsilon \cdot D_\theta X e^{ik \cdot X}
$$

Then the BRST invariant vertex operator is found in the commutator as

$$
\left[ Q, W \right] = D_\theta V
$$

where $Q$ is the BRST operator. To simplify the calculation, we choose units $\alpha' = 2$; $\alpha'$ will be restored in the final result by the replacements $\epsilon \rightarrow \sqrt{\alpha' / 2} \epsilon$ and $k \rightarrow \sqrt{\alpha' / 2} k$. Then,

$$
V = -D_\theta [C(\epsilon \cdot D_\theta X) e^{ik \cdot X(Z)}] + \frac{1}{2} (D_\theta C)(D_\theta X \cdot \epsilon) e^{ik \cdot X(Z)}
$$

$$
-2i(\epsilon \cdot k)(\partial C) e^{ik \cdot X(Z)}.
$$

(32)

In this convention, the propagator

$$
X^\mu(z', \theta') X^\nu(z, \theta) \sim -4 \ln|z' - z - \theta' \theta| \eta^{\mu\nu}
$$

(33)

and the correlation function

$$
\langle 0|C(z_1, \theta_1)C(z_2, \theta_2)C(z_3, \theta_3)|0\rangle = \theta_1 \theta_2 \theta_3 (z_1 + z_2 + z_3 + \theta_1 \theta_2 \theta_3 (z_3 + z_1))
$$

(34)

Then the 4-point amplitude in the superstring can be written as

$$
\mathcal{A}_{4}^{NSR} = -\frac{2g^2 \alpha'}{\alpha'^2} \langle V(Z_1) \int d\theta_2 d\theta W(Z_2) V(Z_3) V(Z_4) \rangle,
$$

(35)

with $z_1 = 0, z_3 = 1, z_4 \rightarrow \infty$ and integrating $z_2$ from 0 to 1.

The vertex operator (32) can also be written as

$$
V = -\frac{1}{2} (D_\theta C)(\epsilon \cdot D_\theta X) e^{ik \cdot X(Z)} + C D_\theta [(D_\theta X \cdot \epsilon) e^{ik \cdot X(Z)}]
$$

$$
-2i(\epsilon \cdot k)(\partial C) e^{ik \cdot X(Z)}.
$$

(36)

Using the anticommutation relation between $C$, $D_\theta$, and $\int d\theta$, move $C$, $D_\theta C$, and $\partial C$ to the left side of $\mathcal{A}_{4}^{NSR}$. To make the calculation simpler, we first set $\theta_1$, $\theta_3$, and $\theta_4$ to zero. Thus we only have to compute the terms independent of $\theta_1$, $\theta_3$, and $\theta_4$. So the amplitude comes only from the parts with two $D_\theta C$’s and one $C$ or $\partial C$.

For the same reason as in the bosonic case, the factor

$$
|y_{14}|^{2\alpha' k^1 \cdot k^4} |y_{24}|^{2\alpha' k^2 \cdot k^4} |y_{34}|^{2\alpha' k^3 \cdot k^4}
$$
appearing in $\mathcal{A}_4^{NSR}$ is just 1 if the rest of the amplitude can be written in gauge-invariant form.

After restoring $\sqrt{\alpha'/2}$’s, we find

$$\mathcal{A}_4^{NSR}(\theta_1 = 0, \theta_3 = 0, \theta_4 = 0) = \alpha'^2 K_0 \frac{\Gamma(-\alpha's)\Gamma(-\alpha't)}{\Gamma(1 - \alpha's - \alpha't)} \quad (37)$$

where $K_0$ is defined by (2).

To check independence from our choice $\theta_1 = \theta_3 = \theta_4 = 0$, we look at conformal invariance of the amplitude. Since the vertex operator $V$ has the weight $\alpha'k^2$, the 4-point amplitude transforms as

$$\langle V'(z_1', \theta_1') \oint WV'(z_3', \theta_3')V'(z_4', \theta_4') \rangle$$

$$= (D_{\theta_1} \theta_1')^{2\alpha'k_1^2}(D_{\theta_3} \theta_3')^{2\alpha'k_3^2}(D_{\theta_4} \theta_4')^{2\alpha'k_4^2} \langle V(z_1, \theta_1) \oint WV(z_3, \theta_3)V(z_4, \theta_4) \rangle \quad (38)$$

Through a conformal transformation $\theta_1 = 0 \rightarrow \theta_1'$, $\theta_2 = 0 \rightarrow \theta_2'$ and $\theta_4 = 0 \rightarrow \theta_4'$,

$$\mathcal{A}_4^{NSR}(\theta_1', \theta_3', \theta_4') \equiv -\frac{2\alpha'^2}{\alpha'^2} \langle V'(z_1', \theta_1') \oint WV'(z_3', \theta_3')V'(z_4', \theta_4') \rangle$$

$$= \left[(D_{\theta_1} \theta_1')^{2\alpha'k_1^2}(D_{\theta_3} \theta_3')^{2\alpha'k_3^2}(D_{\theta_4} \theta_4')^{2\alpha'k_4^2}\right]|_{\theta_1=\theta_3=\theta_4=0} \mathcal{A}_4^{NSR}(0, 0, 0). \quad (39)$$

Using the equation of motion (14),

$$\mathcal{A}_4^{NSR}(\theta_1', \theta_2', \theta_3') = \mathcal{A}_4^{NSR}(0, 0, 0)$$

Then we see the result in (37) is exactly the 4-point tree amplitude for any values of parameters $\theta_1$, $\theta_3$ and $\theta_4$.

Since there is no tachyon in the superstring, it is not surprising that the amplitude doesn’t give the terms associated with tachyon poles in (1).

Expanding the function $\frac{\Gamma(-\alpha's)\Gamma(-\alpha't)}{\Gamma(1 - \alpha's - \alpha't)}$ as in (18), the leading terms correspond to the quadratic Yang-Mills action (21) as in the bosonic case. The absence of $O(\alpha')$ agrees with the absence of $F^3$ terms in the super Yang-Mills action. The $O(\alpha'^2)$ terms represent the higher-derivative $F^4$ action in (30). The complete action for the effective Yang-Mills theory is then given by eq. (8).

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