Entanglement in Random Subspaces

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Abstract. The selection of random subspaces plays a role in quantum information theory analogous to the role of random strings in classical information theory. Recent applications have included protocols achieving the quantum channel capacity and methods for extending superdense coding from bits to qubits. In addition, random subspaces have proved useful for studying the structure of bipartite and multipartite entanglement.

INTRODUCTION

In quantum information theory, we’re fond of saying that Hilbert space is a big place, the implication being that there’s room for the unexpected to occur. You will receive no counterarguments to that homespun wisdom from me for the simple reason that, as far as I can tell, it’s correct. I’m going to present a number of results in quantum information theory that stem from the initially counterintuitive geometry of high-dimensional vector spaces, where subspaces with highly extremal properties are the norm rather than the exception. Peter Shor has shown, for example, that randomly selected subspaces can be used to send quantum information through a noisy quantum channel at the highest possible rate, that is, the quantum channel capacity [1]. More recently, Debbie Leung, Andreas Winter and I demonstrated that a randomly chosen subspace of a bipartite quantum system will likely contain nothing but nearly maximally entangled states, even if the subspace is nearly as large as the original system in qubit terms [2]. This observation has implications for communication, especially superdense coding, as described in [3]. Those results and the intuition behind them will be my focus here.

SURPRISES IN HIGH DIMENSION

Suppose, for the moment, that you are an astronaut orbiting Earth in a space shuttle. Imagine, slightly less plausibly, that you are a also mathonaut, meaning that you observe not our Earth but instead a highly idealized version of it in which the population is evenly distributed over the whole surface of the planet. Bored with the daily routine of gyroscope failures and rebreather malfunctions, you decide to look out the window and count the number of people living within a ten kilometer band of the equator. (You have both a good telescope and lots of time on your hands.) Give or take a few, you find five million people, with the rest of the population of six billion living elsewhere. Now, bold mathonaut that you are, you repeat your observations in higher and higher dimensions, first counting the inhabitants of a ten kilometer thickening about the equator of a 3-sphere version of the Earth (in four dimensions), then of a 4-sphere and so on up. Long before you reach the 50-sphere, however, you discover a great time saver: count the people outside of the band. There aren’t any. Perplexed, you decide to check if your luck was bad by selecting other equators for the 50-sphere, but each time you find that every single inhabitant of the planet lives within ten kilometers of the equator.

What’s going on? Nothing too sophisticated, it turns out. The calculation itself is completely elementary, an exercise in spherical coordinates, but the effect is an example of the broader “concentration of measure” phenomenon: naturally defined random variables on high-dimensional spaces tend to concentrate strongly around their average values [4]. The most familiar example of this is probably the case of the sum of \( n \) independent, bounded random variables. According to Chernoff’s bound, the probability that the sum deviates more than \( \varepsilon \) from its mean value is less than \( \exp(-Cn\varepsilon^2) \)
for some positive constant $C$. The analogous statement for functions on the $k$-sphere is known as Levy’s Lemma:

**Levy’s Lemma** (See [3], Appendix IV, and [4]) Let $f : \mathbb{S}^k \to \mathbb{R}$ be a function with Lipschitz constant $\eta$ (with respect to the Euclidean norm) and a point $X \in \mathbb{S}^k$ be chosen uniformly at random. Then

$$\Pr \{ f(X) - \bar{f} \geq \pm \alpha \} \leq \exp \left( -C(k-1)\alpha^2 / \eta^2 \right)$$

for some constant $C > 0$.

Here $\bar{f}$ is used to denote either the mean value or a median for $f$; the median is actually a more natural quantity in the theory of concentration of measure. The function relevant to our mathematician investigations is simply $f(x_1, \ldots, x_n) = x_1$, which obviously has Lipschitz constant one and both mean and median of zero.

**RANDOM STATES AND RANDOM SUBSPACES**

Quantum states, of course, are represented as unit vectors, so Levy’s Lemma provides a ready-made tool for exploring the properties of random quantum states in high-dimensional systems. We need only choose the function $f$ and plug in its mean value.

The example that will occupy us is the entanglement of a bipartite system. Let $|\phi\rangle$ be a random pure state in $\mathbb{C}^d_A \otimes \mathbb{C}^d_B$, chosen according to the unitarily invariant measure, which in turn corresponds to the uniform measure on the $(2d_A d_B - 1)$-sphere. Assuming without loss of generality that $d_A \leq d_B$, the expected value of the entropy of entanglement is known [6, 7, 8, 9, 10] and satisfies

$$\mathbb{E} (\mathcal{E}(\phi)) = \mathbb{E} (\mathcal{S}(\phi_A)) \geq \log_2 d_A - \frac{d_A}{2 \ln 2 d_B},$$

where $\mathcal{S}$ is the von Neumann entropy. Since any state of this bipartite system can have no more than $\log_2 d_A$ ebits of entanglement, this tells us that on average the entanglement is within one ebit of being maximal. Levy’s Lemma allows us to quantify how likely it is that the entanglement of a random state will fall significantly below the mean. Define $\beta = \frac{1}{\ln 2 d_B}$ Once all the calculations are done, we get the following bound:

$$\Pr \{ \mathcal{S}(\phi_A) < \log_2 d_A - \alpha - \beta \} \leq \exp \left( -\frac{(d_A d_B - 1)\alpha^2}{(\log d_A)^2} \right),$$

for some $C > 0$ provided $d_B \geq d_A \geq 3$. Ignoring the small $(\log d_A)^2$ factor in the denominator of the exponent, this is the same type of exponential convergence to the mean that occurs for population evenly distributed on the $k$-sphere.

The convergence is so rapid, in fact, that it is possible to strengthen these results about random states into statements about random subspaces. The idea is to fix a subspace $\mathcal{N}_0$ of dimension $s$ and choose a very fine net of states $\mathcal{N}_0 \subset \mathcal{N}_0$ so fine that given any state $|\phi\rangle \in \mathcal{N}_0$, there is an approximating $|\tilde{\phi}\rangle \in \mathcal{N}_0$ such that $\|\phi - \tilde{\phi}\|_1 \leq \varepsilon$. If we choose a random unitary $U$ according to the Haar measure, it takes $\mathcal{N}_0$ to a random subspace $US_0$ and it takes the net $\mathcal{N}_0$ to a net $U\mathcal{N}_0$ for the new subspace. The probability that a given state in $U\mathcal{N}_0$ has entanglement less than $\log_2 d_A - \alpha - \beta$ is given by Equation (5) while the probability that any one of them has entanglement less than $\log_2 d_A - \alpha - \beta$ is then bounded above by

$$\Pr \{ |\mathcal{N}_0| \exp \left( -\frac{(d_A d_B - 1)\alpha^2}{(\log d_A)^2} \right) \}. $$

As a net on the unit ball of a subspace of real dimension $2s$, the size of $\mathcal{N}_0$ will scale as $(C/\varepsilon)^{2s}$ for some constant $C > 0$. Proving the existence of a subspace in which all states are highly entangled then becomes a matter of tuning the resolution of the net $\mathcal{N}_0$ and the value of $\alpha$. We find that when $d_B \geq d_A \geq 3$ and $0 < \alpha < 1$, there exists a subspace of $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ of dimension

$$d_A d_B \frac{C \alpha^{2.5}}{(\log d_A)^5} $$

where $C > 0$ is, as always, a constant. From now on, I’ll refer to a subspace having this property (for fixed $\alpha$) as a *maximally entangled subspace*. In qubit terms, in a bipartite system of $n$ by $n + o(n)$ qubits, this is a subspace of size $2n - o(n)$ qubits in which all of the states of entanglement at least $n - o(1)$ ebits. The maximally entangled subspace is nearly as large as the whole system.
For the sake of unfair comparison, we could consider the subspace spanned by any two Bell states of a pair of qubits. Any such subspace will not only fail to contain only nearly maximally entangled states, it will always contain some product states!

**CONSEQUENCES FOR ENTANGLEMENT MEASURES**

Should we be alarmed? Unconcerned? Unconvinced? One way to place the result in context is to reinterpret it in the language of mixed state entanglement measures. Consider the maximally mixed state $\rho$ on one of the maximally entangled subspaces. Because the range of $\rho$ consists only of these states, any convex decomposition of $\rho$ into pure states will again be into these nearly maximally entangled states. Continuing to use the language of qubits, in an $n$ by $n + o(n)$ qubit system, $\rho$ will have entanglement of formation

$$E_f(\rho) = n - o(1),$$

which is nearly maximal. On the other hand, as the maximally mixed state on a subspace of $2n - o(n)$ qubits, $\rho$ will have entropy at least $S(\rho) = 2n - o(n)$. In fact, the parameters can be tuned such that the quantum mutual information satisfies

$$S(\rho_A) + S(\rho_B) - S(\rho_{AB}) = O(\log n).$$

The distillable entanglement of the $\rho$ is therefore also $O(\log n)$ [11]. This leaves a huge gap between the entanglement of formation and the entanglement of distillation, the first being almost as large as it can be with the second simultaneously nearly as small as it can be. Ignoring issues of the additivity of the entanglement of formation, the state $\rho$ provides an example of a state that is nearly as hard to make as a maximally entangled state and yet is nearly useless as a resource. In other words, this $\rho$ would be an example of a state exhibiting near-maximal irreversibility.

**SUPERDENSE CODING OF QUANTUM STATES**

Another way to place the existence of these maximally entangled subspaces in context is to study their applications to communication. Suppose that Bob has in mind a specific state $|\phi\rangle$ on a quantum system $S$ that he would like to send to Alice. (Their roles are reversed from the usual convention in order to be consistent with the rest of the paper.) If $S$ were a bipartite system $C^d_A \otimes C^d_B$ and $|\phi\rangle$ was promised to be maximally entangled, then Bob could take advantage of superdense coding [12]: Alice and Bob would pre-share a fixed maximally entangled state and in order to send $|\phi\rangle$, Bob would apply a local unitary transformation $U_\phi$ before sending his half of the system to Alice. That’s fine, of course, but the promise that $|\phi\rangle$ be maximally entangled would seem to make this a very special case, especially since Alice ends up with both halves of the bipartite system. Actually, thanks to the existence of a maximally entangled subspace, this is essentially the general case. If Alice and Bob pre-share a fixed maximally entangled state and agree on an embedding $S \subset C^d_A \otimes C^d_B$ of a maximally entangled subspace, then Bob can send to Alice any state $|\phi\rangle \in S$ using the simple protocol, up to small errors, since they are all nearly maximally entangled. The qubit accounting then works as follows: Bob can send Alice an arbitrary $2n - o(n)$ qubit state by consuming $n$ ebits of entanglement and sending $n + o(n)$ qubits, achieving the two-for-one savings normally associated with sending only classical information. (An earlier version of the result achieved the same rates but consumed extra shared random bits [13].)

The superdense coding idea can be pushed even further, to the case where the state to be prepared in Alice’s lab is entangled with Bob’s system and, therefore, no longer pure. A quick check of the extremal situation suggests that this should be easier: if the goal is prepare a fixed maximally entangled state between Alice and Bob’s labs, then, provided Alice’s system is no larger than Bob’s, no communication is required at all; Bob need just perform an appropriate local unitary on his own system. The interpolation between the two-for-one of pure states and the no communication of maximally entangled states is analyzed in [3] using techniques similar to those discussed here, with the result that the largest Schmidt coefficient $\lambda_{\text{max}}$ of all the states to be prepared controls the trade-off. To leading order in the asymptotics, $\frac{1}{2}\log_2 s + \frac{1}{2}\log_2 \lambda_{\text{max}}$ qubits and $\frac{1}{2}\log_2 s - \frac{1}{2}\log_2 \lambda_{\text{max}}$ ebits are required. $s$ is defined as before to be the dimension of the system being prepared on Alice’s side alone.
MULTIPARTITE ENTANGLEMENT

The results on bipartite entanglement extend easily to the multipartite realm. For convenience, consider a random state of $n$ qudits, so that $|\phi\rangle \in (\mathbb{C}^d)^\otimes n$ and assume that $n$ is held fixed while $d$ is allowed to increase. The following conclusions about random states are essentially corollaries of what we’ve already seen:

- The pure state entanglement across every bipartite cut is likely to be near maximal simultaneously.
- If $k > n/2$ then the reduced state of any $k$ qudits will likely have near-maximal entanglement of formation. Meanwhile, if $k < n/2$ then it is likely that the entanglement of formation becomes less than any positive constant.
- With the participation of the remaining $n−2$ parties, any pair of parties can distill a nearly maximally entangled pure state.

The last item is at first glance probably the most surprising but no harder to prove than the others. The distillation protocol consists of the remaining $n−2$ parties each measuring in a random local basis. The state shared by the other two conditioned on the outcome of this measurement is essentially random and, therefore, nearly maximally entangled.

CONCLUSION

While it is in retrospect no surprise that techniques for dealing with random subspaces should prove useful in quantum information theory, the ease with which they can be analyzed was certainly a surprise to me. Random subspace techniques have been a mainstay of the “local theory of Banach spaces” ever since Milman [14] gave a proof of Dvoretzky’s Theorem [15] using concentration of measure ideas. It is amusing and perhaps instructive to note that the title of a classic book by Milman and Schechtman on the subject, “Asymptotic theory of finite dimensional normed spaces,” concisely sums up in mathematical terms one of the main goals of quantum information theory.

ACKNOWLEDGMENTS

The work presented here comes from papers I have had the pleasure of working on with my co-authors Anura Abeyesinghe, Debbie Leung, Graeme Smith and Andreas Winter. Also, I would like to thank the Sherman Fairchild Foundation and the NSF, through grant number EIA-0086038, for their support of this research.

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