Bounded solutions of fermions in the background of mixed vector-scalar inversely linear potentials

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Abstract

The problem of a fermion subject to a general mixing of vector and scalar potentials in a two-dimensional world is mapped into a Sturm-Liouville problem. Isolated bounded solutions are also searched. For the specific case of an inversely linear potential, which gives rise to an effective Kratzer potential in the Sturm-Liouville problem, exact bounded solutions are found in closed form. The case of a pure scalar potential with their isolated zero-energy solutions, already analyzed in a previous work, is obtained as a particular case. The behaviour of the upper and lower components of the Dirac spinor is discussed in detail and some unusual results are revealed. The nonrelativistic limit of our results adds a new support to the conclusion that even-parity solutions to the nonrelativistic one-dimensional hydrogen atom do not exist.
1 Introduction

The problem of a particle subject to an inversely linear potential in one spatial dimension ($\sim |x|^{-1}$), known as the one-dimensional hydrogen atom, has received considerable attention in the literature (for a rather comprehensive list of references, see [1]). This problem presents some conundrums regarding the parities of the bound-state solutions and the most perplexing is that one regarding the ground state. Loudon [2] claims that the nonrelativistic Schrödinger equation provides a ground-state solution with infinite eigenenergy and a related eigenfunction given by a delta function centered about the origin. This problem was also analyzed with the Klein-Gordon equation and there it was revealed a finite eigenenergy and an exponentially decreasing eigenfunction [3]. By using the technique of continuous dimensionality the problem was approached with the Schrödinger, Klein-Gordon and Dirac equations [4]. The conclusion in this last work reinforces the claim of Loudon. Furthermore, the author of Ref. [4] concludes that the Klein-Gordon equation provides unacceptable solutions while the Dirac equation, with the interacting potential considered as a time component of a vector, has no bounded solutions at all. On the other hand, in a more recent work [5] the authors use connection conditions for the eigenfunctions and their first derivatives across the singularity of the potential, and conclude that only the odd-parity solutions of the Schrödinger equation survive. The relativistic problem of a fermion in an inversely linear potential was also sketched for a Lorentz scalar potential in the Dirac equation [5], but the analysis is incomplete. In a recent work [6] it was shown that the problem of a fermion under the influence of a general scalar potential for nonzero eigenenergies can be mapped into a Sturm-Liouville problem. Next, the key conditions for the existence of bound-state solutions were settled for power-law potentials, and the possible zero-mode solutions were shown to conform with the ultrarelativistic limit of the theory. In addition, the solution for an inversely linear potential was obtained in closed form. The effective potential resulting from the mapping has the form of the Kratzer potential [7]. It is noticeable that this problem has an infinite number of acceptable bounded solutions, nevertheless it has no nonrelativistic limit for small quantum numbers. It was also shown that in the regime of strong coupling additional zero-energy solutions can be obtained as a limit case of nonzero-energy solutions. The ideas of supersymmetry had already been used to explore the two-dimensional Dirac equation with a scalar potential [8]-[9], nevertheless the power-law potential has been excluded of such discussions.
The Coulomb potential of a point electric charge in a 1+1 dimension, considered as the time component of a Lorentz vector, is linear ($\sim |x|$) and so it provides a constant electric field always pointing to, or from, the point charge. This problem is related to the confinement of fermions in the Schwinger and in the massive Schwinger models [10]-[11], and in the Thirring-Schwinger model [12]. It is frustrating that, due to the tunneling effect (Klein’s paradox), there are no bound states for this kind of potential regardless of the strength of the potential [13]-[14]. The linear potential, considered as a Lorentz scalar, is also related to the quarkonium model in one-plus-one dimensions [15]-[16]. Recently it was incorrectly concluded that even in this case there is solely one bound state [17]. Later, the proper solutions for this last problem were found [18]-[20]. However, it is well known from the quarkonium phenomenology in the real 3+1 dimensional world that the best fit for meson spectroscopy is found for a convenient mixture of vector and scalar potentials put by hand in the equations (see, e.g., [21]). The same can be said about the treatment of the nuclear phenomena describing the influence of the nuclear medium on the nucleons [22]-[30]. The mixed vector-scalar potential has also been analyzed in 1+1 dimensions for a linear potential [31] as well as for a general potential which goes to infinity as $|x| \to \infty$ [32]. In both of those last references it has been concluded that there is confinement if the scalar coupling is of sufficient intensity compared to the vector coupling.

Motived by the success found in Ref. [6] we re-examine the two-dimensional problem of a fermion in the background of an inversely linear potential by considering a convenient mixing of vector and scalar Lorentz structures, that is to say $|V_s| \geq |V_v|$. The problem is mapped into an exactly solvable Sturm-Liouville problem of a Schrödinger-like equation with an effective Kratzer potential. The case of a pure scalar potential with their isolated zero-energy solutions, already analyzed [6], is obtained as a particular case. Those ultrarelativistic zero-eigenmodes emerge despite the well-defined parity of the potential. Our results for $V_v = V_s$ give new support to the conclusion that even-parity solutions to the nonrelativistic one-dimensional hydrogen atom do not exist.
The Dirac equation with mixed vector-scalar potentials in a 1+1 dimension

In the presence of time-independent vector and scalar potentials the 1+1 dimensional time-independent Dirac equation for a fermion of rest mass $m$ reads

$$\mathcal{H}\Psi = E\Psi \quad (1)$$

$$\mathcal{H} = c\alpha p + \beta \left( mc^2 + V_s \right) + V_v \quad (2)$$

where $E$ is the energy of the fermion, $c$ is the velocity of light and $p$ is the momentum operator. $\alpha$ and $\beta$ are Hermitian square matrices satisfying the relations $\alpha^2 = \beta^2 = 1$, $\{\alpha, \beta\} = 0$. From the last two relations it follows that both $\alpha$ and $\beta$ are traceless and have eigenvalues equal to $\pm 1$, so that one can conclude that $\alpha$ and $\beta$ are even-dimensional matrices. One can choose the 2×2 Pauli matrices satisfying the same algebra as $\alpha$ and $\beta$, resulting in a 2-component spinor $\Psi$. The vector and scalar potentials are given by $V_v$ and $V_s$ respectively. The positive definite function $|\Psi|^2 = \Psi^\dagger\Psi$, satisfying a continuity equation, is interpreted as a position probability density and its norm is a constant of motion. This interpretation is completely satisfactory for single-particle states [33]. We use $\alpha = \sigma_1$ and $\beta = \sigma_3$. The subscripts for the terms of potential denote their properties under a Lorentz transformation: $v$ for the time component of the 2-vector potential and $s$ for the scalar term, respectively. It is worth to note that the Dirac equation is covariant under $x \rightarrow -x$ if $V_v(x)$ and $V_s(x)$ remain the same. This is because the parity operator $P = \exp(i\eta)P_0\sigma_3$, where $\eta$ is a constant phase and $P_0$ changes $x$ into $-x$, changes sign of $\alpha$ but not of $\beta$.

Provided that the spinor is written in terms of the upper and the lower components, $\Psi_+$ and $\Psi_-$ respectively, the Dirac equation decomposes into:

$$i\hbar c\Psi_+' = \left[ V_v - E \right] \bar{\Psi}_+ \left( mc^2 + V_s \right) \Psi_+ \quad (3)$$

where the prime denotes differentiation with respect to $x$. In terms of $\Psi_+$ and $\Psi_-$ the spinor is normalized as $\int_{-\infty}^{+\infty} dx \left( |\Psi_+|^2 + |\Psi_-|^2 \right) = 1$, so that $\Psi_+$ and $\Psi_-$ are square integrable functions. It is clear from the pair of coupled first-order differential equations [3] that both $\Psi_+$ and $\Psi_-$ have opposite parities if the Dirac equation is covariant under $x \rightarrow -x$. 

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In the nonrelativistic approximation (potential energies small compared to $mc^2$ and $E \simeq mc^2$) Eq. (3) becomes

$$\Psi^- = \frac{p}{2mc} \Psi^+$$

(4)

$$\left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_v + V_s \right) \Psi^+ = \left( E - mc^2 \right) \Psi^+$$

(5)

Eq. (4) shows that $\Psi^-$ is of order $v/c \ll 1$ relative to $\Psi^+$ and Eq. (5) shows that $\Psi^+$ obeys the Schrödinger equation with binding energy equal to $E - mc^2$, and without distinguishing the contributions of vector and scalar potentials.

It is remarkable that the Dirac equation with a scalar potential, or a vector potential contaminated with some scalar coupling, is not invariant under $V \rightarrow V + \text{const.}$, this is so because only the vector potential couples to the positive-energies in the same way it couples to the negative-ones, whereas the scalar potential couples to the mass of the fermion. Therefore, if there is any scalar coupling the absolute values of the energy will have physical significance and the freedom to choose a zero-energy will be lost. It is well known that a confining potential in the nonrelativistic approach is not confining in the relativistic approach when it is considered as a Lorentz vector. It is surprising that relativistic confining potentials may result in nonconfinement in the nonrelativistic approach. This last phenomenon is a consequence of the fact that vector and scalar potentials couple differently in the Dirac equation whereas there is no such distinction among them in the Schrödinger equation. This observation permit us to conclude that even a “repulsive” potential can be a confining potential. The case $V_v = -V_s$ presents bounded solutions in the relativistic approach, although it reduces to the free-particle problem in the nonrelativistic limit. The attractive vector potential for a fermion is, of course, repulsive for its corresponding antifermion, and vice versa. However, the attractive (repulsive) scalar potential for fermions is also attractive (repulsive) for antifermions. For $V_v = V_s$ and an attractive vector potential for fermions, the scalar potential is counterbalanced by the vector potential for antifermions as long as the scalar potential is attractive and the vector potential is repulsive. As a consequence there is no bounded solution for antifermions. For $V_v = 0$ and a pure scalar attractive potential, one finds energy levels for fermions and antifermions arranged symmetrically about $E = 0$ (see, e.g., Refs. [34] and [35]). For $V_v = -V_s$ and a repulsive vector potential for fermions, the scalar and the vector potentials are attractive for antifermions but their effects are
counterbalanced for fermions. Thus, recurring to this simple standpoint one can anticipate in the mind that there is no bound-state solution for fermions in this last case of mixing.

Introducing the unitary operator

\[ U(\theta) = \exp \left( -i \frac{\theta}{2} \sigma_1 \right) \]  

where \( \theta \) is a real quantity such that \(-\pi \leq \theta \leq \pi\), the transform of the Hamiltonian \( H \), \( H = UHU^{-1} \), takes the form

\[ H = \sigma_1 \phi - \sigma_2 \sin (\theta) (mc^2 + V_s) + \sigma_3 \cos (\theta) (mc^2 + V_s) + V_v \]  

(7)

In terms of the upper (\( \phi \)) and the lower (\( \chi \)) components of the transform of the spinor \( \Psi \) under the action of the operator \( U \), \( \psi = U\Psi \), one has

\[
\begin{align*}
\phi &= \Psi_+ \cos \left( \frac{\theta}{2} \right) - i \Psi_- \sin \left( \frac{\theta}{2} \right) \\
\chi &= \Psi_- \cos \left( \frac{\theta}{2} \right) - i \Psi_+ \sin \left( \frac{\theta}{2} \right)
\end{align*}
\]

(8)

Now, as can be seen by inspection of (8), \( \phi \) and \( \chi \) have mixed parities for a parity-invariant theory unless \( \theta = 0 \) or \( \theta = \pm \pi \). In terms of the components of the new spinor, the Dirac equation becomes

\[
\begin{align*}
\hbar c \phi' + \sin (\theta) (mc^2 + V_s) \phi &= i \left[ E + \cos (\theta) (mc^2 + V_s) - V_v \right] \chi \\
\hbar c \chi' - \sin (\theta) (mc^2 + V_s) \chi &= i \left[ E - \cos (\theta) (mc^2 + V_s) - V_v \right] \phi
\end{align*}
\]

(9)

Choosing \( V_v = V_s \cos (\theta) \) 

(10)

i.e., \( |V_s| \geq |V_v| \), one has

\[
\begin{align*}
\hbar c \phi' + \sin (\theta) (mc^2 + V_s) \phi &= i \left[ E + \cos (\theta) mc^2 \right] \chi \\
\hbar c \chi' - \sin (\theta) (mc^2 + V_s) \chi &= i \left[ E - \cos (\theta) (mc^2 + 2V_s) \right] \phi
\end{align*}
\]

(11)

(12)

Furthermore, using the expression for \( \chi \) obtained from (11), viz.
\[ \chi = -i \frac{\hbar c \phi' + \sin(\theta) (mc^2 + V_s) \phi}{E + \cos(\theta) mc^2}, \quad E \neq -\cos(\theta) mc^2 \]  

(13)

and inserting it in (12) one arrives at the following second-order differential equation for \( \phi \):

\[-\frac{\hbar^2}{2} \phi'' + \left[ \frac{\sin^2(\theta)}{2c^2} V_s^2 + \frac{mc^2 + \cos(\theta) E}{c^2} V_s - \frac{\hbar \sin(\theta)}{2c} V_s' - \frac{E^2 - m^2 c^4}{2c^2} \right] \phi = 0 \]

(14)

Therefore, the solution of the relativistic problem is mapped into a Sturm-Liouville problem for the upper component of the Dirac spinor. In this way one can solve the Dirac problem by recurring to the solution of a Schrödinger-like problem. For the case of a pure scalar potential \((\theta = \pm \pi/2)\) with \(E \neq 0\), it is also possible to write a simple second-order differential equation for \(\chi\), just differing from the equation for \(\phi\) in the sign of the term involving \(V_s'\), namely,

\[-\frac{\hbar^2}{2} \chi'' + \left[ \frac{V_s^2}{2c^2} + mV_s \mp \frac{\hbar}{2c} V_s' \right] \chi = \frac{E^2 - m^2 c^4}{2c^2} \chi \]

(15)

This supersymmetric structure of the two-dimensional Dirac equation with a pure scalar potential have already been appreciated in the literature [8]-[9]. One can check that the Dirac energy levels are symmetrical about \(E = 0\) for a pure scalar potential. This conclusion can be obtained directly from (15) as well as from the charge conjugation. Indeed, if \(\Psi\) is a solution with energy \(E\) then \(\sigma_1 \Psi^*\) is also a solution with energy \(-E\) for the very same potential. It means that the scalar potential couples to the positive-energy component of the spinor in the same way it couples to the negative-energy component. In other words, this sort of potential couples to the mass of the fermion instead of its charge. When a vector potential is present the potentials must undergo the transformations \(V_v \rightarrow -V_v\) and \(V_s \rightarrow V_s\) under the charge conjugation operation, in order to restore the invariance of the theory. In addition to the complex conjugate, the upper and lower components of the Dirac spinor are exchanged by charge conjugation but the position probability density is invariant (see, e.g., [36]).
The solutions for $E = -\cos(\theta) mc^2$, excluded from the Sturm-Liouville problem, can be obtained directly from the Dirac equation (11)-(12). One can observe that only a pure scalar potential ($\theta = \pm \pi/2, E = 0$) might support such a sort of isolated normalizable solutions, with the upper and lower components of the Dirac spinor given by

$$
\phi = N_\phi \exp \left\{ -\frac{\sin(\theta)}{\hbar c} \left[ mc^2 x + v(x) \right] \right\}
$$

$$
\chi = N_\chi \exp \left\{ +\frac{\sin(\theta)}{\hbar c} \left[ mc^2 x + v(x) \right] \right\}
$$

where $N_\phi$ and $N_\chi$ are normalization constants and $v(x) = \int^x V_s(y) \, dy$. One can check that it is impossible to have both components different from zero simultaneously on the same side of the $x$-axis and that $|\phi(\pm x)| = |\chi(\mp x)|$. Furthermore, $\phi$ and $\chi$ change their roles under the substitution $\theta \rightarrow -\theta$. Of course a normalizable zero-mode eigenstate is possible only if $v(x)$ has a distinctive leading asymptotic behaviour.

3 The inversely linear potential

Now let us focus our attention on a scalar power-law potential in the form

$$
V_s = -\frac{\hbar c q}{|x|}
$$

where the coupling constant, $q$, is a dimensionless real parameter.

We begin with the zero-eigenmode solutions. From (16) one sees that the normalizable zero-energy solutions are accomplished only for $q > 0$ and they are expressed by

$$
\psi = |x|^q \left[ \Theta(-x) S_- \exp \left( +\frac{mc}{\hbar c} x \right) + \Theta(+x) S_+ \exp \left( -\frac{mc}{\hbar c} x \right) \right]
$$

where $\Theta(x)$ is the Heaviside step function multiplied by a normalization constant, and the spinors $S_\pm$ are defined by

$$
S_\pm = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad S_\mp = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{for} \quad \theta = \pm \frac{\pi}{2}
$$
It follows that the position probability density of the zero-mode spinor has a lonely hump on each side of the $x$-axis. Furthermore, $q \geq 1$ for obtaining a differentiable spinor at the origin and it means that the scalar inversely linear potential must be strong enough to hold a zero-mode solution. The finding of a zero-mode solution for a scalar potential with the same limit for $x \to +\infty$ and $x \to -\infty$ contradicts the statements made in Ref. [9]. It is intriguing to find Dirac eigenspinors with a vanishing lower component in a theory without a nonrelativistic limit. More surprising is to find a vanishing upper component. Both dramatic circumstances make their appearance due to the particular form assumed by the upper and the lower components of the transform of the spinor $\Psi$ given by (8). In the presence of a pure scalar potential Eq. (8) gives $\phi = (\Psi_+ - i\Psi_-)/\sqrt{2}$ and $\chi = -i(\Psi_+ + i\Psi_-)/\sqrt{2}$. Therefore, in the nonrelativistic regime one obtains $|\phi| \approx |\chi|$. On the other side, in the ultrarelativistic regime one expects that $\Psi_-$ presents a contribution comparable to $\Psi_+$ for nonnegative energies, thus the possibilities $\Psi_+ \approx i\Psi_-$ and $\Psi_+ \approx -i\Psi_-$ imply into $\phi \approx 0$ and $\chi \approx 0$, respectively. Therefore, one can conclude that the zero-mode solutions given by (18) correspond to the ultrarelativistic limit of the theory.

We still need to consider the more general case of solutions, i.e., these ones with $E \neq -\cos(\theta) mc^2$. In this case Eq. (14) becomes

$$-\frac{\hbar^2}{2} \phi'' + V_{\text{eff}}^\varepsilon \phi = E_{\text{eff}} \phi$$

where

$$E_{\text{eff}} = \frac{E^2 - m^2 c^4}{2c^2}$$

$\varepsilon$ stands for the sign function ($\varepsilon = x/|x|$, for $x \neq 0$), and the effective potential is the Kratzer-like potential

$$V_{\text{eff}}^\varepsilon = -\frac{\hbar q_{\text{eff}}}{|x|} + \frac{A^\varepsilon}{x^2}$$

with

$$q_{\text{eff}} = q \frac{mc^2 + \cos(\theta) E}{c^2}, \quad A^\varepsilon = \frac{\hbar^2}{2} \xi (\xi - \varepsilon), \quad \xi = q \sin(\theta)$$

Here, such as for the isolated solutions given by (18), the space is split into two regions, and $\phi_+$ refers to $\phi(x > 0)$ and $\phi_-$ to $\phi(x < 0)$. These last results
tell us that the solution for this class of problem consists in searching for bounded solutions for two Schrödinger equations. Therefore, one has to search for bound-state solutions for \( V_{\text{eff}}^+ \) and \( V_{\text{eff}}^- \) with a common effective eigenvalue. The Dirac eigenvalues are obtained by inserting the effective eigenvalues in (21).

Before proceeding, it is useful to make some qualitative arguments regarding the Kratzer-like potential and its possible solutions. The parameters of the effective Kratzer-like potential are related in such a manner that the change \( \theta \rightarrow -\theta \) induces the change \( V_{\text{eff}}^\pm \rightarrow V_{\text{eff}}^\mp \) \((A^\pm \rightarrow A^\mp)\), meaning that the effective potential for \( \phi^\pm \) transforms into the potential for \( \phi^\mp \). The effective Kratzer-like potential is able to bind fermions on the condition that \( q_{\text{eff}} > 0 \). It follows that \( E_{\text{eff}} < 0 \), corresponding to Dirac eigenvalues in the range \(-mc^2 < E < +mc^2\), and \( q > 0 \). The energies belonging to \( |E| \geq mc^2 \) correspond to the continuum. One can see that \( \phi_{\varepsilon} \) is subject to a potential-well structure for \( V_{\text{eff}}^\varepsilon \) when \( |\xi| > 1 \). For \( 0 < |\xi| \leq 1 \) the effective potential has a potential-well structure on one of the sides of the \( x \)-axis and a singular potential at the origin, with singularity given by \(-1/|x|\) when \( |\xi| = 1 \) and \(-1/x^2\) when \( 0 < |\xi| < 1 \), on the other side of the \( x \)-axis. For \( \xi = 0 \) the singularity \(-1/|x|\) appears on both sides of the \( x \)-axis. It is worthwhile to note at this point that the singularity \(-1/x^2\) never exposes the fermion to collapse to the center [37] because in any condition \( A^\varepsilon \) is never less than the critical value \( A_c = -\hbar^2/8 \). The Schrödinger equation with the Kratzer-like potential is an exactly solvable problem and its solution, for a repulsive inverse-square term in the potential \((A^\varepsilon > 0)\), can be found on textbooks [37]-[39]. Since we need solutions involving a repulsive as well as an attractive inverse-square term in the potential, the calculation including this generalization is presented.

Defining the quantities \( z \) and \( B \),

\[
z = \frac{2}{\hbar} \sqrt{-2E_{\text{eff}}} \ |x|, \quad B = q_{\text{eff}} c \sqrt{-\frac{1}{2E_{\text{eff}}}}
\]

and using (16) and (22) one obtains the equation

\[
\phi''_{\varepsilon} + \left( -\frac{1}{4} + \frac{B}{z} - \frac{2A^\varepsilon}{\hbar^2 z^2} \right) \phi_{\varepsilon} = 0
\]

Now the prime denotes differentiation with respect to \( z \). The normalizable asymptotic form of the solution as \( z \rightarrow \infty \) is \( e^{-z/2} \). As \( z \rightarrow 0 \), when the term \( 1/z^2 \) dominates, the solution behaves as \( z^{s_{\varepsilon}} \), where \( s_{\varepsilon} \) is a solution of the algebraic equation
\begin{align}
  s_\varepsilon (s_\varepsilon - 1) - \frac{2A^\varepsilon}{\hbar^2} &= 0 \quad (26) \\
  \text{viz.} \\
  s_\varepsilon &= \frac{1}{2} \left( 1 \pm \sqrt{1 + \frac{8A^\varepsilon}{\hbar^2}} \right) = \pm \xi + \frac{1 \mp \varepsilon}{2} \geq 0 \quad (27)
\end{align}

The solution for all \( z \) can be expressed as \( \phi_\varepsilon(z) = z^{s_\varepsilon} e^{-z/2} w(z) \), where \( w \) is solution of Kummer's equation \[40\]

\[ zw'' + (b_\varepsilon - z) w' - a_\varepsilon w = 0 \quad (28) \]

with

\[ a_\varepsilon = s_\varepsilon - B, \quad b_\varepsilon = 2s_\varepsilon \quad (29) \]

Then \( w_\varepsilon \) is expressed as \( M(a_\varepsilon, b_\varepsilon, z) \) and in order to furnish normalizable \( \phi_\varepsilon \), the confluent hypergeometric function must be a polynomial. This demands that \( a_\varepsilon = -n_\varepsilon \), where \( n_\varepsilon \) is a nonnegative integer in such a way that \( M(a_\varepsilon, b_\varepsilon, z) \) is proportional to the associated Laguerre polynomial \( L_{n_\varepsilon}^{b_\varepsilon - 1}(z) \), a polynomial of degree \( n_\varepsilon \). This requirement, combined with the first equation of (29), also implies into quantized effective eigenvalues:

\[ E_{\text{eff}} = -\frac{q_{\text{eff}} c^2}{2 (s_\varepsilon + n_\varepsilon)^2}, \quad n_\varepsilon = 0, 1, 2, \ldots \quad (30) \]

with eigenfunctions given by

\[ \phi_\varepsilon(z) = N_{\phi_\varepsilon} z^{s_\varepsilon} e^{-z/2} L_{n_\varepsilon}^{2s_\varepsilon - 1}(z), \quad s_\varepsilon > 0 \quad (31) \]

\( N_{\phi_\varepsilon} \) is a normalization constant and the constraint over \( s_\varepsilon \) is a consequence of the definition of associated Laguerre polynomials which is going to imply that \( q > 0 \), as advertised by the preceding qualitative arguments. If \( A^\varepsilon > 0 \) there is just one possible value for \( s_\varepsilon \) and the same is true for \( A^\varepsilon = A_c \) when \( s_\varepsilon = 1/2 \), but for \( A_c < A^\varepsilon < 0 \) there are two possible values for \( s_\varepsilon \) in the interval \( 0 < s < 1 \). If the inverse-square potential is absent \( (A^\varepsilon = 0) \) then \( s_\varepsilon = 1 \). Note that the behavior of \( \phi_\varepsilon \) at very small \( z \) implies into the Dirichlet boundary condition \( (\phi_\varepsilon(0) = 0) \). This boundary condition is essential whenever \( A^\varepsilon \neq 0 \), nevertheless it also develops for \( A^\varepsilon = 0 \).
The necessary conditions for binding fermions in the Dirac equation with the effective Kratzer-like potential have been put forward. The formal analytical solutions have also been obtained. Now we move on to consider a survey for distinct cases in order to match the common effective eigenvalue on both sides of the \( x \)-axis. As we will see this survey leads to additional restrictions on the solutions, including constraints involving the nodal structure of the Dirac spinor.

For \( \sin (\theta) < 0 \) one has \( n_- = n + 1 \), where \( n = n_+ (s_- = s_+ - 1) \). For \( \sin (\theta) = 0 \) one has \( n_- = n (s_- = s_+) \), and for \( \sin (\theta) > 0 \) one has \( n_- = n - 1 (s_- = s_+ + 1) \). The Dirac eigenvalues can now be written as

\[
E = mc^2 - \frac{\left( \frac{q}{|\xi| + n} \right)^2 \cos (\theta) \pm \sqrt{1 - \left( \frac{\xi}{|\xi| + n} \right)^2}}{1 + \left( \frac{q \cos(\theta)}{|\xi| + n} \right)^2}
\]

and the upper component of the Dirac spinor on the positive half-line is given by

\[
\phi = N e^{-z/2} \left\{ \begin{array}{l}
z L_n^1 (z) \quad \text{for } \xi = 0 \\
z^{\xi |L_n^2|\xi| - 1} (z) \quad \text{for } \xi > 0 
\end{array} \right.
\]

where, as before, \( \xi = q \sin (\theta) \). \( N \) is a normalization constant and \( n = 1, 2, 3, \ldots \) \( (n = 0 \) is to be included for considering the zero-eigenmodes of a pure scalar potential \( (\theta = \pm \pi/2) \) in the event that \( q \geq 1 \). We have used \( L_{n-1}^k (z) = 0 \) for all \( k \). For \( n = 0 \) the solution for \( \chi \) is already embraced in (18) and for \( n \neq 0 \) it can be obtained by using (13). By using some recurrence relations involving the associated Laguerre polynomials [40], one can find that

\[
\chi = \mp i q \frac{n + 1}{n} N e^{-z/2} \left[ L_n^0 (z) + L_{n+1}^0 (z) \right] \left\{ \begin{array}{l}
- \text{ for } \theta = 0 \\
\text{ for } \theta = \pm \pi 
\end{array} \right.
\]

\[
\chi = \mp i q \frac{E \cos (\theta) + mc^2}{\xi + n E + mc^2 \cos (\theta)} N z^{\xi |L_n^0|e^{-z/2} \left[ \frac{mc^2}{E \cos (\theta) + mc^2} \xi + n q \sin (\theta) - 1 \right] L_n^{2|\xi|-1} (z) - 2 L_{n-1}^{2|\xi|} (z) \right\} \text{, for } \xi > 0
\]

It follows from (33) and (35) that for \( \theta = \pi/2 \), \( \phi \) and \( \chi \) are proportional to \( z^{q} e^{-z/2} L_{n-1}^{2q-1} (z) \) and \( z^{q+1} e^{-z/2} L_{n-1}^{2q+1} (z) \), respectively, in agreement with the
result found in Ref. [6]. The upper and lower components of the Dirac spinor for $\xi < 0$ can be obtained from those ones for $\xi > 0$ by changing $\xi$ by $\xi + 1$ and $n$ by $n - 1$ in $\phi$, and $n$ by $n + 1$ in $\chi$. It is worthwhile to note that $\chi$ satisfies the Dirichlet boundary condition, an exception is for $V_v = \pm V_s$, a circumstance when $\chi$ is proportional to the first derivative of $\phi$. Anyway, $|\chi(0)| << 1$ when $q << 1$ in such a manner that the original spinor $\Psi$ has a lower (upper) component suppressed relative to the upper (lower) component for $V_v = V_s$ ($V_v = -V_s$), as can be seen from Eq. (8).

A differentiable spinor at the origin is always possible for $V_v = \pm V_s$, but for $V_v \neq \pm V_s$ an acceptable solution at the origin can be achieved only if $|\xi| > 1$, i.e., $q \geq 1/|\sin(\theta)|$ (remember that such a restriction on the coupling constant has already appeared in the case of the zero-eigenmode for a pure scalar potential). Therefore, an appropriate nonrelativistic limit of the theory becomes possible only if $V_v = V_s$ because only in this case one can consider a weakly attractive potential for fermions. The Dirac eigenergies given by (32) are invariant under the substitution $\theta \rightarrow -\theta$. One can say that this happens because this transformation does not alter the mixing among the vector and scalar potentials. Nevertheless, $\phi$ and $\chi$ are affected by the presence of $\sin(\theta)$ into the Dirac equation. We have already seen its effects for the zero-eigenmode spinor and that for $E \neq -\cos(\theta)mc^2$ it implies into $V_{eff}^\pm \rightarrow V_{eff}^\mp$. Inspection of (8) reveals that $\phi_\pm \rightarrow \phi_\mp^*$ and $\chi_\pm \rightarrow \chi_\mp^*$. Therefore, one can conclude that the change $\theta \rightarrow -\theta$ does not alter the state of a fermion because it just changes $\psi(x)$ by $\psi^*(-x)$ while maintains its eigenenergy. Furthermore, for $V_v = 0$ the transformation $\theta \rightarrow -\theta$ exchanges the upper and lower components of the Dirac spinor. The results for $\theta = \pm \pi/2 + \delta$, where $-\pi/2 < \delta < \pi/2$, can be obtained from the results for $\theta = \pm \pi/2 - \delta$ by changing $E$ by $-E$.

When $\theta = \pm \pi/2$, the case of a pure scalar potential, the energy levels are given by

$$E = \pm mc^2 \sqrt{1 - \left(\frac{q}{q + n}\right)^2}, \quad n = 0, 1, 2, \ldots$$  \hspace{1cm} (36)

so that the energy levels for fermions and antifermions are symmetric about $E = 0$. Note that $E \approx mc^2$ only if $n \gg q$, thus the nonrelativistic limit of the theory would be, in a limited sense, a regime of large quantum numbers. On the other hand, in the regime of strong coupling, i.e., for $q \gg 1$, one has $E \approx mc^2 n/q$ and as the coupling becomes extremely strong the lowest effective eigenvalues end up close to zero. Now one sees clearly that the eigenvalues for a zero-energy solution, in contrast to what is declared in Ref. [35], can be
obtained as a limit case of a nonzero-energy solution. The Dirac eigenenergies are plotted in Fig. 1 for the four lowest bound states as a function of $\theta = \pi/2 + \delta$. Starting from $\pi/2$, as $\theta$ is increased ($\delta > 0$) all the energy levels move toward the upper continuum. On the other side, as $\theta$ decreases ($\delta < 0$) all the energy levels move toward the lower continuum. The mixed vector-scalar Coulomb potential present a continuous transition by starting from $\theta = \pm \pi/2$, with energy levels for fermions and antifermions always present. When $\theta = 0$ or $\theta = \pm \pi$, though, there is a clear phase transition. The phase transition shows its face not only for the energy levels but also for the eigenspinor. This phenomenon is due to the abrupt disappearance of the singularity $1/x^2$ in the effective potential. For $\theta = 0$ ($V_v = V_s$) the energy levels given by

$$E = mc^2 \frac{n^2 - q^2}{n^2 + q^2}, \quad n = 1, 2, 3, \ldots \quad (37)$$

are pushed down from the upper continuum so that these energy levels correspond to bound states of fermions (see Fig. 2). In this case there are no energy levels for antifermions. All the Dirac eigenvalues are positive if $q < 1$, and some negative eigenvalues arise if $q > 1$. One has $E - mc^2 \approx -2q^2/n^2$ as long as $q \ll n$. When $\theta = \pm \pi$ only the energy levels emerging from the lower continuum, the energy levels for antifermions, survive:

$$E = -mc^2 \frac{n^2 - q^2}{n^2 + q^2}, \quad n = 1, 2, 3, \ldots \quad (38)$$

For this case the energy levels are illustrated in Fig. 3 as a function of $q$. Note that $E \approx mc^2$ only in the strong-coupling regime.

In all the circumstances, namely $|V_s| \geq |V_v|$, there is no atmosphere for the spontaneous production of particle-antiparticle pairs. No matter the signs of the potentials or how strong they are, the positive- and negative-energy levels neither meet nor dive into the continuum. Thus there is no room for the production of fermion-antifermion pairs. This all means that Klein’s paradox never comes to the scenario.

Figs. 4, 5 and 6 illustrate the behavior of the upper and lower components of the Dirac spinor, $|\phi|^2$ and $|\chi|^2$, and the position probability density, $|\psi|^2 = |\phi|^2 + |\chi|^2$, on the positive side of the $x$-axis for the positive-energy solutions, with $n = 1$, for $\theta = 0, \pi/4$ and $\pi/2$, respectively. The normalization constant was obtained by numerical computation. Since the inversely linear potential given by (17) is invariant under reflection through the origin $(x \rightarrow -x)$, eigenfunctions of the original Hamiltonian given by (2) with
well-defined parities can be found. For \( \theta \neq 0, \pm \pi \), those eigenfunctions can be constructed by taking symmetric and antisymmetric linear combinations of \( \Psi_+ \) and \( \Psi_- \). These new eigenfunctions are continuous everywhere and possess the same Dirac eigenvalue, then there is a two-fold degeneracy. Nevertheless, the matter is a little more complicated for \( \theta = 0, \pm \pi \). For \( \theta \neq 0, \pm \pi \), the effective potential for \( \phi \) always presents a positive singularity so that it makes sense to consider only the half-line. For \( \theta = 0, \pm \pi \), though, there are attractive singularities on both sides of the \( x \)-axis, so that the behaviour of a fermion on one side of the \( x \)-axis is sensitive to what happens on the other side. Therefore, the entire line has to be considered. In these last circumstances \( \Psi_+ = \phi \) and \( \Psi_- = \chi \), for \( \theta = 0 \), and \( \Psi_+ = \pm i\chi \) and \( \Psi_- = \pm i\phi \), for \( \theta = \pm \pi \). Recall that \( \phi \) vanishes at the origin but \( \chi \) does not, so one of those symmetric and antisymmetric linear combinations of \( \Psi_+ \) and \( \Psi_- \) is discontinuous at the origin. In fact, the pair of first-order differential equations given by (9) implies that \( \Psi_+ \) and \( \Psi_- \) can be discontinuous wherever the potential undergoes an infinite jump. In the specific case under consideration, the effect of the singularity of the potential can be evaluated by integrating (9) from \( -\delta \) to \( +\delta \) and taking the limit \( \delta \to 0 \). Since \( \Psi_+ \) and \( \Psi_- \) have opposite parities, the connection conditions can be summarized in the couple of formula:

\[
\Psi_-(+\delta) = iq \int_{-\delta}^{+\delta} dx \, \frac{\Psi_+}{|x|} , \quad \text{for } \Psi_+ \text{ even}
\]

\[
\int_{-\delta}^{+\delta} dx \, \Psi_- = 0 , \quad \text{for } \Psi_+ \text{ odd}
\]

One can verify that the first connection condition is not satisfied for \( \theta = 0 \), while the second one is not satisfied for \( \theta = \pm \pi \). Therefore, we are forced to conclude that the upper component of the original Dirac spinor (\( \Psi_+ \)) must be an odd-parity function for \( V_v = V_s \), and an even-parity function for \( V_v = -V_s \), so that the bound-state solutions for \( V_v = \pm |V_s| \) are nondegenerate.

4 Conclusions

We have succeed in searching for exact Dirac bounded solutions for massive fermions by considering a convenient mixing of vector-scalar inversely linear potentials in 1+1 dimensions. The satisfactory completion of this task has been alleviated by the methodology of effective potentials which has transmuted the question into Schrödinger-like equations with effective Kratzer-like potentials.
Isolated solutions have also been searched and they have been found only in the special case of a pure scalar potential. We have shown that those isolated zero-energy solutions are consistent with the ultrarelativistic limit of the theory. The existence of such zero-eigenmodes does not conform with the “topological” criterion of Ref. [9], which requires that the scalar potential has different limits for \( x \to +\infty \) and \( x \to -\infty \). From Eq. (16) one can see that there can be other sorts of scalar potentials holding zero-energy solutions which do not satisfy the “topological” criterion. Such a criterion is circumvented because the upper and lower components of the Dirac spinor are normalizable on the entire line, although they are not simultaneously normalizable on each side of the \( x \)-axis, because they can not be simultaneously different from zero there.

For \( -|V_s| < V_v < +|V_s| \), there exist bound-state solutions for fermions and antifermions and the plot of the eigenenergy as a function of the mixing parameter, \( \delta \), looks like a hysteresis loop (Fig. 1). Those two-fold degenerate bounded solutions do not present a nonrelativistic limit because the coupling constant, \( q \), can never be a small quantity.

For the “saturation points”, viz. \( V_v = \pm|V_s| \), there are bound-state solutions either for fermions or for antifermions (Figs. 2 and 3). Those phase transitions manifest not only for the energy levels but also for the eigenspinor as well as for the coupling constant. Furthermore, the solutions become discontinuous at the origin. A careful analysis of those discontinuities shows that the potential can only hold bounded solutions when the upper component of the Dirac spinor behaves as an odd (even)-parity function for \( V_v = +|V_s| \) (\( V_v = -|V_s| \)). Therefore, the phase transitions transform two-fold degenerate solutions into nondegenerate ones. In the “critical points” the coupling constant can assume any value and for the special case \( V_v = +|V_s| \) the theory presents a definite nonrelativistic limit (\( q \ll 1 \) and \( E \simeq mc^2 \)).

Beyond its intrinsic importance as a new solution for a fundamental equation in physics, the problem analyzed in this paper presents unusual results. Moreover, it favors the conclusion that even-parity solutions to the nonrelativistic one-dimensional hydrogen atom do not exist.

Acknowledgments

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References


Figure 1: Dirac eigenvalues for the four lowest energy levels as a function of $\delta$ ($V_v = -V_s \sin(\delta)$, with $-\pi/2 < \delta < \pi/2$). The full thick line stands for $n = 1$, the full thin line for $n = 2$, the heavy dashed line for $n = 3$ and the light dashed line for $n = 4$ ($m = c = 1$ and $q = 10/|\sin(\theta)|$). The isolated point in $\delta = 0$ ($E = 0$) is always present and it corresponds to the zero-eigenmode with $n = 0$. 
Figure 2: Dirac eigenvalues for the four lowest energy levels as a function of $q$ for $\theta = 0$ ($V_v = V_s$). The full thick line stands for $n = 1$, the full thin line for $n = 2$, the heavy dashed line for $n = 3$ and the light dashed line for $n = 4$ ($m = c = 1$).
Figure 3: The same as in Fig. 2 for $\theta = \pi$ ($V_v = -V_s$).
Figure 4: $|\phi|^2$ (full thin line), $|\chi|^2$ (dashed line) and $|\psi|^2 = |\phi_+|^2 + |\chi|^2$ (full thick line) as a function of $x$, corresponding to the ground state ($n = 1$) for $\theta = 0$ with $q = 1/2$ ($m = c = \hbar = 1$).
Figure 5: $|\phi|^2$ (full thin line), $|\chi|^2$ (dashed line) and $|\psi|^2 = |\phi_+|^2 + |\chi|^2$ (full thick line) as a function of $x$, corresponding to the positive-ground-state energy ($n = 1$) for $\theta = \pi/4$ with $q = \sqrt{2}$ ($m = c = \hbar = 1$).
Figure 6: $|\phi|^2$ (full thin line), $|\chi|^2$ (dashed line) and $|\psi|^2 = |\phi_+|^2 + |\chi|^2$ (full thick line) as a function of $x$, corresponding to the positive-first-excited-state energy ($n = 1$) for $\theta = \pi/2$ with $q = 1$ ($m = c = \hbar = 1$).