Infra-red modification of gravity from asymmetric branes

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Abstract

We consider a single Minkowski brane sandwiched in between two copies of anti-de Sitter space. We allow the bulk Planck mass and cosmological constant to differ on either side of the brane. Linearised perturbations about this background reveal that gravity can be modified in the infra-red. At intermediate scales, the braneworld propagator mimics four-dimensional GR in that it has the correct momentum dependence. However it has the wrong tensor structure. Beyond a source dependant scale, we show that quadratic brane bending contributions become important, and conspire to correct the tensor structure of the propagator. We argue that even higher order terms can consistently be ignored up to very high energies, and suggest that there is no problem with strong coupling. We also consider scalar and vector perturbations in the bulk, checking for scalar ghosts.

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1 Introduction

How much do we believe in Einstein’s General Theory of Relativity? We certainly don’t believe it at energies beyond the Planck scale, where a quantum theory is required. In fact, experimental tests of gravity have only ever been conducted at much lower scales. For example, Newton’s law has been tested down to about 0.2 mm, or equivalently up to energies of around $10^{-3}$ eV. Solar system tests of GR are only valid outside of the Schwarzschild radius of the Sun.

Ultra-violet modifications of gravity beyond $10^{-3}$ eV occur in all higher dimensional theories, such as String Theory. Recently, the idea that gravity is also modified in the infra-red has received much interest [1, 2, 3, 4, 5, 6, 7]. This is entirely plausible. Again, Newton’s law has only been tested up to distances of around $10^{26}$ cm, and solar system tests of GR do not extend much beyond the size of the solar system itself.

Infra-red modifications of gravity have some attractive features. They can be used to tackle the cosmological constant problem [8, 9] by weakening gravity at low energies. To understand this in a simple way, suppose we have Newton’s constant $G$ at intermediate scales, and $G_{\text{IR}} \ll G$ in the infra-red. Even for a large cosmological constant $\Lambda$, the contribution to the Hubble rate is considerably suppressed at large values of the scale factor

$$H^2 \sim G_{\text{IR}} \Lambda \ll GA$$

However, that is not to say that these models prohibit late time inflation. On the contrary, these models generically give rise to Friedmann equations that are non-linear in $H^2$. These permit self accelerating solutions that can be tuned to the current Hubble rate [10, 11, 12, 13, 14, 15].

Unfortunately, these models are not without their problems. The most common of these is the vDVZ discontinuity that appears in the graviton propagator at all scales [16, 17]. This represents a deviation from GR that will give the wrong prediction for light bending around the Sun. In some cases, it has been argued that this discontinuity reflects a breakdown in the linearised analysis [18, 19], as opposed to a genuine sickness in the theory. Some theories predict ghost states, and therefore violate unitarity [20]. A sickness as bad as this usually leads to the death of a theory. We should emphasise, however, that some models are indeed ghost-free [1, 7].

In [21] we considered asymmetric branes. By asymmetric, we mean that the gravitational parameters of the theory can differ on either side of the brane. There are at least two ways in which this asymmetry might arise. Firstly, suppose we have some sort of wine bottle shaped compactification down to 5 dimensions. In the effective theory, the Planck scale at the fat end of the bottle will be less than that at the thin end. Secondly, in [22, 23] we showed how to construct a domain wall living entirely on the brane. In some cases [23], the Planck scale on the brane differed on either side of the domain wall. Asymmetric branes admit self accelerating solutions [21]. It is natural to ask, therefore, if asymmetric brane models exhibit infra-red modifications of gravity.

In this paper, we will consider a positive tension brane. To the left of the brane, the
bulk is described by the Einstein-Hilbert action with Planck mass, $M_L$, and negative cosmological constant, $\Lambda_L = -6/l_L^2$. To the right, we have Planck mass, $M_R$, and cosmological constant, $\Lambda_R = -6/l_R^2$. In general, we will take $M_L \gg M_R$ and $l_L \ll l_R$.

We will be particularly interested in Minkowski branes. Because of the asymmetry, we are allowed three classes of solution. We have (i) a Randall-Sundrum like solution where the warp factor in the metric decays away from the brane on both sides \cite{24}, (ii) an “inverse” RS solution where the warp factor grows on both sides, and (iii) a mixed solution where the warp factor decays on one side and grows on the other.

If we place a small matter source on the brane, we can investigate linearised perturbations about the background solutions. For the RS solution and the mixed solution we see the introduction of a new length scale

$$r = \frac{M_R^3 l_L}{M_L^3}$$

At intermediate energies ($1/r \ll p \ll 1/l_L$), the braneworld propagator is proportional to $1/p^2$, and will reproduce Newton’s law. At lower energies ($1/l_R \ll p \ll 1/r$), the momentum dependence changes, so that Newton’s law is modified in the infra-red. However, as in the DGP model \cite{1}, the propagator suffers from a vDVZ discontinuity at intermediate scales, and will predict the wrong results for light bending around the Sun.

It turns out that the vDVZ discontinuity can be removed by including brane bending terms to quadratic order. For a source of mass $m$, we discover yet another scale

$$p_* = \left( \frac{M_L^3 l_L}{m r^2} \right)^{\frac{1}{3}}.$$  (3)

For $p_* \ll p \ll 1/l_L$, the quadratic terms dominate over the linear ones. The result is that the braneworld propagator agrees exactly with four-dimensional General Relativity. We might be worried about the breakdown of the linearised analysis at this scale. In the DGP model, this breakdown has been linked to a strong coupling problem \cite{25, 26, 27, 28}. However, we will argue that there is nothing to worry about. This is because even though quadratic terms become important when $p \sim p_*$, even higher order terms are only important at much higher energies. It seems that we can indeed consistently modify four-dimensional GR in the infra-red.

The rest of this paper is organised as follows. In section 2 we describe our set-up in detail, before deriving background solutions in section 3. In section 4 we derive the linearised equations of motion and solve them. We analyse the solutions and establish the existence of the long distance scale $r$, as well as the vDVZ discontinuity. In section 5 we consider scalar and vector perturbations in the bulk. Vector perturbations only exist for the inverse RS solution. Scalar perturbations exist whenever we include the AdS boundary. We derive the effective action to check that these scalar fields have a well behaved kinetic term. In other words, they are not ghosts. In section 6 we go to quadratic order in the brane bending. We discover the new scale $p_*$, and are able to correct the tensor structure of the propagator. In section 7 we briefly show how the mixed case admits self accelerating solutions. Finally, section 8
contains some discussion. In particular, we argue that the quadratic analysis is valid up to very high scales, and that there is no strong coupling problem.

2 The set-up

Consider two 5 dimensional spacetimes, $\mathcal{M}_L$ and $\mathcal{M}_R$, separated by a domain wall. The domain wall is a 3-brane corresponding to our universe. Our set-up is described by the following action:

$$ S = S_{\text{bulk}} + S_{\text{brane}} $$

where the contribution from the bulk is given by

$$ S_{\text{bulk}} = \sum_{i=L,R} M_i^3 \int_{\mathcal{M}_i} d^5 \sqrt{-g} \left( R - 2\Lambda_i \right) + 2 M_i^3 \int_{\text{brane}} d^4 \sqrt{-\gamma} K^{(i)} $$

Here, $M_i$ is the 5-dimensional Planck mass in $\mathcal{M}_i$. We have also included a negative cosmological constant, $\Lambda_i = -6/l_i^2$. We will not assume that there is $\mathbb{Z}_2$ symmetry across the brane, so that the $M_i$ and $l_i$ can differ on either side of the brane. The bulk metric is given by $g_{ab}$, with corresponding Ricci scalar, $R$. $\gamma_{ab}$ is the induced metric on the brane. The extrinsic curvature of the brane in $\mathcal{M}_i$ is given by

$$ K^{(i)}_{ab} = \gamma^c_a \gamma^d_b \nabla_{(c} n_{d)} $$

where $n^a$ is the unit normal to the brane in $\mathcal{M}_i$, pointing out of $\mathcal{M}_i$. For the most part we will not bother with the index $i$ when referring to bulk quantities, although the reader should understand that they are there.

The brane part of the action is given by

$$ S_{\text{brane}} = \int_{\text{brane}} d^4 x \left( -\sigma \sqrt{-\gamma} + \mathcal{L}_m \right) $$

where $\sigma$ is the brane tension and $\mathcal{L}_m$ describes any additional matter.

The bulk equations of motion are given by the Einstein equations

$$ R_{ab} - \frac{1}{2} R g_{ab} = -\Lambda g_{ab} $$

The boundary conditions at the brane are governed by the Israel equations [29]. This comes from varying the action [11] with respect to the brane metric. Given a quantity $Z_i$ defined in $\mathcal{M}_i$ we shall henceforth write $\langle Z \rangle = (Z_L + Z_R)/2$, for the average across the brane, and $\Delta Z = Z_L - Z_R$, for the difference. The brane equations of motion are given by

$$ 2 \langle M^3 K_{ab} \rangle - \frac{\sigma}{6} \gamma_{ab} = \frac{1}{2} \left( T_{ab} - \frac{1}{3} T \gamma_{ab} \right) $$

where

$$ T_{ab} = -\frac{2}{\sqrt{-\gamma}} \frac{\partial \mathcal{L}_m}{\partial \gamma_{ab}} $$

is the energy momentum tensor for the additional matter on the brane.
3 Background solutions

In this section we will derive the metric, $\bar{g}_{ab}$, for the background spacetime. These correspond to solutions to the equations of motion when no additional matter is present. Let us introduce coordinates $x^a = (x^\mu, z)$, and assume that the brane is located at $z = 0$. The left hand side, $\mathcal{M}_L$ corresponds to $z < 0$, whereas the right hand side, $\mathcal{M}_R$ corresponds to $z > 0$. In order to trust our classical analysis in the bulk, we need to assume

$$M > 1/l \quad (11)$$

Now seek solutions of the form

$$ds^2 = \bar{g}_{ab}dx^d dx^b = a^2(z)\eta_{\mu\nu}dx^\mu dx^\nu + dz^2 \quad (12)$$

where $\eta_{\mu\nu}$ is four-dimensional Minkowski space. The Einstein equations yield the following

$$\left(\frac{a'}{a}\right)^2 = \frac{1}{l^2}, \quad \frac{a''}{a} = \frac{1}{l^2} \quad (13)$$

Without loss of generality we impose the condition $a(0) = 1$ so that

$$a(z) = \exp\left(\frac{\theta z}{l}\right) \quad (14)$$

where $\theta_i = \pm 1$

Given the solution (14), the Israel equations (9) impose the following condition on the brane tension

$$\Delta \left(\frac{M^3 \theta}{l}\right) = \frac{\sigma}{6} \quad (15)$$

Note that this gives the usual fine-tuning of the brane tension for the Randall-Sundrum model [24], where $\Delta M = \Delta l = 0$, and $\theta_L = 1 = -\theta_R$.

4 Linearised perturbations

We shall now consider metric perturbations about the background solutions we have just derived. We will allow additional matter, $T_{ab}$, to be present on the brane, but not in the bulk. We begin by deriving the bulk equations of motion, and the boundary conditions at the brane.

4.1 Bulk equations of motion and boundary conditions

Let us define $g_{ab} = \bar{g}_{ab} + \delta g_{ab}$ to be the perturbed metric. We will work in Gaussian normal (GN) coordinates, so that

$$\delta g_{\mu\nu} = 0 \quad (16)$$
Since we have no additional bulk matter, we can take the metric to be transverse-tracefree in the bulk. In other words, \( \delta g_{\mu\nu} = \chi_{\mu\nu}(x, z) \), where
\[
\partial_{\nu}\chi^{\nu}_{\mu} = 0 = \chi^{\mu}_{\mu}
\]
Here indices are raised and lowered with \( \bar{g}_{\mu\nu} \). In this choice of gauge, the linearised bulk equations of motion are given by
\[
\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{4}{l^2} \right] \chi_{\mu\nu}(x, z) = 0
\]
Unfortunately, we no longer expect the brane to be at \( z = 0 \). The presence of matter causes the brane to bend \([30]\). In fact, we would even expect the brane to be bent by different amounts when viewed on different sides of the brane. In other words, the brane position is given by \( z = f_i(x) \) when viewed in \( M_i \). This makes it very difficult to apply the Israel equations \([39]\). To get round this, we need to apply a gauge transformation in \( M_L \) and another in \( M_R \). The new coordinates should also be GN, with the brane positioned at \( z = 0 \).

To this end, we make the following coordinate transformations
\[
z \to z - f(x), \quad x^\mu \to x^\mu - Q^\mu(x) + \frac{\theta l}{2} (1 - a^{-2}) \partial^\mu f
\]
Note that in addition to \( \theta \) and \( l \), \( f(x) \) and \( Q^\mu(x) \) are understood to have an invisible index \( i = L, R \). In these new coordinates,
\[
\delta g_{\mu\nu} = \chi_{\mu\nu}(x, z) + 2a^2 Q_{(\mu, \nu)} + \theta l (1 - a^2) \partial_\mu \partial_\nu f + \frac{2\theta}{l} f \bar{g}_{\mu\nu}
\]
We can evaluate this at \( z = 0 \) to derive the brane metric
\[
\delta \gamma_{\mu\nu} = \chi_{\mu\nu}(x, 0) + 2Q_{(\mu, \nu)} + \frac{2\theta}{l} f \eta_{\mu\nu}
\]
For this to be well defined, we require that \( \Delta (\delta \gamma_{\mu\nu}) = 0 \). Since the geometry is independent of the pure gauge term, \( 2Q_{(\mu, \nu)} \), we should also demand that the remaining part of \( \delta \gamma_{\mu\nu} \) is well defined. Making use of the fact that \( \chi^\mu_{\mu} = 0 \), this implies the following
\[
\Delta \chi_{\mu\nu}(x, 0) = 0, \quad \Delta Q_{(\mu, \nu)} = 0, \quad \Delta \left( \frac{\theta f}{l} \right) = 0
\]
The last condition suggests that we introduce the function \( F = \theta f / l \). The bulk metric is now given by
\[
\delta g_{\mu\nu} = h_{\mu\nu} = \chi_{\mu\nu}(x, z) + 2a^2 Q_{(\mu, \nu)} + l^2 (1 - a^2) \partial_\mu \partial_\nu F + 2F \bar{g}_{\mu\nu}
\]
and the brane metric by
\[
\delta \gamma_{\mu\nu} = h_{\mu\nu}^{br} = \chi_{\mu\nu}^{br} + 2Q_{(\mu, \nu)} + 2F \eta_{\mu\nu}
\]
where $\chi_{\mu\nu}^{\text{br}} = \chi_{\mu\nu}(x, 0)$. Making use of equation (15), the Israel equations (9) imply the following boundary condition

$$
\Delta \left[ M^3 \chi_{\mu\nu}'(x, 0) \right] - \frac{\sigma}{3} \chi_{\mu\nu}^{\text{br}} = \Sigma_{\mu\nu}(x)
$$

(25)

where

$$
\Sigma_{\mu\nu}(x) = T_{\mu\nu} - \frac{1}{3} T \eta_{\mu\nu} + 2 \Delta (M^3 \theta l) \partial_\mu \partial_\nu F
$$

(26)

Given that $\chi_{\mu\nu}$ is transverse-tracefree, we immediately see that $\Sigma_\mu = 0$. This gives

$$
2 \Delta (M^3 \theta l) \partial^2 F = \frac{T}{3}
$$

(27)

Here we see explicitly how matter on the brane causes it to bend.

### 4.2 Solving the equations of motion

In four dimensional momentum space, the bulk equation (18) becomes

$$
\left[ -\frac{p^2}{a^2} + \frac{\partial^2}{\partial z^2} - \frac{4}{l^2} \right] \tilde{\chi}_{\mu\nu}(p, z) = 0
$$

(28)

where tilde represents the Fourier transform. The general solution is

$$
\tilde{\chi}_{\mu\nu}(p, z) = A_{\mu\nu}(p) I_2 \left( \frac{pl}{a} \right) + B_{\mu\nu}(p) K_2 \left( \frac{pl}{a} \right)
$$

(29)

where $I_n$ and $K_n$ are modified Bessel functions of order $n$ [31]. Naturally, we require that $\chi_{\mu\nu}(p, z) \to 0$ as $|z| \to \infty$. To impose this condition we need to consider three separate cases: (i) the Randall-Sundrum case for which $\theta_L = 1 = -\theta_R$, (ii) the inverse Randall-Sundrum case for which $\theta_R = 1 = -\theta_L$ and (iii) the mixed case for which $\theta_L = 1 = \theta_R$. In figure [1] we can see the generic behaviour of the warp factor in each case. Making use of the fact that [31]

$$
I_2(y) \to 0, \quad K_2(y) \to \infty \quad \text{as} \quad y \to 0
$$

(30)

$$
I_2(y) \to \infty, \quad K_2(y) \to 0 \quad \text{as} \quad y \to \infty
$$

(31)

we see that the boundary condition at infinity is satisfied if and only if

$$
\tilde{\chi}_{\mu\nu}(p, z) = C_2 \left( \frac{pl}{a} \right) \tilde{\chi}_{\mu\nu}^{\text{br}}(p)
$$

(32)

where

$$
C_2 = \begin{cases} 
K_2 & \text{for RS case} \\
I_2 & \text{for inverse RS case}
\end{cases}
$$

(33)

and for the mixed case $C_2^L = K_2$ and $C_2^R = I_2$. 

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(a) The Randall-Sundrum case.

(b) The inverse Randall-Sundrum case.

(c) The mixed case.

Figure 1: The behaviour of the warp factor. For the RS case, $a(z)$ vanishes at infinity. For the inverse RS case, $a(z)$ grows exponentially at infinity. For the mixed case, $a(z)$ vanishes at $-\infty$ but grows exponentially at $+\infty$. 
The Fourier transformed boundary conditions (25) and (27) are given by
\[
\Delta \left[ M^3 \chi'_{\mu\nu}(p, 0) \right] - \frac{\sigma}{3} \tilde{\chi}^{br}_{\mu\nu} = \tilde{\Sigma}_{\mu\nu}(p) \tag{34}
\]
\[
2\Delta(M^3 \theta l)p^2 \tilde{F} = -\frac{\tilde{T}}{3} \tag{35}
\]
Given that (31)
\[
y I'_2(y) + 2 I_2(y) = y I_1(y) \quad y K'_2(y) + 2 K_2(y) = -y K_1(y) \tag{36}
\]
we can insert the solution (32) into (34) to give
\[
\tilde{\chi}^{br}_{\mu\nu} = \frac{\tilde{\Sigma}_{\mu\nu}(p)}{R(p)} \tag{37}
\]
where
\[
R(p) = -p \Delta \left[ M^3 \theta C_1(pl) \right] \tag{38}
\]
Here
\[
C_1 = \begin{cases} 
-K_1 & \text{for RS case} \\
I_1 & \text{for inverse RS case} \end{cases} \tag{39}
\]
and for the mixed case \( C_L^F = -K_1 \) and \( C_R^F = I_1 \). Note that we have also used equation (15) in deriving equation (37).

In momentum space, the metric on the brane is given by
\[
\tilde{h}^{br}_{\mu\nu} = \tilde{\chi}^{br}_{\mu\nu} + 2ip_{(\mu} \tilde{Q}_{\nu)} + 2\tilde{F} \eta_{\mu\nu} \tag{40}
\]
We can choose \( \tilde{Q}_{\mu}(p) \) to cancel off the term proportional to \( p_\mu p_\nu \tilde{F} \) in \( \tilde{\Sigma}_{\mu\nu}(p) \). This leaves
\[
\tilde{h}^{br}_{\mu\nu} = \frac{1}{R(p)} \left[ \tilde{T}_{\mu\nu} - \frac{1}{3} \tilde{T} \eta_{\mu\nu} \right] - \frac{1}{6\alpha p^2} \tilde{T} \eta_{\mu\nu} \tag{41}
\]
where
\[
\alpha = \frac{1}{2} \Delta(M^3 \theta l) \tag{42}
\]
This all makes sense as long as \( R(p) \) and \( \alpha \) do not vanish in \( p > 0 \).

At some energy scale, the solution (41) should agree with the corresponding result from 4-dimensional Einstein gravity
\[
\tilde{h}_{\mu\nu} = \frac{1}{m_{pl}^2 p^2} \left[ \tilde{T}_{\mu\nu} - \frac{1}{2} \tilde{T} \eta_{\mu\nu} \right] \tag{43}
\]
This requires there to be an energy scale for which \( R(p) \sim \alpha p^2 \) and \( \alpha = m_{pl}^2 > 0 \). We shall now investigate whether or not such a scale exists.

At energies that are small compared with the bulk curvature, we have \( pl \ll 1 \). In this limit we can approximate the Bessel functions as follows (31)
\[
I_1(pl) \sim \frac{1}{2} pl, \quad I_2(pl) \sim \frac{1}{8} (pl)^2, \quad K_1(pl) \sim \frac{1}{pl}, \quad K_2(pl) \sim \frac{2}{(pl)^2} \tag{44}
\]
At high energies, \( pl \gg 1 \), we have
\[
\frac{I_1(pl)}{I_2(pl)} \sim 1, \quad \frac{K_1(pl)}{K_2(pl)} \sim 1 \quad (45)
\]

For the RS case (\( \theta_L = 1 = -\theta_R \)) we see that
\[
R(p) = 2p \left\langle \frac{M^3 K_1(pl)}{K_2(pl)} \right\rangle \sim \begin{cases} 
\alpha p^2 & \text{for } p \ll 1/l_R \\
\frac{1}{2} M^3 l_L p^2 + M^3 R & \text{for } 1/l_R \ll p \ll 1/l_L \\
2 \langle M^3 \rangle p & \text{for } p \gg 1/l_L 
\end{cases} \quad (46)
\]

where \( \alpha = \langle M^3 l \rangle \). Note that we have assumed \( l_R \gg l_L \).

At first glance, it appears that we only have Einstein gravity when \( p \ll 1/l_R \). However, if \( M^3 l_L \gg M^3 l_R, R(p) \sim \frac{1}{2} M^3 l_L p^2 \) for \( 1/l_R \ll p \ll 1/l_L \). Furthermore, we now have \( \alpha \approx \frac{1}{2} M^3 l_L \). This means that we also have Einstein gravity whenever \( 1/l_R \ll p \ll 1/l_L \). In any case, there is no modified gravity in the infra-red.

For the inverse RS case (\( \theta_R = 1 = -\theta_L \)), we see that
\[
R(p) = 2p \left\langle \frac{M^3 I_1(pl)}{I_2(pl)} \right\rangle \sim \begin{cases} 
8 \langle M^3 l \rangle & \text{for } p \ll 1/l_R \\
M^3 l_L + 4 \frac{M^3}{l_R} & \text{for } 1/l_R \ll p \ll 1/l_L \\
2 \langle M^3 \rangle p & \text{for } p \gg 1/l_L 
\end{cases} \quad (47)
\]

and \( \alpha = -\langle M^3 l \rangle \). Einstein gravity cannot be produced at any scale.

Finally, for the mixed case (\( \theta_L = 1 = \theta_R \)), we see that
\[
R(p) = p \left[ M^3 \frac{K_1(pl)L}{K_2(pl)L} + M^3 \frac{I_1(pl)L}{I_2(pl)L} \right] \sim \begin{cases} 
\frac{1}{2} M^3 l_L p^2 + 4 \frac{M^3}{l_R} & \text{for } p \ll 1/l_R \\
\frac{1}{2} M^3 l_L p^2 + M^3 R & \text{for } 1/l_R \ll p \ll 1/l_L \\
2 \langle M^3 \rangle p & \text{for } p \gg 1/l_L 
\end{cases} \quad (48)
\]

and \( \alpha = \frac{1}{2} \Delta \langle M^3 l \rangle \). Again, at first glance it appears that Einstein gravity can never be reproduced. However, as in the RS case, if \( M^3 l_L \gg M^3 l_R, \) we have Einstein gravity for \( 1/l_R \ll p \ll 1/l_L \). The difference here is that gravity can be modified in the far infra-red region, \( p \ll 1/l_L \). Unfortunately, as we shall see in section 5, the mixed case contains a radion ghost whenever \( \alpha > 0 \). Here \( \alpha \approx \frac{1}{2} M^3 l_L > 0 \).

These results are a little disappointing. We can only get Einstein gravity in the RS case, without any modifications in the infra-red. This is nothing new 24.

However, all is not lost! Consider the RS case and the mixed case for \( p \gg 1/l_R \). Now suppose \( M^3 l_L \ll M^3 l_R, \) in contrast to what we discussed earlier. This gives
\[
\alpha \approx \begin{cases} 
\frac{1}{2} M^3 R & \text{for the RS case} \\
-\frac{1}{2} M^3 l_R & \text{for the mixed case} 
\end{cases} \quad (49)
\]

Since \( \alpha < 0 \) for the mixed case, there is no radion ghost. In both cases, we see the introduction of a new length scale
\[
\frac{M^3 l_L}{M^3 R} \ll l_R \quad (50)
\]
We now have
\[ R(p) \sim \begin{cases} 
M^3_{R}p & \text{for } 1/l_R \ll p \ll 1/r \\
\frac{1}{2}M^3_{L}l_lp^2 & \text{for } 1/r \ll p \ll 1/l_L \\
M^3_{L} & \text{for } p \gg 1/l_L 
\end{cases} \] (51)

Note that we have also assumed that \( M_L \gg M_R \). In these limits, the brane bending term, \( \tilde{T}/\alpha p^2 \) is always much less than the remaining terms in \( \tilde{h}^{br}_{\mu\nu} \). This implies that
\[ \tilde{h}^{br}_{\mu\nu} \approx \frac{1}{R(p)} \left[ \tilde{T}_{\mu\nu} - \frac{1}{3} \tilde{T} \eta_{\mu\nu} \right] \] (52)

In the intermediate regime (\( 1/r \ll p \ll 1/l_L \)), our solution has the correct momentum dependence, but the wrong tensor structure. As we move into the infra-red (\( 1/l_R \ll p \ll 1/r \)), or the ultra-violet (\( p \gg 1/l_L \)) our momentum dependence changes so that gravity is modified.

Of course, our solution (52) no longer makes sense when \( \tilde{h}^{br}_{\mu\nu} \) is of order one. For a source of mass, \( m \), note that \( |\tilde{T}| \sim mp^3 \). This means that \( \tilde{h}^{br}_{\mu\nu} \) schematically goes like \( mp^3/R(p) \). This becomes of order one when \( p \sim p_{\text{cut-off}} \). For very large masses (\( m > M^3_L l_L r \)), this cut-off occurs in the infra-red. Otherwise we have
\[ p_{\text{cut-off}} = \begin{cases} 
M^3_{L}l_L/m & \text{for } M^3_{L}l_L < m < M^3_{L}l_L r \\
(M^3_{L}/m)^{1/2} & \text{for } m \leq M^3_{L}l_L 
\end{cases} \] (53)

Note that the upper value in (53) lies in the intermediate energy range, whereas the lower value lies in the ultra-violet.

We might think that this cut-off represents the scale at which the linearised theory breaks down. Certainly, the linearised theory makes no sense beyond the cut-off, but is it really valid beforehand? In the DGP model [1], the linearised theory actually breaks down much sooner than expected. This is because higher order terms become important [19]. Curiously, these non-linear terms can sometimes correct the problems with the tensor structure at intermediate energies [32, 33, 34, 35, 36]. In section 6, we will see that the same thing happens here.

## 5 Scalar and vector perturbations

We shall now discuss scalar and vector perturbations in the bulk, when there is no additional matter on the brane (\( T_{ab} = 0 \)).

### 5.1 Scalar perturbations and ghosts

Braneworld models that exhibit infra-red modifications of gravity often contain ghosts (see, for example [1, 20]). Typically, these ghosts appear in the scalar sector. We will be forced to eliminate any model that contains a ghost-like scalar field in the effective theory.
Consider the following scalar perturbations about the background, $\bar{g}_{ab}$,

$$
\delta g_{\mu\nu} = a^2 (2\partial_\mu \partial_\nu E + 2A_{\mu\nu}), \quad \delta g_{\mu z} = \partial_\mu B, \quad \delta g_{zz} = 2\phi
$$

(54)

Under the scalar gauge transformations

$$
x^\mu \to x^\mu + \partial^\mu \xi, \quad z \to z + \xi
$$

(55)

the scalar fields transform as follows

$$
E \to E - \xi, \quad A \to A - \theta l \xi, \quad B \to B - (\xi z + a^2 \xi'), \quad \phi \to \phi - \xi'
$$

(56)

It is easy to check that the following quantities are gauge invariant

$$
X = \phi - l\theta A', \quad Y = B - a^2 E' - l\theta A
$$

(57)

The boundary condition at infinity requires there to be no physical perturbation there. This amounts to the gauge invariants $X$ and $Y$ vanishing at infinity. We can see this most easily by choosing the gauge $E = A = 0$, so that $X = \phi$ and $Y = B$.

In terms of the gauge invariants, the bulk equations of motion (58) are given by

$$
0 = \partial_\mu \partial_\nu \left[ Y' + \frac{2\theta}{l} Y - 2X \right] + \frac{\theta}{l} a^2 \eta_{\mu\nu} \left[ \frac{\partial^2 Y}{a^2} + X' + \frac{8\theta}{l} X \right]
$$

(58)

$$
0 = \frac{3\theta}{l} \partial_\mu X
$$

(59)

$$
0 = \frac{\partial^2 Y'}{a^2} - \frac{\partial^2 X}{a^2} + \frac{4\theta}{l} X' + \frac{8}{l^2} X
$$

(60)

The solution is

$$
X = 0, \quad Y = \frac{U(x)}{a^2}
$$

(61)

where $\partial^2 U = 0$. In principle, $U$ can differ on either side of the brane.

Recall that we require $X, Y \to 0$ as $|z| \to \infty$. For the RS case, this requires that $U_L(x) = U_R(x) = 0$. For the inverse RS case, $U(x)$ can be non-zero on both sides of the brane. Finally, for the mixed case, we must have $U_L(x) = 0$, although $U_R(x)$ can be non-zero.

In order to apply the boundary conditions near the brane we have to choose a gauge. We can choose GN gauge whilst keeping the brane position fixed at $z = 0$. We now have $B = \phi = 0$, and

$$
E = \frac{1}{4} \theta l U(x)(a^{-4} - 1) + \frac{1}{2} l^2 V(x)(a^{-2} - 1) + W(x), \quad A = V(x)
$$

(62)

We can evaluate our solution at $z = 0$ to derive the brane metric

$$
\delta \gamma_{\mu\nu} = 2\partial_\mu \partial_\nu W + 2V \eta_{\mu\nu}
$$

(63)

The pure gauge part, $2\partial_\mu \partial_\nu W$, and the remainder, $2V \eta_{\mu\nu}$, should both be well defined. This means that $\Delta V = \Delta W = 0$. In particular, since $\Delta W = 0$, it is easy to see that
$W$ can be continuously gauged away on both sides of the brane. We therefore set $W = 0$.

The Israel equations (9) are now given by

$$2 \partial_\mu \partial_\nu \Delta (M^3 E') = 0$$

where we have made use of equation (15). This implies that

$$\Delta (M^3 U) = - \Delta (M^3 \theta l) V = - 2 \alpha V$$  \hspace{1cm} (65)

For the RS case ($\theta_L = 1 = - \theta_R$), we have deduced that $U_L = U_R = 0$. Since $\alpha = \langle M^3 l \rangle \neq 0$, equation (65) clearly implies that $V = 0$. This means that there is no scalar perturbation in the RS case.

In contrast, for the mixed case and the inverse RS case, we conclude that there is at least one remaining degree of freedom, $V$, say. This means that there is at least one scalar perturbation, which we will refer to as the radion.

We need to check whether or not the radion is a ghost. This involves calculating the radion effective action. Again, we need to choose a gauge. Our gauge choice must correspond to the brane being fixed at $z = 0$, and the perturbation vanishing at infinity.

Let us begin by choosing the gauge $E = A = 0$, for which

$$B = \frac{U}{a^2}, \quad \phi = 0$$

This vanishes at infinity. Near the brane, we begin in GN gauge, but transform to $E = A = 0$ by choosing

$$\xi = \frac{1}{4} \theta l U(x)(a^{-4} - 1) + \frac{1}{2} l^2 V(x)(a^{-2} - 1), \quad \xi_z = l \theta V$$

The brane is now positioned at $z = l \theta V$. We need to move it back to $z = 0$ without introducing a perturbation at infinity. This can be done by choosing $\xi_z = P(z)$, where $P(0) = - l \theta V$ and $P \to 0$ as $|z| \to \infty$. The bulk perturbation is now given by the following

$$E = 0, \quad A = - \frac{\theta}{l} P(z), \quad B = \frac{U}{a^2} - P(z), \quad \phi = - P'(z)$$  \hspace{1cm} (68)

To quadratic order, the effective action is

$$S_{\text{eff}} = - \frac{1}{2} \int_{\text{bulk}} \sqrt{-g} M^3 h^{ab} \delta E_{ab} - \frac{1}{2} \int_{\text{brane}} \sqrt{-\gamma} h^{ab} \delta \Theta_{ab}$$

where $\delta E_{ab}$ and $\delta \Theta_{ab}$ are the expansions, to linear order, of the bulk and boundary equations of motion respectively

$$E_{ab} = R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab}$$

$$\Theta_{ab} = 2 \langle M^3 (K_{ab} - K \gamma_{ab}) \rangle + \frac{\sigma}{2} \gamma_{ab}$$  \hspace{1cm} (71)
We find that

$$S_{\text{eff}} = -\frac{3}{2} \int d^4x \left[ \frac{M^3_{L} \theta_{L} \partial^2 U_{L}}{l_{L}} \int_{-\infty}^{0} dz P'(z) + \frac{M^3_{R} \theta_{R} \partial^2 U_{R}}{l_{R}} \int_{0^+}^{\infty} dz P'(z) \right]$$

$$= \frac{3}{2} \int d^4x V \partial^2 \Delta(M^3U) = -3\alpha \int d^4x V \partial^2 V$$

(72)

where we have made use of equation (65). We immediately see that the radion is a ghost whenever \( \alpha > 0 \). This is not a problem for the inverse RS case, as \( \alpha = -\langle M^3l \rangle < 0 \). However, as we saw in the last section, this can be a problem for the mixed case.

5.2 Vector perturbations

Now consider vector perturbations in the bulk. Since brane bending represents a scalar fluctuation we can set the brane to be at \( z = 0 \). In general, the vector perturbation is given by

$$\delta g_{\mu\nu} = 2a^2 F_{(\mu,\nu)}, \quad \delta g_{\mu z} = B_{\mu}, \quad \delta g_{zz} = 0$$

(73)

where \( F_{\mu} \) and \( B_{\mu} \) are divergence-free.

Under the vector gauge transformation,

$$x^\mu \rightarrow x^\mu + \xi^\mu, \quad z \rightarrow z$$

(74)

the vector fields transform as follows

$$F_{\mu} \rightarrow F_{\mu} - \xi_{\mu}, \quad B_{\mu} \rightarrow B_{\mu} - a^2 \xi'_{\mu}$$

(75)

where \( \xi_{\mu} = \eta_{\mu\nu} \xi^\nu \). Note that \( \xi^\mu \) is divergence-free. It is clear that we have the following gauge invariant

$$C_{\mu} = B_{\mu} - a^2 F'_{\mu}$$

(76)

The bulk equations of motion (8) are given by

$$0 = \left( \partial_z + \frac{2\theta}{l} \right) C_{(\mu,\nu)}$$

(77)

$$0 = -\frac{1}{2a^2} \partial^2 C_{\mu}$$

(78)

These have the following solution

$$C_{\mu} = \frac{\lambda_{\mu}(x)}{a^2}$$

(79)

where \( \partial^2 \lambda_{\mu} = 0 \). Of course \( \lambda_{\mu} \) can differ on either side of the brane. In analogy with the scalar gauge invariant, we require that \( C_{\mu} \rightarrow 0 \) as \( |z| \rightarrow \infty \). For the RS case, this requires that \( \lambda^L_{\mu} = \lambda^R_{\mu} = 0 \). For the inverse RS case, \( \lambda_{\mu} \) can be non-zero on both sides.
of the brane. Finally, for the mixed case, we must have $\lambda^L_\mu = 0$, although $\lambda^R_\mu$ can, in principle, be non-zero.

To apply the boundary conditions we go to GN gauge ($B_\mu = 0$), so that

$$F_\mu = \frac{1}{4} \theta l \lambda_\mu (a^{-4} - 1) + G_\mu(x)$$

Continuity at the brane implies that $\Delta G_\mu = 0$, so we can continuously gauge away $G_\mu$. The Israel equations (9) give

$$2\Delta [M^3 F'_{(\mu,\nu)}] = 0$$

This implies that

$$\Delta [M^3 \lambda_\mu] = 0$$

It is easy to see that if $\lambda_\mu$ is zero anywhere, it is zero everywhere. We conclude that vector perturbations can only exist for the inverse RS case.

### 6 Beyond the linearised analysis

At the end of section 4, we entertained the possibility of the linearised analysis breaking down at an unexpectedly low scale. In this section, we will show that this is indeed the case. We will go to quadratic order to gain insight into the full non-linear theory.

Let us begin with the transverse-tracefree + GN gauge in the bulk

$$\delta g_{\mu \nu} = \chi_{\mu \nu}, \quad \delta g_{\mu z} = \delta g_{zz} = 0$$

The brane is now bent. We need to fix the position of the brane to be at $z = 0$, whilst remaining in GN gauge to quadratic order. To this end, we make the following coordinate transformation

$$z \rightarrow z - f(x) - \frac{\theta l}{4} (1 - a^{-2}) \partial_\mu f \partial^\mu f$$

$$x^\mu \rightarrow x^\mu - Q^\mu(x) + \frac{\theta l}{2} (1 - a^{-2}) \left[ \partial^\mu f + \partial_\nu f \partial^\mu Q^\nu + \frac{2 \theta}{l} f \partial^\mu f \right] + \int_0^z \chi^{\mu \nu} \partial_\nu f$$

Note that this agrees with the linearised transformations (19) to first order. The bulk metric is now given by

$$\delta g_{\mu \nu} = h_{\mu \nu} = \chi_{\mu \nu}(x,z) + 2a^2 Q_{(\mu,\nu)} + l^2 (1 - a^2) \partial_\mu \partial_\nu F + 2F \bar{g}_{\mu \nu} - l^2 \partial_\mu F \partial_\nu F + \delta h_{\mu \nu}$$

where $F = \theta f/l$. The terms in $\delta h_{\mu \nu}$ are always much smaller than the other terms in $h_{\mu \nu}$, so we will neglect them. For the linearised analysis to be valid, we require that the linear terms in (86) are much larger than the remaining quadratic term, $l^2 \partial_\mu F \partial_\nu F$. 

15
Suppose $p \gg 1/l_R$, $M_L \gg M_R$ and $M_L^3 l_L \ll M_R^3 l_R$. For a source of mass $m < M_L^3 l_L r$, we have shown that the linearised analysis certainly breaks down when $p \sim p_{\text{cut-off}}$, where $p_{\text{cut-off}}$ is given by equation (53). We will now show that the quadratic term, $l^2 \partial_{\mu} F \partial_{\nu} F$, actually becomes important much sooner.

From section 4, we know that the linear terms go like $m p^3 / R(p)$. Schematically, the quadratic term becomes important when $p^2 l^2 F^2 \sim m p^3 / R(p)$. To identify when this happens we need to know the size of $F$. From equation (35), we see that

$$|F_L| = |F_R| \sim \frac{|T|}{|T|} \sim \frac{m p}{M_L^3 l_R} \sim \frac{m p}{M_L^3 l_R}$$

It turns out that the quadratic term becomes important when $p \sim p^* \ll p_{\text{cut-off}}$. For the remainder of this section we will restrict our attention to sources with the following mass

$$M_L^3 l_L r < m < M_L^3 l_L$$

When $r$ is large, this covers a huge range of masses. By taking our mass to lie in this range we ensure that $1/r < p^* < 1/l_L$. Specifically,

$$p^* = \left( \frac{M_L^3 l_L}{m r^2} \right)^{\frac{1}{3}}$$

We conclude that the linearised analysis only makes sense for $p \ll p^* \ll p_{\text{cut-off}}$. For $p^* \ll p \ll p_{\text{cut-off}}$, we cannot ignore the quadratic term, $l^2 \partial_{\mu} F \partial_{\nu} F$. Let us proceed with this in mind.

It is convenient to decompose the remaining quadratic as follows

$$-l^2 \partial_{\mu} F \partial_{\nu} F = F_{\mu \nu} + 2 F_{(\mu, \nu)} + 2 \partial_{\mu} \partial_{\nu} E + 2 A \eta_{\mu \nu}$$

where $F_{\mu \nu}$ is transverse-tracefree and $F_{\mu}^\nu$ is divergence free. It is easy to show that

$$\partial^2 A = \frac{l^2}{6} \left[ \partial^\mu \partial^\nu - \eta_{\mu \nu} \partial^2 \right] \partial_{\mu} F \partial_{\nu} F$$

Since $F_{\mu \nu}$ is transverse-tracefree, let us absorb it into a redefinition of $\chi_{\mu \nu}$. Furthermore, let

$$Q_{\mu} \to Q_{\mu} - F_{\mu} - \partial_{\mu} E$$

so that the bulk metric is given by

$$h_{\mu \nu} = \chi_{\mu \nu}(x, z) + 2 a^2 Q_{(\mu, \nu)} + l^2 \left( 1 - a^2 \right) \partial_{\mu} \partial_{\nu} F + 2 F g_{\mu \nu}$$

$$\qquad \quad + 2 \left( 1 - a^2 \right) \left[ F_{(\mu, \nu)} + \partial_{\mu} \partial_{\nu} E \right] + 2 A \eta_{\mu \nu}$$

We can evaluate this at $z = 0$ to derive the brane metric,

$$\delta \gamma_{\mu \nu} = h_{\mu \nu}^{\text{br}} = \chi_{\mu \nu}(x, 0) + 2 Q_{(\mu, \nu)} + 2 (F + A) \eta_{\mu \nu}$$

The pure gauge part, $2 Q_{(\mu, \nu)}$, and the remainder must be well defined. Making use of the fact that $\chi_{\mu}^\nu = 0$, we find

$$\Delta \chi_{\mu \nu}(x, 0) = 0, \quad \Delta Q_{(\mu, \nu)} = 0, \quad \Delta (F + A) = 0$$
It turns out that the Israel equations take very nearly the same form as in the linearised theory, the only subtly being that $\Delta F \neq 0$. Once again we have equation (25) but with

$$\Sigma_{\mu\nu}(x) = T_{\mu\nu} - \frac{1}{3} T \eta_{\mu\nu} + 2 \partial_\mu \partial_\nu \Delta(M^3 \theta l F)$$  \hspace{1cm} (96)$$

Note that we have used the fact that $F(\mu,\nu) + \partial_\mu \partial_\nu E + A \eta_{\mu\nu} \sim p^2 l^2 F^2 \ll p^2 l^2 F \sim l^2 \partial_\mu \partial_\nu F$  \hspace{1cm} (97)$$

Given that $\chi_{\mu\nu}$ is transverse-tracefree, we have that

$$2 \partial^2 \Delta(M^3 \theta l F) = \frac{T}{3}$$  \hspace{1cm} (98)$$

In momentum space, the metric on the brane is given by

$$\tilde{h}^{\text{br}}_{\mu\nu} = \tilde{\chi}^{\text{br}}_{\mu\nu} + 2 i p_\mu \tilde{Q}_\nu + 2 (\tilde{F} + \tilde{A}) \eta_{\mu\nu}.$$  \hspace{1cm} (99)$$

As in section 4, we see that

$$\chi^{\text{br}}_{\mu\nu} = \frac{\tilde{\Sigma}_{\mu\nu}(p)}{R(p)}$$  \hspace{1cm} (100)$$

We choose $\tilde{Q}_\mu(p)$ to cancel off the term proportional to $p_\mu p_\nu \Delta(M^3 \theta l F)$ in $\tilde{\Sigma}_{\mu\nu}(p)$. This leaves

$$\tilde{h}^{\text{br}}_{\mu\nu} = \frac{1}{R(p)} \left[ T_{\mu\nu} - \frac{1}{3} T \eta_{\mu\nu} \right] + 2 (\tilde{F} + \tilde{A}) \eta_{\mu\nu}$$  \hspace{1cm} (101)$$

For the RS case and the mixed case, recall that

$$R(p) \sim \begin{cases} M^3_R p & \text{for } 1/l_R \ll p \ll 1/r \\ \frac{1}{2} M^3_L l_R p^2 & \text{for } 1/r \ll p \ll p_{uv} \end{cases}$$  \hspace{1cm} (102)$$

provided $M_L \gg M_R$ and $M^3_L l_L \ll M^3_R l_R$. Here

$$p_{uv} = \min\{1/l_L, p_{\text{cut-off}}\} = \min\{1/l_L, M^3_L l_L/m\}.$$  \hspace{1cm} (103)$$

Our goal is to reproduce Einstein gravity at certain scales. $R(p)$ already has the correct momentum dependance when $1/r \ll p \ll p_{uv}$. To ensure that our solution also has the right tensor structure (see equation (43)), we require that

$$\tilde{F} + \tilde{A} = - \frac{T}{6 M^3_L l_L p^2}$$  \hspace{1cm} (104)$$

We will now see that this is indeed the case when $p_* \ll p \ll p_{uv}$.

For $p \ll p_*$, we know that the linearised analysis can be trusted. However, for $p_* \ll p \ll p_{uv}$, we have

$$F_L \gg A_L, \quad F_R \ll A_R$$  \hspace{1cm} (105)$$

Making use of boundary condition (95), we deduce that

$$F_L \approx F_L + A_L = F_R + A_R \approx A_R$$  \hspace{1cm} (106)$$
Now consider the trace equation (98). In principle, we have two possibilities: (i) $M_3^3 l_L |F_L| \gg M_3^3 l_R |F_R|$ or (ii) $M_3^3 l_L |F_L| \ll M_3^3 l_R |F_R|$. For $p_* \ll p \ll p_{uv}$, it turns out that case (i) is the only one to give a self-consistent solution. The trace equation (95) now implies that

$$
\bar{F}_L \approx -\frac{T}{6M_3^3 l_L p^2}
$$

so that equation (104) holds.

To briefly summarise, we have shown that the RS case and the mixed case mimic four-dimensional General Relativity when $p_* \ll p \ll p_{uv}$. This happens because the linearised results break down, and we have to include quadratic contributions. At lower energies, the linearised results hold, and we deviate from four dimensional GR. At first, the tensor structure of our propagator changes. As we go even deeper into the infra-red, our momentum dependence also changes.

It is important to ask whether or not the radion analysis of section 5 is significantly affected by non-linear effects. The answer is clearly “no”. This is because the radion field is massless $p^2 = 0$, so the linearised analysis can be trusted. Recall that there is no radion in the RS case, and the radion is not a ghost for the mixed case when $M_3^3 l_L < M_3^3 l_R$. We conclude that the radion is not a problem here.

7 A word on cosmology

Infra-red modifications of gravity generically have far reaching implications for cosmology, particularly at late times. In this section, we will briefly examine the cosmological solutions that arise in our models. We will omit most of the details, as they can be found in [21]. We should note that in [21], $\theta_R$ is defined with the opposite sign to here.

Let us consider a spatially flat Friedmann-Robertson-Walker brane sandwiched in between two copies of anti-de Sitter space. The dynamics of the brane are governed by the following equation [21]

$$
\Delta \left[ M^3 \theta \sqrt{\frac{1}{l^2} + H^2} \right] = \frac{\rho}{6}
$$

(108)

where $H$ is the Hubble parameter on the brane, and $\rho$ is the energy density. Here $\rho$ is made up of brane tension and additional matter

$$
\rho = \sigma + \rho_m
$$

(109)

Note that $\sigma$ is given by equation (15). For $1/l_R \ll H \ll 1/l_L$, equation (108) approximates to

$$
\theta_L H^2 - 2\theta_R \frac{|H|}{r} \approx \frac{\rho_m}{3M_3^3 l_L}
$$

(110)

At early times, $H \gg 1/r$, so that

$$
H^2 \approx \frac{\theta_L \rho_m}{3M_3^3 l_L}
$$

(111)
For the RS and the mixed case, $\theta = 1$, so this simply gives the standard 4d cosmology with $m_{pl}^2 = \frac{1}{2} M_L^3 l_L$.

We might expect to see a de Sitter phase of expansion at later times, as $\rho_m \to 0$. For the RS case ($\theta_L = -\theta_R = 1$), it is clear from equation (108) that this will not happen: $H^2 = 0$ is the only solution for $\rho_m = 0$. In contrast, for the mixed case ($\theta_L = \theta_R = 1$), the solution $H \approx \frac{2}{r}$ is permitted.

We conclude that late time de Sitter expansion occurs for the mixed case, but not the RS case. The RS result is a surprise as we have modified gravity in the range $1/l_R \ll p \ll 1/r$.

8 Discussion

We have considered in detail perturbations about a flat brane embedded in between two copies of anti-de Sitter space. We have abandoned $Z_2$ symmetry across the brane, giving rise to three distinct cases: (i) the RS case where the warp factor decays away from the brane on both sides, (ii) the inverse RS case where the warp factor grows on both sides and (iii) the mixed case where the warp factor decays to the left of the brane, and grows to the right.

Let us now summarize our main results. The inverse RS case is a dead end: four dimensional Einstein gravity can never be achieved. In contrast, take the RS case and the mixed case when $M_L \gg M_R$ and $M_L^2 l_L \ll M_R^2 l_R$. Consider a source of mass, $m \in (M_L^3 l_L/\rho^2, M_L^3 l_L)$, where $r = M_L^3 l_L/M_R^3$. The metric on the brane is given by

$$\tilde{h}^{br}_{\mu\nu} = \begin{cases} 
\frac{1}{m_{pl}^2 p^2} \left[ \tilde{T}_{\mu\nu} - \frac{1}{2} \tilde{T} \eta_{\mu\nu} \right] & \text{when } p_s \ll p \ll p_{uv} \\
\frac{1}{m_{pl}^2 p} \left[ \tilde{T}_{\mu\nu} - \frac{1}{3} \tilde{T} \eta_{\mu\nu} \right] & \text{when } 1/r \ll p \ll p_s \\
\frac{1}{M_R^2 p} \left[ \tilde{T}_{\mu\nu} - \frac{1}{3} \tilde{T} \eta_{\mu\nu} \right] & \text{when } 1/l_R \ll p \ll 1/r 
\end{cases} \quad (112)$$

where $m_{pl}^2 = \frac{1}{2} M_L^3 l_L$, $p_s = (M_L^2 l_L/m r^2)^{1/3}$, and $p_{uv} = \min\{1/l_L, M_L^2 l_L/m\}$. Gravity on the brane mimics four-dimensional GR at intermediate scales, but is modified in the infra-red. This modification begins with a change in the tensor structure of the propagator. At even lower energies the momentum dependence also changes. Crucially, these models are free from ghosts.

We should also note that a cosmological brane in the mixed case will approach a de Sitter phase at late times, even when there is no additional matter. This does not happen for the RS case.

To illustrate our results more clearly, let us put some numbers in. Consider the mixed case with $M_L \sim 1/l_L \sim m_{pl} \sim 10^{19}$ GeV, $M_R \sim 10 - 100$ MeV, and $1/l_R \ll 10^{-34}$ eV. Firstly, note that the current Hubble rate, $H_0 \approx 2/r \sim 10^{-34}$ eV, as desired. Now consider our source to be the Sun, with mass $m \sim 10^{66}$ eV. Four-dimensional GR is reproduced at distances between $r_{\text{Schw}}$ and $10^{21}$ cm. Here, the Schwarzschild radius of the Sun is given by $r_{\text{Schw}} \sim 1$ km. The large distance limit, $10^{21}$ cm, extends well beyond the size of the solar system, so we have no conflict with experiment.
Of course, the results \(^{112}\) were derived by going to quadratic order in perturbation theory. When \(p_\ast \ll p \ll p_{uv}\), we found that the quadratic brane bending term dominated over the linear terms. We might be worried that this means we have lost control of perturbation theory. However, we will now argue that we do have control of perturbation theory, because terms beyond quadratic order are always small, right up to \(p_{uv}\).

Suppose, we have two GN coordinate systems, \(x^a\) and \(\hat{x}^a\), where
\[
\hat{x}^\mu = x^\mu + \xi^\mu(x, z), \quad \hat{z} = z + \xi^z(x, z)
\]  

(113)

In the “unhatted” coordinates, the brane is positioned at \(z = 0\). For simplicity we will demand that \(\xi^\mu(x, 0) = 0\). This just corresponds to a choice of gauge on the brane. In the “hatted” coordinates the brane is positioned at \(\hat{z} = \xi^z(x, 0) = f(x)\).

Non-perturbatively, the metrics in each coordinate system are related as follows
\[
g_{ab}(x, z) = \hat{g}_{cd}(\hat{x}, \hat{z}) \frac{\partial \hat{x}^c}{\partial x^a} \frac{\partial \hat{x}^d}{\partial x^b}
\]  

(114)

We shall now evaluate the \((\mu\nu)\) component at \(z = 0\) to derive the metric on the brane
\[
g_{\mu\nu}(x, 0) = \hat{g}_{\mu\nu}(x, f) + \partial_\mu f \partial_\nu f
\]  

(115)

Suppose we write \(\hat{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \chi_{\mu\nu}\), and Taylor expand the right-hand side of (115). We get \(g_{\mu\nu}(x, 0) = g^{\text{quad}}_{\mu\nu}(x) + \delta g_{\mu\nu}(x)\), where
\[
g^{\text{quad}}_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x, 0) + f \bar{g}'_{\mu\nu}(x, 0) + \frac{1}{2} f^2 \bar{g}''_{\mu\nu}(x, 0) + \chi_{\mu\nu}(x, 0) + f \chi'_{\mu\nu}(x, 0) + \partial_\mu f \partial_\nu f
\]  

(116)

and
\[
\delta g_{\mu\nu}(x) = \sum_{n=3}^{\infty} \frac{f^n}{n!} \bar{g}^{(n)}_{\mu\nu}(x, 0) + \sum_{n=2}^{\infty} \frac{f^n}{n!} \chi^{(n)}_{\mu\nu}(x, 0)
\]  

(117)

Here prime denotes partial differentiation with respect to \(z\). \(g^{\text{quad}}_{\mu\nu}(x)\) includes all those terms that were considered in the analysis to quadratic order. \(\delta g_{\mu\nu}(x)\) contains all those terms that were assumed to be of higher order, and were consequently ignored. We shall now ask whether or not it was safe to ignore them.

Roughly speaking, we can associate each \(z\) derivative with either a factor of \(p\) or a factor of \(1/l\). It is obvious that as long as \(p f \ll 1\) and \(f/l \ll 1\), then \(\delta g_{\mu\nu} \ll g^{\text{quad}}_{\mu\nu}\). In the quadratic analysis it is clear that \(p f\) and \(f/l\) are indeed small whenever \(p \ll p_{uv}\). We conclude that it was entirely consistent to ignore all terms beyond quadratic order.

In analogy with the DGP model, we shall assume that a break down in perturbation theory gives rise to strong coupling. To derive the strong coupling scale we consider a source of mass \(m \sim m_{pl}\), and ask at what scale do we lose perturbative control \(^{28}\). If this scale were \(p_\ast\) we would be in trouble. We would expect strong coupling below distances of around 1000 km. However, we have just shown that we don’t expect to lose control until at least \(p_{uv}\), which typically corresponds to the Planck scale! These results suggest that there is no strong coupling problem here, although a more thorough analysis is clearly required.
Finally, note that the mixed case in particular appears to have much in common with the DGP model \[1\]. The expressions for \( r \) and \( p_* \) seem to have obvious DGP analogues. The cosmological behaviour is also very similar (see equation \[110\]). One can’t help thinking that this is more than mere coincidence. With this in mind, consider the brane to be the common boundary to the decaying bulk and to the growing bulk. If we were to calculate the boundary stress-energy tensor on the decaying side, we would expect there to be divergences. These must be cancelled by boundary counter terms that will probably take the form of localised curvature on the brane. If this is indeed the case, it would represent a new mechanism for obtaining induced curvature on a DGP brane.

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**References**


