Dark Matter Prediction from Canonical Quantum Gravity with Frame Fixing

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We show how, in canonical quantum cosmology, the frame fixing induces a new energy density contribution having features compatible with the (actual) cold dark matter component of the Universe. First we quantize the closed Friedmann-Robertson-Walker (FRW) model in a synchronous reference and determine the spectrum of the super-Hamiltonian in the presence of ultra-relativistic matter and a perfect gas contribution. Then we include in this model small inhomogeneous (spherical) perturbations in the spirit of the Lemaitre-Tolman cosmology. The main issue of our analysis consists in outlining that, in the classical limit, the non-zero eigenvalue of the super-Hamiltonian operator is calculated via a perturbation theory of the wave functional. In particular in [3, 4] it was outlined the non-relativistic Planck gas (i.e. a perfect gas of particles with the Planck mass) is reliably postulated. In section 3 we investigate the behavior of small (spherical) perturbations, by the framework of a Lemaitre-Tolman cosmology. The inhomogeneous corrections to the super-Hamiltonian operator, can be restated, in the classical limit, in terms of Einstein equation with a perfect dust fluid source. More precisely it outcomes a dust fluid whose energy density reads as

\[ \varepsilon \equiv -\frac{\Omega(x^i)\varepsilon}{2K\sqrt{\hbar}} \]

where \( K \) denotes the Einstein constant. The non-relativistic nature of such energy density and its quantum origin lead (see [2]) to candidate it as a possible explanation for the Universe dark matter component.

In this work we show how it is possible to have a closed FRW model for which the classical limit of the super-Hamiltonian eigenvalue has the appropriate features to account of the cold dark matter (actual) energy density. In section 2 we quantize the closed FRW model in agreement to the equations [3] and including in the quantum dynamics two phenomenological terms, one accounting for the ultra-relativistic matter and another one corresponding to a perfect gas component. The former term describes the thermal bath of fundamental particles, while the latter is included because the quantum behavior of the Universe is expected in the Planck era and then a non-relativistic Planck gas (i.e. a perfect gas of particles with the Planck mass) is reliably postulated. In section 3 we investigate the behavior of small (spherical) perturbations, by the framework of a Lemaitre-Tolman cosmology. The inhomogeneous corrections to the super-Hamiltonian operator is calculated via a perturbation theory of the eigenvalues problem. In section 4 we show how the actual value of the Universe dark matter energy density [3] can be fitted by our analysis.

A relevant issue of the perturbation theory is shown to be the direct correlation between the inhomogeneities of

\[ \hat{H}_i\psi = 0. \]
the ultra-relativistic matter and those ones emerging in the dust fluid energy density, i.e. our candidate for dark matter.

**CANONICAL QUANTIZATION OF THE CLOSED FRW MODEL**

In this section we will quantize the closed FRW model into a synchronous reference and in the presence of ultra-relativistic matter and a perfect gas contributions. The line element of such a model takes the form:

\[ ds^2 = -N^2dt^2 + R_c^2(t)[d\xi^2 + \sin^2\xi(d\eta^2 + \sin^2\eta d\phi^2)] , \]  

where \( N \) the lapse function, \( R_c \) the Universe radius of curvature and with \( 0 < \xi < \pi, 0 < \eta < \pi, 0 < \phi < 2\pi \). The dynamics associated to this line element is summarized by the action:

\[ S = \int_{\Sigma_t^0 \times R} dt d^3x \left\{ p_{R_c} \frac{\partial R_c}{\partial t} - NH \right\} , \]

where the super-hamiltonian reads

\[ H = - \frac{l_{pl}^2}{3\pi\hbar} \frac{\mu^2}{R_c} + \frac{\mu^2}{R_c} - \frac{3\pi\hbar}{4l_{pl}^2} R_c + \frac{\sigma^2}{R_c} , \]  

where \( \mu^2, \sigma^2 = const \). Here we introduced the term \( \mu^2/\sigma^2 = const \). This allows that the spectrum of the total energy of the Universe is unbounded. However such a feature is removed as soon as we include in our dynamical scheme a cut-off on admissible physical lengths; indeed there are clear indications (coming either from the Loop Quantum Gravity theory \([11]\), either from the String approaches \([12]\)) that the space-time must have a “lattice” structure on the Planckian scale; in this respect we will require that every minimal length has to be equal to the Planck

The solution of the above equation is the following Fuchs function

\[ \omega = \sum_{n=0}^{\infty} c_n(\varepsilon) R_c^{n+2} , \]  

whose coefficients obey to the following relations

\[ c_n = - \frac{1}{n(n+2)} \left[ (n+1) \frac{6\pi l_{pl}^2}{\sigma^2} c_{n-1} + \left( \frac{n}{l_{pl}^2} + \frac{6\pi}{l_{pl}^2} \sigma^2 \right) c_{n-2} \right] . \]  

Since \( R_c \in [0, \infty) \), the series radius of convergence has to diverge and therefore we have

\[ \lim_{n \to \infty} \frac{c_{n+2}}{c_n} = 0 ; \]

hence we see in such a limit the relation \([12]\), in the leading order, rewrites

\[ \frac{c_{n+2}}{c_n} \sim \frac{3\pi}{l_{pl}^2 n} \]  

Comparing this behaviour to the corresponding one for the exponential term in \([4]\), we conclude that the Fuchs series has to be truncated in correspondence to a given integer \( m \) and the following relation holds

\[ m + 2 = \frac{l_{pl}^2 \varepsilon^2}{3\pi\hbar^2} + \frac{\mu^2}{\hbar} . \]  

Equation \([14]\) comes out taking into account that the wave function \([9]\) is solution of equation \([10]\) if the key relation takes place

\[ \varepsilon = - \frac{3\pi c}{l_{pl}^2} \sigma^2 . \]  

Finally we stress how a notion of probability distribution can be associated to this solution, in agreement to the discussion presented in \([1]\).
one \( l_{PL} = \sqrt{\frac{Gh}{c^3}} \). Now we observe that the ultra
relativistic energy density can be written as follows:

\[
\rho_{ur} = \frac{\mu^2 G}{R_c^4} = \frac{3}{2} \frac{\hbar c}{d^2 l_{PL}^2},
\]

where \( d \) denotes the cosmological horizon. On the other hand the
length per particle of a perfect gas reads as

\[
\ell^3 = \frac{V}{N} = \frac{3}{2} m_{Pl} \frac{k}{\rho_{Pl} l_{Pl}^2},
\]

being \( m_{Pl} \) the Planck mass, \( \lambda \) the thermal length of
the particles and \( \rho_{Pl} \) the perfect gas energy density, i.e.
\( \rho_{Pl} = \sigma^2 c/RL^6 \). Thus if we require \( d > l_{Pl} \) and \( l > l_{Pl} \) we
get the following inequalities for the quantum numbers \( \sigma^2 \) and \( \mu^2 \) respectively

\[
\sigma^2 \leq \frac{3h^2}{2m_{Pl} c} \left( \frac{R_c}{l_{Pl}} \right)^5
\]

(19)

\[
\mu^2 \leq \frac{3h}{2} \left( \frac{R_c}{l_{Pl}} \right)^4.
\]

(20)

Taking the minimal radius of curvature as that one at the Planckian time, i.e.

\[
R_{cPL} = \frac{R_{today}}{1 + z_{PL}}
\]

(21)

(being \( z_{Pl} \) the Planckian redshift), then by [13, 14, 15] we
arrive to the constraint

\[
\varepsilon \geq \frac{9\pi}{2} \frac{h^2}{m_{Pl} l_{Pl}} \left( \frac{R_{c today}}{1 + z_{Pl}} \right)^5.
\]

(22)

Since \( z_{Pl} \) must have a finite value we see how requiring
a cut-off on the admissible lengths, yields a lower
bounded spectrum.

**INHOMOGENEOUS SPHERICAL PERTURBATIONS**

In order to make account for the quantum dinamics of
inhomogeneous perturbations we consider the spherically
symmetric Lemaître-Tolman [13, 14, 15] model, whose
spatial line element reads

\[
dl^2 = e^{2\alpha} \sigma^2 dr^2 + e^{2\beta} (d\theta^2 + \sin^2 \theta d\phi^2),
\]

(23)

where the functions \( \alpha(t, r), \beta(t, r) \) take the explicit form

\[
e^{\alpha} = \frac{(a(r, t)r')'}{\sqrt{1 - r^2/R^2}}, \quad e^{\beta} = ra(r, t)
\]

(24)

being \( R \) a given constant and with \( (\cdot)' \equiv d(\cdot)/dr \). In order to study inhomogeneous perturbations [16] we write the function \( a(t, r) \) as follows

\[
a(t, r) = R_c(t) + \xi(t, r)
\]

(25)

with \( |\xi| \ll |R_c| \). Expanding the Einstein-Hilbert-
matter action up to first order in \( \xi \) we get in a synchronuous
reference an action of the form (the spherical simmetry
prevents to have a non zero shift vector)

\[
S = \int dt \left\{ p_{Rc} \frac{\partial R_c}{\partial t} + \int_0^R dr p_{\xi} \frac{\partial \xi}{\partial t} - (H_0 + H_1) \right\},
\]

(26)

where \( p_{Rc} \) and \( p_{\xi} \) denote the conjugate momenta respectively to \( R_c \) and \( \xi \), \( H_0 \) is the FRW Hamiltonian and \( H_1 \) reads

\[
H_1 = H_1^g + H_1^M
\]

(27)

with

\[
H_1^g = \int_0^R dr A(R_c, r) \xi
\]

(28)

\[
H_1^M = \int_0^R dr \left[ B(R_c, r) \delta \mu^2(r) + C(R_c, r) \delta \sigma^2(r) \right];
\]

(29)

the above functions \( A, B, C \) correspond to the expressions

\[
A(R_c, r) = -\frac{2G P_{Rc}^2}{(3\pi)^2 Nc R_c} \left( F_3' - F_2 \right) - \frac{Nc^3}{2G} M_0 +
+ 4\pi^{3/2} N \frac{\sigma^2}{R_c^2} + 3 \sqrt{\pi} N \frac{\mu^2}{R_c^2} \Sigma,
\]

(30)

\[
B(R_c, r) = 3 \sqrt{\pi} N \frac{F_2}{R_c},
\]

(31)

\[
C(R_c, r) = 2 \pi^{3/2} N \frac{F_2}{R_c},
\]

(32)

where we used the definitions

\[
F_n(r) = \frac{r^n}{\sqrt{1 - r^2/R(r)^2}}, \quad \Sigma(r) = \frac{(F_2 r')'}{3} - F_2.
\]

(33)

\( H_1^g \) corresponds to the pure geometrical contribution
to the perturbed Hamiltonian, while \( H_1^M \) makes account
for the corresponding perturbation in the matter fields (both the ultra-relativistic matter and the perfect gas are taken, in the leading order, comoving to the expansion).

Now we observe that from the action \( \mathcal{S} \) it follows \( \delta \xi = 0 \), i.e. \( \xi = \xi(r) \); such a behaviour allows to identify \( \xi \) with the matter fields perturbations and, by comparing the same inverse powers of \( R_c \) appearing in \( H_t^2 \) and \( H_1^M \), we get the key relation

\[
\xi(r) = -\frac{2}{3} \frac{\delta \sigma^2(r)}{\delta \mu^2}.
\]

The eigenvalue problem \( \psi \) takes, for such perturbed FRW model, the smeared form

\[
\left( \hat{H}_0 + \hat{H}_1 \right) \psi(R_c, r) = \left[ \varepsilon + \int_0^r 4\pi r^2 \Delta \rho(r)dr \right] \psi(R_c, r) = 0,
\]

where

\[
\hat{H}_0 = \frac{1}{2} \left( \frac{h}{3\pi l_p} \right)^2 \left( \frac{\mu^2}{\sigma^2} \right) I_2 - \frac{2}{2l_p^2} \frac{h}{3\pi l_p} I_0 + \frac{h}{3\pi l_p} \frac{\sigma^2}{\epsilon} I_3 - \frac{2}{3} \frac{\mu^2}{\sigma^2} \sigma^2 I_3 F_0 + \frac{c}{12} \frac{1}{\sigma^2} I_0
\]

being

\[
D(r) = -\frac{\rho}{3c^2} (F_3 - F_2) \frac{F_0}{\mu^2} \frac{1}{\Sigma} \frac{1}{\sigma^2} I_2 - \frac{2}{2l_p^2} \frac{h}{3\pi l_p} I_0 + \frac{h}{3\pi l_p} \frac{\sigma^2}{\epsilon} I_3 - \frac{2}{3} \frac{\mu^2}{\sigma^2} \sigma^2 I_3 F_0 + \frac{c}{12} \frac{1}{\sigma^2} I_0
\]

\[
E(r) = \frac{3}{4} \frac{c I_1}{\sqrt{\pi}} F_0.
\]

Above by \( I_k, k = 1, 2, 3... \) we denoted the corresponding mean values

\[
I_k = \int_0^\infty dR_c \frac{\chi_0 \chi_0}{R_c^k}.
\]

we stress that that \( I_1, I_2, I_3 \) appearing in \( \mathcal{S} \) and \( \mathcal{S} \) correspond to finite values because the first contribution in \( \omega(R_c) \) is of order \( R_c^2 \).

PHENOMENOLOGICAL IMPLICATIONS

As shown in \( \mathcal{E} \) in the classical limit, as taken in the WKB approach, the critical parameter \( \Omega_{dm} \) for our dark matter candidate reads in term of \( \varepsilon \) (which behaves like a constant of the motion) in the form

\[
\Omega_{dm} = \frac{4l_p^2 c \varepsilon}{3\pi h H^2 (R_c^{today})^3}.
\]

The value of \( \varepsilon \) which ensures that \( \Omega_{dm} \) today of order unity is estimated to be \( \varepsilon \sim \mathcal{O}(10^{-8}) \text{GeV} \). Now if we take \( \varepsilon \) from the inequality \( \mathcal{Q} \), we arrive to the final inequality for \( \Omega_{dm} \)

\[
\Omega_{dm} \geq \frac{6c h}{H^2 m_p} \left( \frac{1}{l_p (1 + z^p)} \right)^5 (R_c^{today})^2.
\]

In agreement to the standard interpretation of the quantum mechanics we expect that the Universe settled down into the state of minimal energy and this is equivalent to take in equations \( \mathcal{Q} \) and \( \mathcal{R} \) the equality sign. Being \( R_c^{today} \) estimated of the order \( \mathcal{O}(10^{28}) \text{cm} \) \( \mathcal{R} \) we get \( \Omega_{dm} \sim \mathcal{O}(1) \) as soon as \( z^p \sim \mathcal{O}(10^{80}) \); such a value of the Planckian redshift is compatible with a smooth inflationary scenario whose \( \text{e-folding} \) can be estimated of the order \( \sim 50 \).

We remark that the case in which the Universe did not undergo an inflationary scenario, would correspond to \( z^p \sim \mathcal{O}(10^{30}) \); such a value of \( z^p \) fixes the lower boundary for \( \varepsilon \) in \( \mathcal{Q} \) like \( \mathcal{O}(10^{13}) \text{GeV} \).

In such a case the value of \( \varepsilon \) which provides today \( \Omega_{dm} \sim \mathcal{O}(1) \) is no longer the minimum of the spectrum but it lies within it. The case of the Standard Cosmological Model (SCM) \( (z^p \sim \mathcal{O}(10^{30})) \) of particular interest here because if we take the value of \( \sigma^2 \) corresponding to \( \varepsilon \sim \mathcal{O}(10^{-82}) \text{GeV} \) and evaluate \( \mu^2 \) via its maximum value (for such values in the spectrum relation \( \mathcal{Q} \) the \( \varepsilon \)-term dominates the \( \mu^2 \)-one), then the perturbation density \( \mathcal{E} \) rewrites, in the leading order, as

\[
\Delta \rho = \frac{3}{4} \frac{c I_1}{\sqrt{\pi}} F_0 \delta \mu^2.
\]

The above expression acquires particular interest because it outlines the expected direct correlation between the perturbations in our dark matter candidate and the ultra-relativistic matter. Summarizing we constructed an evolutitive quantum cosmology which leads to a non-zero super-Hamiltonian eigenvalue and outlined how it can be a satisfactory dark matter candidate. In particular the perturbation theory of the eigenvalue problem allowed us to fix the dark matter perturbation in terms of those ones in the ultra-relativistic matter and in the perfect gas, with the relevant issue \( \mathcal{E} \) for the SCM case. Our result calls attention for a deep investigation of the semi-classical limit which allows to precise the mechanism by which the quantum perturbation \( \mathcal{E} \) are frozen out and approach a classical limit.