QCD Sum Rules and Models for Generalized Parton Distributions

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Key words Quantum Chromodynamics, Generalized Parton Distributions, QCD Sum Rules
PACS 13.40.Gp, 13.60.Fz, 12.38.Lg

Dedicated to Klaus Goeke on occasion of his 60th birthday

I use QCD sum rule ideas to construct models for generalized parton distributions. To this end, the perturbative parts of QCD sum rules for the pion and nucleon electromagnetic form factors are interpreted in terms of GPDs and two models are discussed. One of them takes the double Borel transform at adjusted value of the Borel parameter as a model for nonforward parton densities, and another is based on the local duality relation. Possible ways of improving these Ansätze are briefly discussed.

1 Introduction

The concept of Generalized Parton Distributions \[1,2,3,4\] is a modern tool to provide a more detailed description of the hadron structure. The need for GPDs is dictated by the present-day situation in hadron physics, namely: \(i\) The fundamental particles from which the hadrons are built are known: quarks and gluons. \(ii\) Quark-gluon interactions are described by QCD whose Lagrangian is also known. \(iii\) The knowledge of these first principles is not sufficient at the moment, and we still need hints from experiment to understand how QCD works, and we need to translate information obtained on the hadron level into the language of quark and gluonic fields.

One can consider projections of combinations of quark and gluonic fields onto hadronic states \(|P\rangle\langle0|q_\alpha(z_1)q_\beta(z_2)\rangle P\rangle\), etc., and interpret them as hadronic wave functions. In principle, solving the bound-state equation \(H|P\rangle = E|P\rangle\) one should get complete information about the hadronic structure. In practice, the equation involving infinite number of Fock components has never been solved. Moreover, the wave functions are not directly accessible experimentally. The way out is to use phenomenological functions. Well known examples are form factors, usual parton densities, and distribution amplitudes. The new functions, Generalized Parton Distributions \[1,2,3,4\] (for recent reviews, see \[5,6\]), are hybrids of these “old” functions which, in their turn, are the limiting cases of the “new” ones.

2 Generalized parton distributions

Generalized parton distributions parametrize nonforward matrix elements of lightcone operators. For example, the twist-2 part of the vector operator built of quark fields \(O^\mu(z) = \bar{\psi}(-z/2)\gamma^\mu\psi(z/2)\) in the simplest case of a (pseudo)scalar hadron, e.g., pion can be parametrized in two ways. The first one is in...
terms of the off-forward parton distribution \[1\] 2 3

\[
\langle P - \frac{r}{2} | O^\mu(z) | P + \frac{r}{2} \rangle = 2 P^\mu \int_{-1}^{1} dx e^{-ix(Pz)} H(x, \xi, t) ,
\]

(1)

(where \( \xi = (rz)/2(Pz) \) is the skewness of the matrix element and \( t = r^2 \)) or in terms of two double distributions (DDs) \[1\] 3 7

\[
\langle P - \frac{r}{2} | O^\mu(z) | P + \frac{r}{2} \rangle = \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{-1-|\beta|} d\alpha e^{-i\beta(Pz)-i\alpha(rz)/2} \left\{ 2 P^\mu F(\beta, \alpha, t) + r^\mu G(\beta, \alpha, t) \right\} .
\]

(2)

The variables \( x, \xi \) of OFPDs (\( \beta, \alpha \) of DDs) can be interpreted as momentum fractions: initial and returning quarks carry the momenta \( (x + \xi)P^+ \) and \( (x - \xi)P^+ \), \((\beta P^+ + (1 + \alpha)r^+) / 2 \) and \( \beta P^+ - (1 - \alpha)r^+ / 2 \), respectively. The functions \( H(x, \xi, t), F(\beta, \alpha, t), G(\beta, \alpha, t) \) are related by

\[
H(x, \xi, t) = \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{-1-|\beta|} d\alpha \delta(x - \beta - \xi \alpha) \left\{ F(\beta, \alpha, t) + \xi G(\beta, \alpha, t) \right\} .
\]

(3)

The resolution of the apparent discrepancy of describing the same object in terms of one or two functions is based on the observation that the choice of two DDs \( F, G \) is not unambiguous \[7\] 8: one can perform transformations which do not change the combination \( \partial F/\partial \beta + \partial G/\partial \alpha \) \[8\]. In particular, there exists a DD representation in terms of a single function \[7\]

\[
\langle P - \frac{r}{2} | O^\mu(z) | P + \frac{r}{2} \rangle = \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{-1-|\beta|} d\alpha e^{-i\beta(Pz)-i\alpha(rz)/2} \left\{ 2 \beta P^\mu + \alpha r^\mu \right\} h(\beta, \alpha, t)
\]

(4)

\[
= 2i \frac{\partial}{\partial z^\mu} \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{-1-|\beta|} d\alpha e^{-i\beta(Pz)-i\alpha(rz)/2} h(\beta, \alpha, t).
\]

The generalized parton distribution functions provide a very detailed description of the hadronic structure. They include information contained in simpler functions, like usual parton densities \( f(x) \) and form factors \( F(t) \), reducing to them in particular limits. The forward limit gives \( H(x, \xi = 0, t = 0) = f(x) \), while the local one produces the reduction formula

\[
\int_{-1}^{1} H(x, \xi, t) \, dx = F(t).
\]

(5)

Equivalent relations between DDs, parton densities and form factors also can be written. Intermediate in complexity are nonforward parton densities \[10\] \( F(x, t) = H(x, \xi = 0, t) \), or GPDs at zero skewness. They reduce to parton densities for zero \( t \), and give form factors after integration over \( x \). The functions \( F(x, t) \) can be also obtained from the \( F \)-DDs by integration over \( \alpha \):

\[
F(x, t) = \int_{-1+|x|}^{-1-|x|} F(x, \alpha, t) \, d\alpha.
\]

(6)

Note, that the \( \alpha \)-integral of \( G \)-DDs is zero \[11\] 12 because they are odd functions of \( \alpha \). Interplay between \( x \) and \( t \) dependence of \( F(x, t) \) is an interesting and nontrivial problem. In particular, it is closely associated with the question \[13\] of interrelation between large-\( t \) behavior of hadron form factors and \( x \to 1 \) shape of parton densities.

GPDs accumulate information about long-distance interactions, hence, they are nonperturbative functions. Possible ways to get theoretical estimates for them include lattice QCD \[14\] and QCD-inspired
models \[15\] \[16\] \[17\] \[18\]. Building self-consistent models of GPDs is, however, a rather difficult problem, because one needs to satisfy many constraints which should be obeyed by GPDs. They include spectral properties, polynomiality condition, positivity, relations to parton densities and form factors \[14\] \[15\] \[16\]. Most of these conditions are satisfied, of course, in perturbation theory, and there were attempts \[21\] \[22\] to use perturbative expressions for modeling GPDs. Still, there remains a question about relation of perturbative results to functions describing nonperturbative dynamics. In this respect, QCD sum rules \[23\] look as an attractive possibility, being an approach which is closely related to Feynman diagrams. Its basic concept, quark-hadron duality, provides a tool for translating perturbative results into statements about nonperturbative functions. In the past, QCD sum rules were used to get information about form factors \[24\] \[25\] \[26\] \[27\] \[28\] and parton densities (see \[29\] and references therein, and \[30\]). A natural idea is to apply the QCD sum rule techniques to model GPDs. This idea in the pion case was already elaborated in some detail by the Bochum/Dubna group \[11\]. My goal in the present paper is to give the idea with recent developments.

3 QCD Sum Rule for Pion Form Factor

Basic objects of QCD sum rule analysis \[23\] are correlators of local currents with quantum numbers of the hadrons one intends to study. The usual choice for the pion is the axial current \(j_\alpha = \bar{d}\gamma_\alpha u\). Its projection on the single-pion state \(|\langle 0|j_\beta(0)|p\rangle| = i f_\pi p_\alpha\) is specified by the pion decay constant \(f_\pi\). To study the pion form factor, one should consider correlator of three currents \[24\] \[25\]

\[
T^{\mu}_{\alpha\beta}(p_1, p_2) = i \int e^{-ip_1z_1+ip_2z_2} \langle 0 | T\{j_\beta(z_2), J^\mu(0)j_\alpha^+(z_1)\} | 0 \rangle d^4z_1d^4z_2,
\]

where \(J^\mu = e_u\bar{u}\gamma^\mu u + e_d\bar{d}\gamma^\mu d\) is the electromagnetic current. The pion-to-pion transition term corresponds to

\[
\langle 0 | j_\beta(z_2)|p_2\rangle\langle p_2|J^\mu(0)|p_1\rangle\langle p_1|j_\alpha^+(z_1)|0 \rangle,
\]

where the pion form factor contribution

\[
\langle p_2|J^\mu(0)|p_1 \rangle = 2P^\mu F_\pi(Q^2)
\]

appears in the middle matrix element. As usual, \(Q^2 = -q^2\) or \(Q^2 = -t\), if the GPD notation \(t\) is used. The relevant invariant amplitude can be extracted by taking the projection \[23\]

\[
T^\mu(p_1, p_2) \equiv n^\alpha n^\beta T^{\mu}_{\alpha\beta}(p_1, p_2) / (nP)^2,
\]

with \(n^\alpha\) chosen to be a lightlike vector with equal projections on \(p_1\) and \(p_2\), \((np_1) = (np_2) \equiv (nP)\). Since \((nq) = 0\) and \(n^2 = 0\), the projection kills the structures containing \(q_\alpha, q_\beta\) and \(g_{\alpha\beta}\). Still, the projection may contain \(n^\mu\) terms, which cannot be directly related to the form factor contribution. In what follows, we will always omit them without explicit notice.

The starting point of the QCD sum approach \[23\] is the dispersion relation for invariant amplitudes that appear in the correlator, in fact, the double dispersion relation in case of a three-point function \[24\] \[25\]

\[
T(p_1^2, p_2^2, Q^2) = \frac{1}{\pi^2} \int_0^\infty ds_1 \int_0^\infty ds_2 \frac{\rho(s_1, s_2, Q^2)}{(s_1 - p_1^2)(s_2 - p_2^2)},
\]

One should find the expression for the perturbative version of the correlator and nonperturbative corrections which modify the spectrum in the \(p_i^2\) channels, converting the free-quark spectral density into a function containing physical hadrons. From a practical point of view, it is more convenient to consider the double Borel transform \[24\] \[25\]

\[
\Phi(\tau_1, \tau_2, Q^2) = \frac{1}{\pi^2} \int_0^\infty ds_1 \int_0^\infty ds_2 \rho(s_1, s_2, Q^2) e^{-s_1\tau_1 - s_2\tau_2}
\]
in which power weights are substituted by the exponential ones. Formally, the action of the Borelization operator is given by
\[ B(p^2 \to \tau) \{ 1/(s - p^2) \} = e^{-\alpha \tau}. \]
QCD sum rule for the pion form factor then has the structure
\[ f^2 \pi^p(Q^2)e^{-m_\pi^2}(\tau_1 + \tau_2) + \text{higher states} = \frac{1}{\pi^2} \int_0^\infty ds_1 \int_0^\infty ds_2 \rho^{\text{pert}}(s_1, s_2, Q^2) e^{-s_1\tau_1 - s_2\tau_2} + \tau^2 A(\tau/\tau)(\alpha s G^2) + \tau^3 B(\tau/\tau, Q^2) \alpha_s(\bar{q}q)^2 + \text{higher condensates}, \]
where \( \tau \equiv \tau_1 + \tau_2 \). We kept the pion mass on the left hand side, but it will be neglected from now on.

4 Structure of perturbative term

In the case of \( T^\mu(p_1, p_2) \) amplitude, the lowest order perturbative term is given by a triangle diagram. It is very convenient to write it in the \( \alpha \)-representation (some details can be found in Refs. [4, 31])
\[ T^{\mu \text{pert}}(p_1, p_2) = \frac{3}{2\pi^2} \int_0^\infty \frac{1}{\lambda^2} d\alpha_1 d\alpha_2 d\alpha_3 \frac{\alpha_3(\alpha_1 + \alpha_2)}{\lambda^2} \left\{ 2 P^\mu \frac{\alpha_3}{\lambda} + q^\mu \frac{\alpha_1 - \alpha_2}{\lambda} \right\} \exp \left\{ -Q^2 \frac{\alpha_1 \alpha_2 + p_1^2 \alpha_1 \alpha_3 + p_2^2 \alpha_2 \alpha_3}{\alpha_1 + \alpha_2 + \alpha_3} \right\}, \]
where \( \lambda = \alpha_1 + \alpha_2 + \alpha_3 \). Using the formula [25, 26]
\[ B(p_1^2 \to \tau_1) \{ e^{A_1 i} p_1^2 \} = \delta(\tau_1 - A_1) \]
one can obtain the double Borel transform
\[ \Phi^\mu(\tau_1, \tau_2, Q^2) = B(p_1^2 \to \tau_1)B(p_2^2 \to \tau_2)T^\mu(p_1^2, p_2^2, Q^2) \]
of the perturbative amplitude
\[ \Phi^{\mu \text{pert}}(\tau_1, \tau_2, Q^2) = \frac{3}{2\pi^2} \int_0^\infty \frac{1}{\lambda^2} d\alpha_1 d\alpha_2 d\alpha_3 \frac{\alpha_3(\alpha_1 + \alpha_2)}{\lambda^2} \left\{ 2 P^\mu \frac{\alpha_3}{\lambda} + q^\mu \frac{\alpha_1 - \alpha_2}{\lambda} \right\} \frac{\delta(\tau_1 - \alpha_1 \alpha_3)}{\lambda} \frac{\delta(\tau_2 - \alpha_2 \alpha_3)}{\lambda} \exp \left\{ -Q^2 \frac{\alpha_1 \alpha_2}{\lambda} \right\}. \]
It is instructive to rewrite this expression using new variables
\[ x = \frac{\alpha_3}{\lambda}, \quad \frac{\alpha_1}{\lambda} = \rho(1 - x) \equiv \rho\bar{x}, \quad \frac{\alpha_2}{\lambda} = (1 - \rho)\bar{x} \equiv \bar{\rho}\bar{x}. \]
This gives the following integral representation
\[ \Phi^{\mu \text{pert}}(\tau_1, \tau_2, Q^2) = \frac{3}{2\pi^2} \int_0^1 dx \int_0^1 d\rho x(1 - x) \left\{ 2 x P^\mu + \bar{x}(\rho - \bar{\rho}) q^\mu \right\} \frac{\delta(\tau_1 - \rho\lambda x\bar{x})}{\lambda} \frac{\delta(\tau_2 - \bar{\rho}\lambda x\bar{x})}{\lambda} \exp \left\{ -\lambda \rho \bar{\rho} x^2 Q^2 \right\}. \]
Two delta functions can be used to perform integration over \( \lambda \) and \( \rho \). The result is
\[ \Phi^{\mu \text{pert}}(\tau_1, \tau_2, Q^2) = \frac{3}{2\pi^2(\tau_1 + \tau_2)} \int_0^1 \bar{x} dx \left\{ 2 x P^\mu + \bar{x} q^\mu \frac{\tau_1 - \tau_2}{\tau_1 + \tau_2} \right\} \exp \left\{ -Q^2 \frac{(1 - x)\tau_1 \tau_2}{x(\tau_1 + \tau_2)} \right\}. \]
5 GPD interpretation of perturbative term

In case of GPDs, we should substitute the local current by the bilocal operator $O_\mu(z)$, in which quark fields are separated by a lightlike distance $z$, and then use parametrization in terms of OFPDs or DDs. In fact, nothing prevents us from taking $z$ equal to the projection vector $n$. Then calculation of the triangle diagram contribution completely parallels that for the purely scalar case discussed in Refs. [4, 21]. The conversion to DD variables is especially transparent. One should just use the fact that the spinor factors $z$ fields are separated by a lightlike distance $O(\lambda^2)$.

The restriction $\beta > 0$ reflects the obvious fact that the triangle diagram involves only valence quarks. Now it is straightforward to calculate the double Borel transform

$$
\Phi^\mu_{\text{pert}}(\tau_1, \tau_2; \alpha, \beta) = \frac{3\theta(\beta)}{2\pi^2(\tau_1 + \tau_2)} (1 - \beta) \left\{ 2\beta P^\mu + \alpha r^\mu \right\} \delta \left( \alpha - \frac{\tau_2 - \tau_1}{\tau_1 + \tau_2} (1 - \beta) \right) e^{-Q^2(\tau_1 + \tau_2) / (1 - \beta)}.
$$

(17)

Note, that the spectral property $|\alpha| \leq 1 - \beta$ is manifest in this expression.

6 “Borel” model for nonforward densities

Integrating DD over $\alpha$, we should get nonforward parton density (see Eq. (6)). Performing the integral, we obtain

$$
\int_{-1 - \beta}^{1 - \beta} \Phi^\mu_{\text{pert}}(\tau_1, \tau_2; \alpha, \beta) \, d\alpha = \frac{3\theta(\beta)}{2\pi^2(\tau_1 + \tau_2)} (1 - \beta) \left\{ 2\beta P^\mu + (1 - \beta) r^\mu \frac{\tau_2 - \tau_1}{\tau_1 + \tau_2} \right\}
$$

(18)
Comparing this result with Eq. (14), we see that the integration variable \( x \) in that equation has the meaning of the momentum fraction, and the integrand can be treated as a nonforward parton density \( F(x, t = -Q^2) \).

Writing QCD sum rule for the pion form factor, we should treat symmetrically both pions, the initial and the final one, i.e. take the Borel parameters \( \tau_1, \tau_2 \) equal to each other, as it was done in Refs. [24, 25]. Such a choice corresponds to DD \( \Phi^\mu \) proportional to \( \delta(\alpha) \). In other words, each quark takes exactly a half of the \( t \)-channel momentum \( r \): there is no spread in the distribution of \( r \) among constituents. Furthermore, the \( G \) part of DD vanishes for \( \tau_1 = \tau_2 \). As a result, the perturbative term for the off-forward distribution in this case

\[
H^\text{pert}_n(x, \xi, t)|_{\tau_1=\tau_2=\tau} = 6N_c x(1-x) \exp \left[ \frac{(1-x)}{2x} t \right]
\]  

(19)

coincides with that for the nonforward parton density, or in the forward \( t = 0 \) limit, with the perturbative term for the usual parton density \( f(x) \). The perturbative term suggests \( 6x(1-x) \) for the shape of the normalized parton density. This does not look realistic, even for a valence distribution. However, as shown in Ref. [50], nonperturbative corrections shift the maximum of the distribution to smaller \( x \), and then DGLAP evolution from a low normalization point \( \mu^2 \sim 0.25 \text{ GeV}^2 \) (to which QCD sum rules refer) produces acceptable valence distributions for the pion. In general, it is a rather popular idea that there are no skewness effects at a low normalization point, and one can start with the forward approximation, generating nontrivial \( \xi \) dependence through evolution [52, 53].

Our result (19) gives an example of a nontrivial interplay between \( x \) and \( t \) dependence of a nonforward parton density. It has the same form as the result of calculation of the overlap contribution of two lightcone wave functions \( \Psi(x, k_\perp) \) with the Gaussian \( \sim \exp[-a^2k_\perp^2/x(1-x)] \) dependence on transverse momentum. The parallel between the Borel transform and Gaussian wave functions is well known (see, e.g., Refs. [26, 34]). It can be explained by the exponential weight \( e^{-s^\perp} \) and the lightcone form \( s = (k_\perp^2 + m_q^2) x(1-x) \) for the invariant mass of the \( \bar{q}q \) system. This analogy with the wave function description suggests to take the Borel transform \( \Phi^\mu \text{pert}(\tau, \tau; \alpha, \beta) \) at a particular (adjusted) value of the Borel parameter as a model for the pion DD, treating \( \tau \) as the width scale of a Gaussian wave function. From the QCD sum rule point of view, the model

\[
f^2 F^B_\pi(Q^2) = \frac{1}{\pi} \int_0^\infty ds_1 \int_0^\infty ds_2 \rho^{\text{pert}}(s_1, s_2, Q^2) e^{-(s_1 + s_2)\tau}
\]

(20)

corresponds to taking such a value of \( \tau \) for which the condensate corrections and the subtraction of contributions due to higher states perfectly cancel. For each fixed \( Q^2 \), such a value of \( \tau \) exists, but in principle it may depend on \( Q^2 \). The absence of such a dependence can be expected only if the description of the pion vertex by a Gaussian wave function is a very good approximation of reality. However, inside the QCD sum rule approach, the wave function backing of this model faces difficulties. In particular, one can try to check the normalization condition \( F_\pi^n(0) = 1 \) by using the Ward identity relation

\[
\rho^{\text{pert}}(s_1, s_2, Q^2 = 0) = \pi \delta(s_1 - s_2) \rho^{\text{pert}}(s_1).
\]

between three-point and two-point function densities. The resulting expression

\[
f^2 F^B_\pi(0) = \frac{1}{\pi} \int_0^\infty \rho^{\text{pert}}(s) e^{-2s\tau} ds
\]

(21)

matches the expression for \( f^2 \) derived from the two-point sum rule

\[
(f^B_\pi)^2 = \frac{1}{\pi} \int_0^\infty \rho^{\text{pert}}(s) e^{-s\tau} ds
\]

(22)

only if one decreases the \( \tau \) parameter of the three-point function by factor 2 compared to that used in the two-point function relation.
7 Local quark-hadron duality model

Another approach to obtaining predictions for hadronic characteristics from their analogues calculated for free-quark systems is suggested by the local quark-hadron duality hypothesis. For the pion form factor, it gives

$$f_\pi^2 F_\pi^{LD}(Q^2) = \frac{1}{\pi^2} \int_0^{s_0} ds_1 \int_0^{s_0} ds_2 \rho^{pert}(s_1, s_2, Q^2),$$  
(23)

where $s_0$ is the duality interval. This relation corresponds to the $\tau = 0$ limit of the full QCD sum rule \[10\] in which the higher states are modeled by the perturbative spectral density starting at $s_0$ in both $s_1$ and $s_2$ directions. When $\tau \to 0$, the condensate corrections vanish, and the exponential weight $e^{-(s_1+s_2)\tau}$ converts into 1. The local duality approach has no problems with the $Q^2 = 0$ limit. It gives for $f_\pi^2 F_\pi^{LD}(0)$ the same expression as the local duality relation

$$\left(f_\pi\right)^2 = \frac{1}{\pi} \int_0^{s_0} \rho^{pert}(s) ds$$  
(24)

based on two-point sum rule gives for $f_\pi^2$. Numerically, the result $\rho^{pert}(s) = 1/4\pi$ \[25\] fixes $s_0$ at $4\pi^2 f_\pi^2 \approx 0.7 \text{ GeV}^2$ \[26\]. This value is also the result \[23\] of fitting full QCD sum rule, with condensates included.

The duality interval $s_0$ has the meaning of the effective threshold for the onset of higher states. Since the location of higher states is fixed, $s_0$ has good chances to be $Q^2$-independent. This hope is supported by the fact that the local duality prediction \[25, 35\] for the pion form factor

$$F_\pi(Q^2) = \left(1 - \frac{1}{1 + 6s_0/Q^2} \right) + \frac{\alpha_s/\pi}{1 + Q^2/2s_0}$$  
(25)

(with $\alpha_s/\pi = 0.1$ and $s_0 = 4\pi^2 f_\pi^2$) is in perfect agreement with the latest Jefferson Lab measurements \[36\]. The $O(\alpha_s)$ term in Eq. \[25\] is the simplest interpolation \[35\] between the $Q^2 = 0$ value fixed by the Ward identity and the large-$Q^2$ asymptotic behavior $F_\pi(Q^2) \to 8\pi\alpha_s f_\pi^2/Q^2$ \[37, 38\] due to the one-gluon exchange.

Knowing the double Borel transform one can obtain the spectral density by the inverse transformation. It is convenient to write the result in a form similar to the lightcone representation \[39\]

$$\rho^{pert}(s, s_2, Q^2) = \frac{3}{2\pi} \int_0^1 x\bar{x} \, dx \int d^2k_{\perp} \delta(s_1 - \kappa_{\perp}^2) \delta(s_2 - (\kappa_{\perp} + \bar{q}_{\perp})^2),$$  
(26)

where $\bar{q}_{\perp}$ is a two-dimensional vector with $\bar{q}_{\perp}^2 = \bar{x}Q^2/x$. From this representation, it is evident that the “Borel” model \[20\] corresponds to Gaussian wave functions, while the local duality \[25\] corresponds to step-like effective wave functions $\sim \theta(\kappa_{\perp}^2 \leq s_0)$ (note, that $\kappa_{\perp}$ differs from the usual $k_{\perp}$ by $\sqrt{xx}$ rescaling). Thus, to get the form factor through the local duality formula, one needs to calculate the area of overlap of two circles having equal radii. This gives the representation \[39\]

$$F_\pi^{LD}(Q^2) = \frac{12}{\pi} \int_0^1 x\bar{x} \, dx \left\{ \arccos \left( \frac{xQ^2}{4xs_0} \right) - \sqrt{\frac{xQ^2}{4xs_0}} \left( \frac{1 - xQ^2}{4xs_0} \right) \right\}.$$  
(27)

Its integrand can be treated as the local duality model for the nonforward parton density $F(x, t = -Q^2)$. Simple structure of Eq. \[26\] allows to write form factor also in the impact parameter representation

$$F_\pi^{LD}(Q^2) = \frac{6}{\pi} \int \frac{d^2b_{\perp}}{b_{\perp}^2} e^{i(q_{\perp}b_{\perp})} \int_0^1 x\bar{x} \left[ J_1 \left( \sqrt{\frac{xs_0}{1 - x}} b_{\perp} \right) \right]^2 dx,$$  
(28)

where $J_1(z)$ is the Bessel function. Again, the integrand gives a model for the nonforward density in the impact parameter space.
8 Nucleon form factors

QCD sum rule analysis of the nucleon form factors is based on the study of the 3-point correlator

\[ T_{\alpha\beta}^{\mu}(p_1, p_2) = \int \langle 0 \mid T(\eta_{\beta}(z_2), J^{\mu}(0), \bar{\eta}_{\alpha}(z_1)) \mid 0 \rangle \delta^{(4)}(p_1 + z_2 - p_2) d^4z_1 d^4z_2 \]  

(29)

of the electromagnetic current \( J^{\mu} \) and two Ioffe currents \( \eta, \bar{\eta} \) with the nucleon quantum numbers,

\[ \eta = \varepsilon^{abc} (u^a C^{-1} \gamma_\rho u^b) \gamma_\rho \gamma_5 d^c. \]

Here, \( C \) is the charge conjugation matrix, \( \{a, b, c\} \) refer to quark colors, the absolutely antisymmetric tensor \( \varepsilon^{abc} \) ensures that the currents are color singlets, and \( \alpha, \beta \) are Dirac indices. The amplitude \( T_{\alpha\beta}^{\mu} \) is the sum of various structures: \( P^{\mu} P \equiv V^{\mu}(P) \), \( q^{\mu} P \), \( i \epsilon^{\mu\lambda\sigma\tau} P_{\lambda} q_{\rho} \gamma_\tau \gamma_\sigma \equiv A^{\mu}(P, q) \), \( Q^2 \gamma^{\mu} \), etc. To compare contributions of different structures, one should specify the reference frame. A natural choice is the infinite momentum frame (IMF), where \( q^{\mu} \) is in the plus direction and \( P^+ \to \infty \), while \( q^\mu \equiv q_\perp^\mu \) is fixed. Note, that neglecting \( q^\mu \) compared to \( P^\mu \) is exactly what we did in the pion case by taking projection on the light-like vector \( \eta \) orthogonal to \( q \). The leading IMF structure is clearly \( V^{\mu}(P) \): it does not contain the "small" parameter \( q \). For \( p_1^2 = p_2^2 \), this structure satisfies the transversality condition \( q_{\rho} V^{\mu}(P, q) = 0 \). Another structure possessing this property is \( A^{\mu}(P, q) \). These two structures have the most direct connection with the \( P \) component of the two-point function defined through the correlator \( \langle 0 \mid T(\eta_{\beta}(z), \bar{\eta}_{\alpha}(0)) \mid 0 \rangle \). In the decomposition of the proton-to-proton transition part of the correlator \( T_{\alpha\beta}^{\mu}(p_1, p_2) \), the structure \( V^{\mu}(P) \) is accompanied by the nucleon \( F_1(Q^2) \) form factor, while the \( A^{\mu}(P, q) \) structure is accompanied by the form factor \( G_M(Q^2) \). The double Borel transform of the invariant amplitude \( T_{\nu}^{\mu}(p_1^2, p_2^2, Q^2) \) related to \( V^{\mu}(P) \) structure has the form [41]

\[ \Phi_1(\tau_1, \tau_2, Q^2) = \frac{1}{(2\pi)^4(\tau_1 + \tau_2)^3} \int_0^1 dx \left[ 3e_u \vec{x}^2 - (2e_u - e_d) \vec{x}^2 \right] \exp \left[ -Q^2 \frac{x \tau_1 \tau_2}{x(\tau_1 + \tau_2)} \right]. \]  

(30)

Taking \( \tau_1 = \tau_2 \equiv \tau \), one obtains the Borel model for the \( F_1(Q^2) \) form factor. Similarly, the double Borel transform of the \( T_{\nu}^{\mu}(p_1^2, p_2^2, Q^2) \) amplitude [41]

\[ \Phi_M(\tau_1, \tau_2, Q^2) = \frac{1}{(2\pi)^4(\tau_1 + \tau_2)^3} \int_0^1 dx \left[ 3e_u \vec{x}^2 \right] \exp \left[ -Q^2 \frac{x \tau_1 \tau_2}{x(\tau_1 + \tau_2)} \right] \]  

(31)

gives the Borel model for the magnetic form factor \( G_M(Q^2) \). Using \( G_M(Q^2) = F_1(Q^2) + F_2(Q^2) \) we obtain

\[ \Phi_2(\tau_1, \tau_2, Q^2) = \frac{1}{(2\pi)^4(\tau_1 + \tau_2)^3} \int_0^1 dx \left[ 2(e_u - e_d) \vec{x}^2 \right] \exp \left[ -Q^2 \frac{x \tau_1 \tau_2}{x(\tau_1 + \tau_2)} \right] \]  

(32)

for the double Borel transform of the amplitude related to the \( F_2(Q^2) \) form factor. The integrands of these representations can be treated (for \( \tau_1 = \tau_2 = \tau \)) as models for the corresponding nonforward parton densities. In the \( Q^2 = 0 \) limit, one obtains models for forward parton densities. The use of local duality changes the exponential factor \( \exp[-\vec{x}Q^2/2x] \) into some function of \( \vec{x}Q^2/xs_0 \) without changing the pre-factors, i.e., forward parton densities. Let us discuss main features of the models for forward densities.

- In case of \( \Phi_1 \), the forward densities correspond to usual parton densities. The model gives:

\[ f_u^{mod}(x) = 4 \vec{x}^2(3 - 2 \vec{x}) = 4(1 - x)^3, \quad f_d^{mod}(x) = 4 \vec{x}^3 = 4(1 - x)^3. \]

Just like in the pion case, the condensate corrections and then DGLAP evolution will shift the distributions towards smaller \( x \). One may expect that these effects will modify both distributions in a similar way. So, it is interesting to compare relative shapes of the model \( u \) and \( d \) distributions. The main feature is that \( d \) distribution has an extra power of \( (1 - x) \) compared to \( u \) distribution. The extra power of \( (1 - x) \) in
\(d\) distribution is a well known phenomenological observation. For comparison, GRV parametrization for normalization point \(\mu^2 \sim 1\ \text{GeV}^2\) can be rather accurately reproduced by \cite{10}

\[
f_u^\text{phen}(x) = 1.89x^{-0.4}(1 - x)^{3.5}(1 + 6x), \quad f_d(x)^\text{phen} = 0.54x^{-0.6}(1 - x)^{1.2}(1 + 8x),
\]

with \(d\) distribution having extra power \((1 - x)^{0.7}\) for \(x \to 1\). It is easy to check that the ratios \(f_d^\text{mod}(x)/f_u^\text{mod}(x)\) and \(f_d^\text{phen}(x)/f_u^\text{phen}(x)\) are very close to each other.

• In case of \(\Phi_2\), the densities \(e_u(x)\) correspond to the forward limit of the spin-flip GPDs

\[
e_u(x) = E_u(x, \xi = 0, t = 0).
\]

They are inaccessible in deep inelastic scattering and other inclusive processes. However, the normalization integrals \(\kappa_n\) for these functions are related to the anomalous magnetic moments of the nucleons:

\[
e_u\kappa_u + e_d\kappa_d = \kappa_p, \quad e_d\kappa_u + e_u\kappa_d = \kappa_n,
\]

and, hence, they are known:

\[
\kappa_u = 2\kappa_p + \kappa_n = 1.65, \quad \kappa_d = 2\kappa_n + \kappa_p = -2.07.
\]

The model densities are

\[
e_u^\text{mod}(x) = 8(1 - x)^3, \quad e_d^\text{mod}(x) = -4(1 - x)^3.
\]

The normalization integrals for these functions are \(\kappa_u^\text{mod} = 2\) and \(\kappa_d^\text{mod} = -1\). The correct nontrivial sign of \(\kappa_d\) is a very encouraging indication that the model is a reliable starting point. Another feature of the model \(e_u,d(x)\) distributions is that they both have an extra power of \((1 - x)\) compared to \(f_u(x)\). As a result, the \(F_2(Q^2)\) form factor, as we will see later, decreases faster with \(Q^2\) than \(F_1(Q^2)\) and \(G_M(Q^2)\), again in agreement with experimental observations.

### 9 Form factors at large momentum transfer

In the Borel model, the form factors are given by the integrals

\[
F^B(Q^2) = \int_0^1 dx f^B(x) \exp\left[-Q^2 \frac{1 - x}{2x}\right]
\]

(33)

involving the model parton density \(f^B(x)\) and the exponential factor \(e^{-(1-x)Q^2\tau/2x}\). Hence, at large \(Q^2\), the form factors are dominated by integration over the region where \(Q^2 \bar{\tau} \bar{x} \sim 1\) or \(1 - x \sim \tau/Q^2\). For the local duality model, the whole integration region over \(x\) is restricted to \((1 + 4s_0/Q^2)^{-1} \leq x \leq 1\), i.e., \(1 - x < 4s_0/Q^2\) for large \(Q^2\).

Hence, the result of integration in both models is completely determined by the behavior of parton densities at \(x\) close to 1. Namely, if \(f(x) \sim (1 - x)^\nu\) for \(x \to 1\), then the relevant form factor drops like \(1/(Q^2)^{\nu+1}\) at large \(Q^2\). This gives \(1/Q^4\) for the asymptotics of the pion form factor, \(1/Q^6\) for the large-\(Q^2\) behavior of the nucleon form factors \(F_1(Q^2)\) and \(G_M(Q^2)\), and \(1/Q^8\) for \(F_2(Q^2)\). All these results seem to be in contradiction with the experimentally established exponents of the power-law behavior of these form factors, so one may be tempted to conclude that our models have no chance to describe the data.

In fact, as already mentioned, the local duality prediction \cite{25} is in excellent agreement with the results of recent JLab data. In the nucleon case, the local duality calculation of \(G_M(Q^2)\) performed in Ref. \cite{26, 41} agrees with the data up to \(Q^2 \sim 20\ \text{GeV}^2\) (see also Ref. \cite{42}, where the curve based on Refs. \cite{26, 41} is compared with the results of other approaches). The ratio \(F_2^p(Q^2)/F_1^p(Q^2)\) as calculated in the local duality model agrees with the data based on Rosenbluth separation \cite{43}, though at the highest \(Q^2\) it is
somewhat lower than the results of the polarization transfer experiments [44]. In all cases, deviations do not exceed 20-30%. This means that the model curves mimic the “canonic” $Q^2$ behavior of these form factors ($1/Q^2$ for $F_2(Q^2)$, $1/Q^4$ for $F_1^p(Q^2)$ and $G_{M1}^p(Q^2)$, and $1/Q^6$ for $F_2^p(Q^2)$).

The resolution of the paradox is based on a trivial observation that the model curves are more complicated functions than just pure powers of $1/Q^2$. Their nominal large-$Q^2$ asymptotics is achieved only at very large values of $Q^2$, well beyond the accessible region. Thus, conclusions made on the basis of asymptotic relations might be of little importance in practice: a curve with “wrong” large-$Q^2$ behaviour might be quite successful phenomenologically in a rather wide range of $Q^2$.

10 Improved Ansätze

The models discussed above, of course, have some drawbacks. In particular, the small-$x$ behavior of the model parton densities is unsatisfactory: it does not have the standard Regge-type behavior $f(x)|_{x \to 0} \sim x^{-\alpha(0)}$. The excuse is that such a behavior cannot result from a single lowest-order triangle diagram: to get it, one should add an infinite number of diagrams with the ladder structure in the $Q^2$ channel, and perform summation of all contributions. A simple solution of this hopeless problem is to take experimental forward distributions in Eq. (33) instead of the model ones. Such an approach was successfully used in Ref. [11] for the pion form factor and in Refs. [10, 17] for the proton $F_2^p(Q^2)$ form factor. The modified Borel (or Gaussian) model for $F_2^p(Q^2)$ was able to successfully describe the data up to $Q^2 \sim 10$ GeV$^2$.

Next question is about the exponential factor. The Regge picture suggests $x^{-\alpha(t)}$ behavior at small $x$ or

$$F(x, t) = f(x)x^{-(\alpha(t)-\alpha(0))}$$

(34)

model for the nonforward densities. Assuming a linear Regge trajectory with the slope $\alpha'$, one gets

$$F^R(x, t) = f(x)x^{-\alpha't} = f(x)e^{-\alpha't \ln x}.$$  

(35)

This Ansatz was already discussed in Refs. [5] and [45]. It also provides finite mean squared radii

$$\langle r^2 \rangle = -6 \alpha'_1 \int_0^1 dx f(x) \ln x,$$

(36)

which are in good agreement with experimental values [45]. For large $t$, experimental data support Drell-Yan (DY) relation [13] and Bloom-Gilman duality [46]. According to DY, if the parton density behaves like $(1-x)^\nu$, then the relevant form factor should decrease as $1/t^{(\nu+1)/2}$ for large $t$. The simplest idea is to attach an extra $(1-x)$ factor in the exponential [47], i.e. to take the model

$$F^{\text{remod}}_a(x, t) = f_a(x)x^{-\alpha'_1(1-x)t}.$$  

(37)

To calculate $F_2$, we need an Ansatz for the spin-flip nonforward parton densities $E_a(x, t)$. One can assume the same model $E^{\text{remod}}_a(x, t) = e_a(x)x^{-\alpha'_2(1-x)t}$ as for $F_a(x, t)$, with possibly a slightly different slope $\alpha'_2$. To model the forward magnetic densities $e_a(x)$, we can use the lesson from the triangle diagram calculation that $e_a(x)$ has an extra power of $(1-x)$ compared to $f_a(x)$. In fact, one can take the extra factors in the form $(1-x)^{\eta_a}$ with $\eta_a$’s being fitting parameters. Within this approach, it is possible to get a rather good description of all four nucleon form factors [48] (see Ref. [49] for a similar analysis).

11 Discussion

In this paper, we discussed basics of an approach that uses QCD sum rule ideas to build models for generalized parton distributions. The underlying idea is to consider three-point functions in which the hadrons are represented by local currents with appropriate quantum numbers. The necessary nonlocality bringing in
parameters having the meaning of the hadron size can be introduced in several ways: by nonzero virtualities $p_i^2$ of the momenta associated with the currents, by taking the double Borel transformation from $p_i^2$’s to $\tau_i$’s, or through the local duality prescription within the duality square $s_0 \times s_0$. In terms of the basic function, the spectral density $\rho(s_1, s_2, t)$, these possibilities correspond to integration with different weights: a) $(s_1 - p_1^2)^{-1}(s_2 - p_2^2)^{-2}$ (producing the original amplitude $T(p_1^2, p_2^2, t)$), b) $\exp[-s_1 \tau_1 - s_2 \tau_2]$, and c) $\theta(s_1 \leq s_0)\theta(s_2 \leq s_0)$ (local duality prescription). We have considered only the lowest approximation for $\rho(s_1, s_2, t)$ ($O(\alpha_s)$ terms for pion form factor were discussed in refs. [25, 50]). One of the expected effects of higher-order corrections is the emergence of the Regge-type behavior at small $x$. Since there is no doubt about this outcome, such a behavior can be introduced in the model expressions by using experimental parton densities instead of those generated by the lowest order term.

In higher order diagrams, one would also obtain contributions corresponding to the leading large-$Q^2$ asymptotics of perturbative QCD, like the one-gluon-exchange term for the pion form factor. All such higher-order contributions are suppressed by $\alpha_s/\pi \sim 1/10$ factor per each extra loop. Such a suppression is manifest in the local duality result [25] for the pion form factor. Note, that the local duality prediction is in perfect agreement with the data despite the fact that the $O(\alpha_s)$ term containing the hard gluon exchange is insignificant compared to the lowest order term. For the nucleon form factors, the leading pQCD two-gluon exchange term has a priori suppression by a factor of 100, so it is unlikely to be relevant at any accessible momentum transfer.

Another pQCD prediction is about the $x \rightarrow 1$ behavior of parton densities. In the nucleon case, the leading $(1 - x)^3$ term corresponds to four-gluon-exchange diagrams and $\sim 10^{-4}$ suppression compared to the $O(\alpha_s^3)$ term. The latter has $(1 - x)^2$ behavior for $u$ quarks and $(1 - x)^3$ for $d$ quarks. Nonperturbative effects and DGLAP evolution are undoubtedly capable to shift these densities towards the experimentally observed shapes. Furthermore, there is no need to make extra efforts to bring in the relative $(1 - x)$ suppression of the $d$ density: it is present in the starting approximation.

The lowest order term also implies a faster fall-off of the $F_2(Q^2)$ form factor compared to $F_1(Q^2)$. This effect results from the extra $(1 - x)$ power of $e_{u}^{\text{mod}}(x)$ compared to $f_{u}^{\text{mod}}(x)$. It should be emphasized that, in general, the large-$Q^2$ behavior of form factors in the $O(\alpha_s^3)$ approximation is completely governed by Feynman mechanism, i.e., by $x \sim 1$ integration, so that the large-$Q^2$ behavior of form factors is always determined by the $x \rightarrow 1$ behavior of the parton densities. The specific correlation pattern between the $\nu$ power in $(1 - x)^\nu$ and $n$ in $1/(Q^2)^n$ depends on the structure of a factor like $\exp[-Q^2 \tau(1 - x)/2x]$ in the Borel model. As we discussed, it should be modified to $\exp[Q^2 \alpha(1 - x) \ln x]$ to impose the $n = (\nu + 1)/2$ correlation dictated by the Drell-Yan relation. In contrast, in pQCD the large-$Q^2$ behavior of form factors is governed by configurations in which all the valence quarks carry finite momentum fractions $x_i$, i.e., the $1/Q^2$ behavior of $F_1(Q^2)$ is not a consequence of the $(1 - x)^3$ behavior of the parton densities. The relation $n = (\nu + 1)/2$ in pQCD is just an accidental correlation between two parameters. In other words, there is a correlation between $\nu$ and $n$ in pQCD because both are determined by the same hard gluon exchange mechanism, but there is no causal connection between these two numbers. As noted in the pioneering paper [38], there is no Drell-Yan/Feynman mechanism in pQCD.

Summarizing, the gross features of generalized parton distributions are dominated by nonperturbative dynamics, and, hence, we need nonperturbative approaches to build models for GPDs. The models motivated by QCD sum rule ideas have already made several successful predictions, and they also have the advantage of being closely related to perturbative calculations, which allows to satisfy nontrivial constraints imposed on GPDs. This makes the QCD sum rule based approach an attractive possibility for building realistic models of GPDs.

Acknowledgements  I use this opportunity to thank my friend and colleague Prof. Dr. Klaus Goeke for (in)numerous invitations to visit the Institute of Theoretical Physics II in Bochum and conferences in Germany; his warm hospitality, inspiring discussions and support. This paper was strongly influenced by my contacts with the members of his group M.V. Polyakov, C. Weiss, P.V. Pobyliutsa, N.G. Stefanis, W. Schroers, N. Kivel, and visitors to Bochum: A.P. Bakulev,
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