Detecting a stochastic background of gravitational waves by correlating $n$ detectors

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We discuss the optimal detection strategy for a stochastic background of gravitational waves in case $n$ detectors are available. In literature so far, only two cases have been considered: the $n(n-1)/2$ 2-point correlators and the single $n$-point correlator. We generalize these analyses to $m$-point correlators (with $m < n$) built out of the $n$ detector signals, obtaining the result that the optimal choice is to combine 2-point correlators. Correlating $n$ detectors in this optimal way will improve the (suitably defined) signal-to-noise ratio with respect to the $n=2$ case by a factor equal to the fourth root of $n(n-1)/2$.

PACS numbers: 04.80.Nn,04.80.-y,95.55.Ym,07.05.Kf

As it is well known [1], the sensitivity to a stochastic background signal can be greatly enhanced by correlating the output of two detectors. To show how this works it is useful to consider the cross correlation $S_{12}$ [2,3] between two detectors outputs $S_1$ and $S_2$, defined by

$$S_{12} \equiv \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' S_1(t)S_2(t')Q(t-t') = \int_{-\infty}^{\infty} df \tilde{S}_1^*(f)\tilde{S}_2(f)\tilde{Q}(f),$$

(1)

where the filter function $Q(t)$ has been introduced. The cross correlation $S_{12}$ depends only on the time difference $t - t'$ as stationarity in both the signal and the noise is assumed. In the last equality the Fourier transform of the signal and the limit $T \to \infty$ have been taken. For any finite $T$, $S_{12}$ is made of the sum of statistically independent random variable involving $\tilde{S}_1(f)$ and $\tilde{S}_2(f')$, which are correlated only over a frequency range $|f-f'| < 1/T$. Thus, as $S_{12}$ is the product of random variables it is a random variable itself and it can be approximated by a Gaussian variable by virtue of the central limit theorem, even in the case of narrow band detectors, provided that $T$ is much larger than the inverse of the
bandwidth. The same will be true in the case the product of more than two random variables
is considered, as it will be later.

The outputs of two detectors can be split as

\[ S_{1,2} = s_{1,2} + N_{1,2}, \]

being \( s_i \) the physical signal and \( N_i \) the noise. The signal-to-noise ratio for the correlation of the 2 detectors at our disposal (this redundant notation will be useful later, where \( m \)-point correlators out of \( n \) detectors will be considered) is given by

\[
[\text{SNR}(2|2)]^2 \equiv \frac{\langle S_{12} \rangle}{\sigma_{12}} = \frac{\langle S_{12} \rangle}{\langle (S_{12}^2 - \langle S_{12} \rangle)^{1/2} \rangle} = \frac{\langle s_1 s_2 \rangle}{\langle N_1^2 N_2^2 \rangle^{1/2}},
\]

where \( \langle S_{12} \rangle \) and \( \sigma_{12} \) are respectively the average and the square root of the variance of the cross correlation. We have adopted the convention which makes the signal-to-noise ratio proportional to the metric perturbation \( h \), so that in our notation \( \text{SNR} \propto h \), as in \([4]\), differently from \([3]\) where \( \text{SNR} \propto h^2 \). To obtain the last equality in \((2)\) we have made the basic assumptions that we will never drop throughout this Letter: both the signal and the noise are Gaussian, they are statistically independent, stationary and with zero mean, \( N_i \ll s_i \) and finally the noises of different detectors are completely uncorrelated.

The filter function \( Q(t) \) appearing in \((1)\) can be freely chosen in order to maximize the signal-to-noise ratio. The best choice is obtained in the standard way by imposing the functional variation of \((1)\) with respect to \( Q(t) \) equal to zero and solving it for \( Q(t) \). To write down the explicit form of the filter function it is necessary to introduce some further quantity. The signal can be usefully written as

\[
s_i(t_i, x_i) = \int_{-\infty}^{\infty} df_i \int d\Omega_i \tilde{h}_A(f_i, \Omega_i) e^{2\pi i f_i (t_i - \Omega_i x_i)} F_A(\Omega_i),
\]

where \( \tilde{h}_A \) is the Fourier transform of the metric perturbation with polarization \( A \) and \( F_A \) is the pattern function of the detectors, which encodes the information on its angular sensitivity. Given the stochastic nature of the signal, the 2-point correlator (ensemble average of the Fourier components) of the metric perturbation can be parameterized as

\[
\langle \tilde{h}(f_1, \Omega_1) \tilde{h}(f_2, \Omega_2) \rangle = \delta(f_1 + f_2) \frac{1}{4\pi} \delta^2(\Omega_1, \Omega_2) \frac{1}{2} S_h(f_1),
\]

where the spectral function \( S_h \) has been introduced. Analogously a noise spectral function \( S_{N,i} \) for the \( i \)-th detector can be defined through

\[
\langle N_i(f_1) N_j(f_2) \rangle = \delta_{ij} \delta(f_1 + f_2) \frac{1}{2} S_{N,i}(f_1).
\]
The filter function which maximizes the signal-to-noise ratio is

\[ \tilde{Q}(f) \propto \frac{S_h(f) \Gamma(f, x_{12})}{S_{N,1}(f) S_{N,2}(f)}, \tag{6} \]

where the overlap function \( \Gamma \) has been introduced. It takes care of the relative distance and orientation of the two detectors and its explicit form (which anyway is not important for our purposes) reads

\[ \Gamma(f_i, x_{ab}) = \frac{1}{4\pi} \int d^2 \Omega e^{2\pi i f_i \Omega(x_a - x_b)}. \]

Inserting the optimal filter function (6) in (1) and in (2) the explicit form of the signal-to-noise ratio for the correlation of two detectors is obtained

\[ \text{SNR}(2, 2) = \left( T \int_{-\infty}^{\infty} df \, \Gamma^2(f, x_{12}) \frac{S_h^2(f)}{S_{N,1}(f) S_{N,2}(f)} \right)^{1/4}, \tag{7} \]

which gains in the case of two identical detectors with respect to the single detector case, as it is well known, a factor of \((T\Delta f)^{1/4}\), being \(T\) the experiment time and \(\Delta f\) the bandwidth.

Now one might well ask what can be gained by the correlation of several such detectors. A partial answer is obtained by generalizing (7) to the case of \(2n\) detectors (the number of detectors must be even for the correlator not to vanish)

\[ \text{SNR}(2n|2n) = \frac{\left( \prod_{i=1}^{n} S_h(f_i) \Gamma(f_i, x_{i,n+i}) \right)^2 + \text{perm}}{\prod_{i=1}^{n} S_{N,i}(f_i) S_{N,i+n}(-f_i)} \left( T \int_{-\infty}^{\infty} df_1 \ldots \int df_n \right)^{1/2n}, \tag{8} \]

where in our notation \(\text{SNR}(i|j)\) is the signal-to-noise ratio given by \(i\)-point correlators taken out of \(j\) detectors. To obtain (8) the explicit form of the optimal filter function has been used. The permutations come from all the different pairings of \(2n\) signals: as the detector output is Gaussian its \(n\)-point correlator can be computed from the product of 2-point ones. In \(\text{SNR}(2n|2n)\) we indicated only the terms with leading behavior in \(T\) for large \(T\), which are \((2n - 1)!!\). Eq. (8) can be rewritten as

\[ [\text{SNR}(2n|2n)]^{4n} = \prod_{i=1}^{n} [\text{SNR}(i, n + i|2)]^4 + \text{perm}, \tag{10} \]

i.e. it is the sum of permutations of products 2-point correlators.
We now show that there exist a better way to treat data obtained from 2\(n\) detectors, as out of 2\(n\) detectors, \(2m\)-correlators can be considered, for any \(m < n\). For \(m = 1\) we can follow the analysis of \([1]\) or \([3]\) and consider all the possible pairs taken out of 2\(n\) detectors. For each detector pair a mean value and a variance can be defined as usual

\[
\bar{S}_{ij} \equiv \langle S_{ij} \rangle = \bar{S}_2 \quad \sigma_{ij}^2 = \langle S_{ij}^2 \rangle - \bar{S}_{ij}^2 ,
\]

(11)

where the optimal filter function has been normalized so to make the theoretical mean \(\langle S_{ij} \rangle = \bar{S}_2\) equal for every pair. A SNR\((i,j|2)\) of the type \((7)\) can thus be assigned to each pair

\[
[\text{SNR}(i,j|2)]^2 = \frac{\bar{S}_{ij}}{\sigma_{ij}} .
\]

(12)

The best way to gather the information from all the pairings is to take a weighted average with weights \(\lambda_{ij}\)

\[
S_2 \equiv \frac{\sum_{i<j} \lambda_{ij} S_{ij}}{\sum_{i<j} \lambda_{ij}} ,
\]

(13)

whose variance

\[
\sigma_{S_2}^2 \equiv \langle S_2^2 \rangle - \langle S_2 \rangle^2 = \frac{\sum_{i<j} \lambda_{ij}^2 \sigma_{ij}^2}{\left(\sum_{i<j} \lambda_{ij}\right)^2} ,
\]

which is justified by large noise approximation we are using, that allows to neglect non diagonal terms like \(\sigma_{ij}\sigma_{kl}\) (for \(\{i,j\} \neq \{k,l\}\)) compared to \(\sigma_{ij}^2\). The signal-to-noise ratio obtained by combining in pairs the 2\(n\) detector outputs in this way is given by

\[
[\text{SNR}(2|2n)]^4 = \left(\frac{\langle S_2 \rangle}{\sigma_{S_2}}\right)^2 = \frac{\left(\sum_{i<j} \lambda_{ij} \bar{S}_{ij}\right)^2}{\sum_{i<j} \lambda_{ij}^2 \sigma_{ij}^2} .
\]

(14)

The best signal-to-noise ratio is obtained by choosing \(\lambda_{ij} \propto \sigma_{ij}^{-2}\) (which correspond to weighing less the more noisy data) and it is

\[
[\text{SNR}(2|2n)]^4 = \sum_{i<j} \frac{\bar{S}_{ij}^2}{\sigma_{ij}^2} = \sum_{i<j} [\text{SNR}(i,j|2)]^4 .
\]

(15)

The optimal signal-to-noise ratio is thus given by the sum of terms like \((7)\) (to the fourth power); note that we recover the time dependence of \((8)\): \(\text{SNR}(2|2n) \propto T^{1/4}\). For 2\(n\) detectors with equal noise level and data collection time we have

\[
\text{SNR}(2|2n) \propto \left[n(2n - 1)\right]^{1/4} .
\]

(16)
We now generalize the analysis of the combination of 2-points correlators to the case of 2m-point correlators. Analogously to (13) we can define a linear combination $S_{2m}$ of the $(2n)!/[(2m)!(2n - 2m)!]$ 2m-point correlators $S_{i_1...i_{2m}}$ that is possible to build out of 2n detectors. Defining a signal-to-noise ratio of the type (8)

$$[\text{SNR}(i_1\ldots i_{2m}|2m)]^{2m} \equiv \frac{\langle S_{i_1...i_{2m}} \rangle}{\sigma_{i_1...i_{2m}}},$$

as a natural generalization of (12), we are led to consider the combination of the 2m-correlators analogous to (13)

$$S_{2m} = \sum_{i_1 < \ldots < i_{2m}} \frac{\lambda_{i_1...i_{2m}} S_{i_1...i_{2m}}}{\lambda_{i_1...i_{2m}}}, \quad \sigma_{S_{2m}}^2 = \sum_{i_1 < \ldots < i_{2m}} \frac{\lambda_{i_1...i_{2m}}^2 \sigma_{i_1...i_{2m}}^2}{\left(\sum_{i_1 < \ldots < i_{2m}} \lambda_{i_1...i_{2m}}\right)^2},$$

(with $i_k \in \{1\ldots2n\}$) so that the signal-to-noise ratio for 2m-correlators can be written, for the optimal choice of weights $\lambda_{i_1...i_{2m}} \propto \sigma_{i_1...i_{2m}}^{-2}$, as

$$[\text{SNR}(2m|2n)]^{4m} \equiv \frac{\langle S_{2m} \rangle^2}{\sigma_{S_{2m}}^2} = \sum_{i_1 < \ldots < i_{2m}} \frac{\langle S_{i_1...i_{2m}} \rangle^2}{\sigma_{i_1...i_{2m}}^2} = \sum_{i_1 < \ldots < i_{2m}} [\text{SNR}(i_1\ldots i_{2m}|2m)]^{4m}.$$

(17)

Each of the terms in the sum in the most rhs of (17) is on its own the sum of $(2m - 1)!!$ terms as shown in (8).

For equal noises and observation times the scaling of the signal-to-noise ratio with respect to the number of detectors $2n$, and with the order of the correlator $2m$, is given by

$$\text{SNR}(2m|2n) \propto (2m - 1)!! \times \left(\begin{array}{c} 2n \\ 2m \end{array}\right)^{\frac{1}{4m}},$$

(18)

where the first factor comes from the number of contribution in each $\text{SNR}(i_1\ldots i_{2m}|2m)$ and the binomial coefficient from the possible choices of 2m-ple out of 2n detectors.

For any fixed $n$ the maximum is obtained always for $m = 1$ implying that the optimal signal-to-noise ratio is obtained by combining the detectors in pairs as in (15). In particular for a network made of a large number of detectors the signal-to-noise is expected to scale with the square root of the number of detectors as in (16).

We thank Michele Maggiore, Florian Dubath, Stefano Foffa and Alice Gasparini for useful discussions. The work of R. S. is supported by the Fonds National Suisse.

