Semclassical limit of the entanglement in closed pure systems

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We discuss the semiclassical limit of the entanglement for the class of closed pure systems. By means of analytical and numerical calculations we obtain two main results: (i) the short-time entanglement does not depend on Planck’s constant and (ii) the long-time entanglement increases as more semiclassical regimes are attained. This result is in contrast with the common belief that the entanglement should be destroyed when the macroscopic limit is reached. We also found that, for Gaussian initial states, the entanglement dynamics may be described by an entirely classical entropy in the semiclassical limit.

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I. INTRODUCTION

Entanglement is one of the greatest resources offered "exclusively" by the quantum world. It allows for the realization of the most challenging ideas like quantum information processing and quantum computation [1-3]. Nonetheless, up to nowadays, finding a general and practical measure of entanglement remains an open important task. For globally pure bipartite systems, however, there exist suitable entropic quantities obeying all requirements for an adequate entanglement measure [4, 5]. Particularly useful is the reduced linear entropy, which offers in general the same informations given by the von Neumann entropy but with a much less computational effort.

Since entanglement is regarded as an intrinsically quantum property with no classical analog, natural questions arise concerning its semiclassical behavior. Closed pure systems present a well-behaved semicalssical limit concerning its expectation values in coherent states, which asymptotically recover the classical trajectories as \( \hbar \to 0 \). In this case, the breakdown of the correspondence between the quantum centroid and the underlying classical trajectory is well understood in terms of classical time scales \( \gg 2 \), although other quantities may have different scales \( \ll 1 \). Then we may ask about the behavior of the entanglement in such situations in which an underlying classical dynamics is well-known. This question motivates us to analyze the semiclassical limit of the entanglement dynamics for closed pure systems in which classicality is achieved by means of macroscopic coherent states, i.e., minimal uncertainty states \( |\alpha\rangle \) with \( |\alpha|^2 \gg 1 \).

The common belief induces us to expect a decrease in the entanglement in the semiclassical limit. However we show by some numerical examples that this is not a precise idea. Our starting point is showing analytically that the short-time entanglement is \( \hbar \)-independent.

This is demonstrated also numerically in terms of the reduced linear entropy, by means of simulations in the N-atom Jaynes-Cummings model without rotating wave approximation (RWA) or Dicke model (DM) [11], which has recently received renewed attention from solid-state community working on phonon cavity quantum dynamics [12, 13], Josephson Junctions and quantum dots [13], quantum chaos [13] and quantum phase transitions [14]. We thus proceed to show analytical results of the entanglement dynamics in a second model, for which it is possible to understand the behavior of the entanglement in the presence of an effect with no classical analog, namely, the quantum interference. This is done for a model of two coupled non-linear oscillators which in certain regimes describe a coupled Bose-Einstein condensates. Our argumentation will be supported by a classical entropy, which is defined within the classical theory of ensembles. It describes the classical correlation dynamics of initially separable probabilities. This quantity has been shown to evolve very closely to its quantum counterpart, the reduced linear entropy, in cases that are initially separable and in which intrinsic quantum effects are absent [17]. Similar approaches may be found in [18].

This paper is organized as follows: in section II we discuss a short time expansion for the entanglement in an arbitrary pure bipartite system. In Section III a semiclassical analysis of the whole dynamics of the entanglement is numerically investigated for two different models and an explanation in terms of mechanisms that are classically understandable as well, is given. Section IV is reserved for some concluding remarks.

II. THE SHORT-TIME ENTANGLEMENT

The reduced linear entropy (RLE) for a bipartite system belonging to the Hilbert space \( \mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2 \) is defined...
by
\[ S(t) = 1 - \text{Tr}_k [\rho_k^2(t)], \]
in which \( \rho_{1,2}(t) = \text{Tr}_2,1 \rho(t) \) and \( k = 1, 2 \). The density operator \( \rho \) satisfies the von Neumann equation \( i\hbar \dot{\rho} = [H, \rho] \), being \( H \) the Hamiltonian operator. This is a dynamical measure of the purity of the subsystem \( k \) for unitary bipartite dynamics of initially pure states. For this case, the Schmidt decomposition \[12\] guarantees that RLE is also a measure of the entanglement between the parts \[17\]. Furthermore, the Araki-Lieb inequality \[20\] guarantees that RLE of the subsystems are identical for pure states.

Consider now the following expansion for the RLE \[22\]
\[ S(t) = S_0 + S_1 t + S_2 t^2 + \cdots \]
For a separable pure initial state \( |\psi_0\rangle = |\psi_{10}\rangle \otimes |\psi_{20}\rangle \) under the unitary dynamics governed by a time-independent Hamiltonian, a simple calculation leads us to \( S_0 = S_1 = 0 \), indicating that the first contribution to the short-time entanglement is proportional to \( t^2 \). Defining the dimensionless Hamiltonian
\[ \mathbb{H} = \frac{H t}{\hbar}, \]
we re-write (2) in the following compact form:
\[ S(t) \simeq 2 \left[ C_{00}(\mathbb{H}) + C_{11}(\mathbb{H}) - C_{10}(\mathbb{H}) - C_{01}(\mathbb{H}) \right], \]
in which he have defined the correlations
\[ \begin{align*}
C_{00} &= \left( \langle \psi_0 | \mathbb{H} | \psi_0 \rangle \right)^2, \\
C_{01} &= \langle \psi_0 | \mathbb{H} \rho_1(0) \mathbb{H} | \psi_0 \rangle, \\
C_{10} &= \langle \psi_0 | \rho_2(0) \mathbb{H} | \psi_0 \rangle, \\
C_{11} &= \langle \psi_0 | \mathbb{H}^2 | \psi_0 \rangle.
\end{align*} \]
In this result we have considered only terms up to order \( t^2 \). Interestingly, by \[13\] we may put the conditions for the existence of short-time entanglement in terms of the inequality
\[ 2 \left( C_{00} + C_{11} - C_{10} - C_{01} \right) > 0. \]

Now we look for a semiclassical expansion for the short-time entanglement given by \[24\] in the basis of coherent states. We start by defining a general two-degree of freedom classical Hamiltonian.
\[ \mathcal{H}(q_1, p_1, q_2, p_2) = \sum_{n,m,l,k} c_{nmkl} q_1^n p_1^m q_2^l p_2^k. \]
Then, following the ordered quantization procedure \[9\] we obtain the corresponding Hamiltonian operator
\[ H = S_1 S_2 \sum_{n,m,l,k} c_{nmkl} Q_1^n P_1^m Q_2^l P_2^k, \]
being \( S_k = \exp \left( -\frac{i\hbar}{2} \partial_{Q_k} \partial_{P_k} \right) \) the symmetric ordering operator \[9\]. As may be guessed by \[4\], the semiclassical expansion have the following aspect
\[ S(t) \approx t^2 \left[ \mathcal{O}(\hbar^{-2}) + \mathcal{O}(\hbar^{-1}) + \mathcal{O}(\hbar^0) + \mathcal{O}(\hbar^1) + \cdots \right], \]
in which the first orders in \( \hbar \) are indicated. Notice that the classical limit \( \hbar \to 0 \) seems to yield a divergence of the short-time entanglement, what would be a quite unexpected result. However, an exhaustive calculation shows that \( \mathcal{O}(\hbar^{-2}) = \mathcal{O}(\hbar^{-1}) = 0 \) and \( \mathcal{O}(\hbar^0) \neq 0 \), in general. Then, in a semiclassical regime, in which \( \hbar \) is finite, but arbitrarily smaller than a typical action of the system, we get
\[ S(t) \approx t^2 \mathcal{O}(\hbar^0), \]
stressing the fact that the short-time entanglement does not depend on \( \hbar \) in a semiclassical regime. The constant \( \mathcal{O}(\hbar^0) \) is constituted by a sum of terms involving first-order derivatives (in phase space variables) of the Hamiltonian function \( \mathcal{H} \). The possibility of expressing the first contributions to the entanglement dynamics in terms of classical quantities is a direct consequence of the special basis used in the calculation.

Result \[10\], which is general within the class of bipartite bosonic systems in pure states, attests that the entanglement does not achieve asymptotically its expected classical limit \( S(t) = 0 \), as does well-behaved quantities like expectation values of canonical operators like \( Q \) and \( P \).

Also it has to be remarked that the quadratic dependence in time found for the short-time entanglement is independent of the integrability of the underlying classical system, i.e., it is always algebraic no matter whether the classical system is chaotic or regular. In an approach based on the Loschmidt echo dynamics \[22\] it has been predicted different behavior for the entanglement in chaotic and regular regimes. However, this is not done for the time scale which we are considering, and furthermore, in that case the entanglement generation is obtained by means of two distinct unitary evolutions (echo operator). In this sense, there is no conflict with our results. In fact, the quadratic dependence of the short-time linear entropy has already been found in \[22\]. It is just as a consequence of the initial separability of the quantum state. Recent studies also confirm our results concerning the algebraic dependence \[23\] and the \( \hbar \)-independence \[24\] of the short-time entanglement.

Next we present two numerical examples confirming the above result and discuss the long-time behavior of the entanglement.

**III. NUMERICAL ANALYSIS**

**A. Classical States**

Since we are interested in the quantum-to-classical aspect, we avoid initial states that are initially entangled,
mixed or delocalized. The natural choice is the minimum uncertainty coherent states defined by
\[ |v\rangle = D(v) |r\rangle, \]  
\[ D(v) = \begin{cases} \exp\left[ \frac{v a^+ - v^* a}{\sqrt{2}} \right], \\ \exp\left[ \frac{\arctan|v|}{|v|} (v J_+ - v^* J_-) \right]. \end{cases} \]  

\( D \) was defined respectively for the harmonic oscillator and for the angular momentum (spins). The reference state \( |r\rangle \) stands for the ground states \( |0\rangle \), in the Fock basis, and \( |J, -J\rangle \), in the angular momentum basis.

The coherent state label, \( v \), has the following common parametrization in terms of classical phase space variables:
\[ v = \begin{cases} \frac{q + i p}{\sqrt{2\hbar}}, \\ \frac{q + i p}{\sqrt{\hbar J - (q^2 + p^2)}}. \end{cases} \]  

For bosonic coherent states, the classicality is attained by taking \(|v|^2\) sufficiently macroscopic to guarantee that the average energy \( \langle v | H | v \rangle \) be much greater than a typical spectral distance \( E_{n+1} - E_n \). In this case, taking \(|v|^2 \rightarrow \infty \) is mathematically equivalent to requiring \( \hbar \rightarrow 0 \). On the other hand, for spin coherent states, the classicality emerges by means of a more sophisticated limit \( J \rightarrow \infty, \hbar \rightarrow 0 \) and \( \hbar \sqrt{J} \approx \hbar J = 1 \).

We consider Hamiltonian operators like
\[ H = H_1 \otimes 1_2 + 1_1 \otimes H_2 + H_{12}, \]  
the last term being a nonlinear coupling which is able to entangle initially separable coherent states.

**B. Dicke Model**

First, we investigate the entanglement in one of the most important models in quantum optics (and more recently in quantum computation), namely, the \( N \)-atom Jaynes-Cummings Model (or Dicke model) \[11\], which describes the coupling of a monochromatic field (or a solid state resonator) of frequency \( \omega \) with \( N \) non-interacting two-level atoms (or qubits) with level separation \( \epsilon \). The Hamiltonian is composed by the following terms
\[ H_1 = \epsilon J_z, \quad H_2 = \hbar \omega a^\dagger a, \]  
\[ H_{12} = \frac{G}{\sqrt{2}J} (a J_+ + a^\dagger J_-) + \frac{G'}{\sqrt{2}J} (a^\dagger J_+ + a J_-), \]  
in which the operators \( a \) and \( a^\dagger \) are the usual harmonic oscillator creation and destruction operators, and \( J_+ \), \( J_- \), \( J_z \) are angular momentum ladder operators.

Here, the spin algebra is associated to the atoms, being the total spin given by \( J = \frac{N}{2} \). \( G \) and \( G' \) are real coupling constants, and for convenience we take them generally unequal.

This model possesses a well-known classical counterpart \( \mathcal{H}_{N-JCM}(J) \) \[26\] which presents several interesting properties, e.g., chaotic behavior. Another peculiar feature is the scaling property
\[ \mathcal{H}_{N-JCM}(J) \rightarrow \mathcal{H}_{N-JCM}(J'). \]  

If this relation is satisfied for the pairs \( [E(J), J] \) and \( [E(J'), J'] \), then the associated dynamics will be totally equivalent.

The initial state \( |\psi_0\rangle = |v_a\rangle \otimes |v_f\rangle \) is thus obtained by means of the following process. We use the formula \( r_J = r_1 \sqrt{J} \) in order to rescale the vector of initial conditions \( r_1 = (q_a, p_a, q_f, p_f) \), keeping the dynamics for \( J = 1 \) as a reference. This automatically guarantees that we obtain a dynamics \( [E(J), J] \) equivalent to the unitary \( [E(1), 1] \). Then, we construct the coherent initial state by putting \( r_J \) in \[19\] and \[20\]. Resuming, we chose initial quantum states with different \( J \), but producing always the same classical dynamics. The semiclassical conditions in this case are satisfied by increasing \( J \) and keeping \( \hbar \) constant (as it indeed occurs in nature). It may be shown that this is totally equivalent to the mathematical condition \( \hbar = 1/J \), with \( J \rightarrow \infty \) and fixed initial conditions.

In Fig.\[12\] we show the numerical result for the entanglement as a function of time for several values of \( J \). Actually, for long times, the quantity plotted is a kind of mean entanglement of the states ‘associated’ to a given classical trajectory. We calculated it as follows: given a certain classical trajectory, we choose a set of \( M \) initial states \( |\psi_i(0)\rangle \) centered at points along the trajectory, and calculate the respective RLE’s, \( S_i(t) \). Then, the averaged quantity is given by
\[ S_m(t) \equiv \frac{1}{M} \sum_{i=1}^{M} S_i(t). \]  

Such a calculation allows us to observe a smooth mean behavior for entanglement without the characteristic oscillations associated to the border effect \[27\]. The mean long-time entanglement is then suitably described by a fitting expression \[28\], namely,
\[ S_m(t) = A_0 \left( 1 - e^{-A_1 t} \right), \]  
where \( A_0 \) and \( A_1 \) are fitting parameters.

At least two aspects are remarkable in this numerical result: (i) the short-time behavior, proportional to \( t^2 \), is \( J \)-independent and (ii) the plateau value increases with \( J \).

The aspect (i) has already been predicted in section \[11\] but for a different class of Hamiltonians. The results shown in Fig.\[12\] are, therefore, a numerical verification of
in a more general context, in which the Hamiltonian operator $H$ is not obtained from a classical function by means of a quantization based on bosonic coherent (Gaussian) states. Such a $\hbar$-independence induces us to believe that the rising in the entanglement is determined just by classical sources, as the local departure of neighboring phase space points. This is indeed suggested by previous studies on quantum chaos \cite{29}. But the main point is that entanglement does not diminish as more macroscopic regimes are attained.

The second aspect is even more surprising, since it means that the entanglement increases as we tend to a more semiclassical limit. We will retake this analysis after our second example.

C. Coupled Nonlinear Oscillators

Our next example is constituted by a pair of resonant oscillators coupled nonlinearly. The terms are

\begin{equation}
H_k = \hbar \omega \left( a_k^\dagger a_k + \frac{1}{2} \right) \quad (k = 1, 2),
\end{equation}

\begin{equation}
H_{12} = \hbar \lambda \left( a_1^\dagger \hat{a}_2 + \hat{a}_1^\dagger a_2^\dagger \right) + \hbar^2 g \left( \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + 1 \right)^2.
\end{equation}

This Hamiltonian has been applied to describe an ideal two-mode BEC in the equal scattering length situation \cite{30} with internal Josephson coupling \cite{31}. Recently, the entanglement dynamics between the modes for initially disentangled coherent states, was exactly calculated \cite{32}. We reproduce here the analytical result for the RLE:

\begin{equation}
S(t) = 1 - e^{-2|\beta_1(t)|^2} \sum_{n,m} \frac{|\beta_n(t)|^2 n!}{m!} e^{-4|\beta_2(t)|^2} \sin^2 \left( \frac{\hbar g t(n-m)}{2} \right),
\end{equation}

in which $\beta_{1,2}(t) = (\alpha_{1,2} e^{-i\omega t} \cos \lambda t - i \alpha_{2,1} e^{-i\omega t} \sin \lambda t)$ and $\alpha_i = (q_i + i p_i)/\sqrt{2\hbar}$. Result \cite{19} predicts that the system recovers totally its purity (disentangled states) at

\begin{equation}
t = \frac{\pi}{g \hbar},
\end{equation}

after passing by an intricate interference process (see \cite{32} for details).

An useful tool in our discussion is the classical reduced linear entropy (CRLE), defined by \cite{17}

\begin{equation}
S_{\text{cl}}(t) = 1 - \frac{\text{tr}_k[P_k(t)]}{\text{tr}_k[P_k(0)]},
\end{equation}

in which the partial classical trace is given by $\text{tr}_k = \int dq_k dp_k$, for $k = 1, 2$. $P_{1,2}$ denotes the reduced probability function obtained by $\text{tr}_{2,1}[P(t)]$, in which the joint probability $P(t)$ is a function satisfying the Liouville equation

\begin{equation}
\frac{\partial P}{\partial t} = \{\mathcal{H}, P\}.
\end{equation}

$\mathcal{H}$ is the classical Hamiltonian function. The normalization in \cite{21} is necessary to guarantee both an adequate dimensional unity and $S_{\text{cl}}(0) = 0$ for separable initial distributions $P(0) = P_1(0) P_2(0)$. The connection between quantum and classical worlds is established by taking

\begin{equation}
P(0) = \frac{\langle v_1, v_2 | \rho_0 | v_2, v_1 \rangle}{2\hbar},
\end{equation}

where $\rho_0$ is the pure density operator at $t = 0$. The normalization is chosen in such a way that $\text{tr} P = 1$. The initial distribution \cite{28} is the unique source of $\hbar$ in the CRLE dynamics. Thus, the semiclassical regime in the classical Liouvillian formalism is associated to strongly localized initial distributions.
The numerical results for both the quantum and classical reduced linear entropies are shown in Fig. 2. The dependence of CRLE in time reflects the fact that classical probabilities also become correlated, i.e., initially independent probability distributions are transformed into conditional ones as the dynamical evolution takes place. In some cases, similarities between these quantum and classical quantities indicate that entanglement has indeed a strong statistical character [15]. Here, however, remarkable differences emphasize the role played by an intrinsic quantum effect, as is the interference. In fact, it has been shown that interference is the major responsible for such flagrant differences, since there is no analog effect in the classical formalism [32].

Concerning the semiclassical limit, we see that quantum recurrences, and consequently, quantum interferences, are postponed by the dynamics as is made small as compared with the classical action $|\beta_1|^2 + |\beta_2|^2$. This is indeed predicted analytically by (20). Accordingly, the resemblance between quantum and classical entropies tends to increase in semiclassical regimes. In this sense, these results allow us to consider the CRLE as an indicator of the entanglement due solely to spreading effects, which may be observed, e.g., in the classical phase space representation of the density operator.

The entanglement dynamics of the system (18) presents the same qualitative general features of our precedent example, namely: short-time behavior independent of $\hbar$ (see [11] for the analytical demonstration) and higher plateaus for more classical regimes. Actually, this scenario has been shown to be quite general for a wide class of nonlinear systems [36].

These results induce us to separate the entanglement dynamics in two main time scales. The first one, namely the short-time scale, may be regarded as a classical scale in which entanglement is determined essentially by the vicinity of the initial condition in classical phase space. Accordingly, it has been shown that this scale is connected with the Ehrenfest time for the system (18) [32]. The second scale, associated to long times (equilibrium), contains exclusively quantum effects, like interferences, and points out to the fact that the entropy is extensive with the number of pure states accessible to the dynamics. These remarkably different time scales and the extensivity of the entropy have already been mentioned in slightly different contexts in [32] and [34] respectively. Also, other time scales have been established in [10].

All results shown here point out to the same important conclusion: in closed pure bipartite systems, entanglement exists even in arbitrarily semiclassical regime. This contradicts the common belief about the existence of appreciable entanglement only in exclusively quantum regime. On the other hand, in the strictly classical limit ($\hbar = 0$), entanglement is indeed expected to disappear completely, since classical points have no statistics associated. But this is a peculiar mathematical limit which will never be accessible in physical world. In this sense, our results also indicate that the classical limit of entanglement is rather singular.

IV. CONCLUDING REMARKS

We have shown by two numerical examples and by analytical calculations that entanglement does not decrease as the semiclassical regime is attained asymptotically. In fact, the results pointing out to an increasing amount of entanglement as the equilibrium is attained. This unexpected behavior may be understood by noticing that semiclassical regime avoids quantum interferences, but it is not able to eliminate the spreading of the wave packet. Even extremely localized coherent states perceive the local spreading caused by the classical flux of trajectories associated. Furthermore, since we note that the entropy is extensive with the number of pure quantum states of the density operator, the increase in entanglement in the semiclassical limit becomes just a natural fact.
Recently, it was shown that entanglement can always arise in the interaction of an arbitrarily large system in any mixed state with a single qubit in a pure state [35]. This result stresses the role played by the subsystem purity as an enforcer of entanglement, even in a thermodynamic limit of high temperatures. Here, on the other hand, we are concerned with a different kind of classicality, namely, that one attained by $\hbar \to 0$. Furthermore, we study only initially pure states for the subsystems. The common point in these investigations is that the regarded systems do not interact with the environment. In this sense, unitarity seems to be a key word in such apparent contradicting semiclassical limits. Therefore, a natural question follows: Is decoherence able to provide the “expected” semiclassical limit for the entanglement?

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