THE PARTICLE PROBLEM IN CLASSICAL GRAVITY
A HISTORICAL NOTE ON 1941

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ABSTRACT. This historical note is mainly based on a relatively unknown paper published by Albert Einstein in Revista de la Universidad Nacional de Tucumán in 1941. Taking the ideas of this work as a leitmotiv, we review the discussions about the particle problem in the theory of gravitation within the historical context by means of the study of seminal works on the subject. The revision shows how the digressions regarding the structure of matter and the concise problem of finding regular solutions of the pure field equations turned out to be intrinsically unified in the beginning of the programme towards a final theory of fields.

The mentioned paper (Einstein 1941a) represents the basis of the one written by the same author in collaboration with Wolfgang Pauli in 1943, in which, following analogous lines, the proof of the non-existence of regular particle-type solutions was generalized to the case of cylindrical geometries in Kaluza-Klein theory (Einstein & Pauli, 1943). Besides, other generalizations were subsequently presented.

The (non)-existence of such solutions in classical unified field theory was undoubtedly an important criterion leading Einstein’s investigations. This aspect was gathered with expertness by Jeroen van Dongen in a recent work, though restricting the scope to the particular case of the investigations on Kaluza-Klein theory (van Dongen, 2002). Here, we discuss the particle problem within a more general context, presenting in this way a complement to previous reviews.

1. INTRODUCTION: PUBLISHING AFFAIRS

In 1941, Guido Fubini interceded in asking Albert Einstein for submitting a contribution to the Argentinian journal Revista de la Universidad Nacional de Tucumán, which in the early 40’s was being founded by the Italian mathematician Alessandro Terracini (Terracini 1941; 1944). Einstein kindly agreed in submitting an article in which a novel demonstration of the non-existence of regular static spherically symmetric solution in the general relativity was given.

Einstein’s proof, which was described in the Italian translation of (Einstein, 1941b) as “una graziosa dimostrazione”, represents an ingenious way to show that no such a solution is admitted in the theory of gravitation $R_{\mu\nu} = 0$. The original title of the Einstein’s handwritten German draft reads Beweis der Nichteexistenz von Singularitaetsfreien Gravitationsfeldern mit nicht Verschwindender Gesamtmasse¹ and it is in possession of the Biblioteca Mathematica of the University of Turin.

The Einstein’s proof acquires particular relevance, not due to its particular application to the general theory of relativity itself, but because of its property of being suitable of generalizations to other theories of gravitation. Works of this sort were the one written by Einstein and Pauli in 1943, in which the case of Kaluza-Klein theory was studied (Einstein & Pauli, 1943), and Papapetrou’s article of 1948, treating the case of the non-symmetric unified field theory (Papapetrou, 1948b) (see also (Lichnerowicz, 1955)).

¹Literal translation of the titles of (Einstein, 1941a; 1941d).
The work (Einstein, 1941a), which is certainly hardly known, is cited\(^2\) in (Fölsing, 1993) and in other few articles devoted to the study of the historical context (see for instance (van Dongen, 2002; Earman & Eisenstaedt, 1999)).

It was A. Taub who reviewed Einstein’s paper for the *Mathematical Review* (Taub, 1942) and the translation of the article (from the original in German) was supervised by Terracini who, in a letter addressed to Einstein (Terracini, 1941), wrote “Ihrem Wunsch nach, anstatt des deutschen Textes Ihres Aufsatzes, werden wir nebst der spanischen eine englische Übersetzung veröffentlichen\(^3\).” The Spanish translation (Einstein, 1941d) and the English version (Einstein, 1941a) were published, as we refer in the bibliography, in the series A (devoted to mathematics and physics) of the *Revista*. In the subsequent editions, the journal continued publishing articles related to (unified) field theories (see for instance (Santaló, 1954; 1959)).

The original aim of our study was the question about what were the motivations that Einstein might have had to publish his *Beweis* in the Argentinian journal in 1941. Thus, after this brief introduction, we describe in the following section the historical context in which Einstein’s paper appeared. This concerns to the search for non-singular particle-like solutions in gravitational field theory. In the third section we comment on Einstein’s paper specifically. We conclude in sections 4 and 5 with remarks on the subsequent investigations on the topic.

### 2. The historical context

Having undertaken the task of studying the development of Einstein’s ideas on the theory of gravitation in the second third of the 20th century, one inevitably arrives to the conclusion that the particle problem\(^4\) in general relativity is, indeed, inseparably related to other two problems in the theory of the continuum, namely the problem of motion and that of the unification of the fields.

The pioneer’s conviction about the idea that these aspects should be entailed, is particularly illustrated by an assertion belonging to the introduction of a quoted work (Einstein & Rosen, 1935) written by A. Einstein and N. Rosen in 1935 in which, among the arguments sketching ideas which later would become a solid statement, it is claimed that “[o]n the basis of the description of a particle without singularity one has the possibility of a logically more satisfactory treatment of the combined problem: the problem of the field and that of the motion coincide”. In fact, the quest for regular solutions in classical unified field theory played a crucial role as a criterion leading Einstein’s investigations.

The usually called “particle problem”, within the context of Einstein’s investigations between 1919 and 1955, comprised two basic degrees of speculation: first, the representation of particles as regular solutions of a pure field theory (i.e. free of sources); secondly, the pious attempt to derive the atomistic nature\(^5\) of matter as a property encoded in the non-linearity of the theory of field. Let us review some aspects of these.

\(^2\)though with a mistake in the year of publication referred, since 1942 is referred instead 1941.

\(^3\)manifesting that both English and Spanish translations were going to be prepared.

\(^4\)referring to the question about how to describe a particle in the framework of the theory of gravitation in terms of the pure field theory representation (i.e. free of sources).

\(^5\)even in what is referred to the quantum behavior of matter: this aim was based on the idea that the non-linearity of the gravitational (or unified) field theory would lead to mimic the conditions yielding to the quantum behavior of atoms. Clearly, the second stadium of the programme results more ambitious and seems to be more unlikely than the first one.
The period 1919-1935. Concerning earlier periods, and by attempting to answer what was the influence of Einstein’s conviction about the fact that particles should be represented as nonsingular solutions of the gravitational (unified) field, we could refer to preliminary stages of the unification programme where, instead, “Einstein had considered the possibility of interpreting the elementary electric and light quanta as certain singular points of generalized field equations”\(^6\) (Vizgin, 1994). On the other hand, if one analyzes the “principles” which are presents as starting points in the subsequent literature (cf. (Einstein & Rosen, 1935)), it turns out to be evident that such a picture of the reality resulted to be abandoned (substantially overcome) in the ulterior periods of Einstein’s digressions on the structure of matter.

It was pointed out by V. Vizgin in (Vizgin, 1994) (refering specifically to the research activities by 1923) that “Einstein did not associate the problem of explaining the quantum behavior of [...] the electron with the maximum problem in the first stage of its solution, which consisted merely of proving that the field equations had nonsingular static centrally symmetric solution that could be interpreted as the electron.”\(^6\)

Moreover, independently of how strongly were Einstein’s researches focused on this aspect by the beginning of the decade of 1920, the question about the existence of regular solutions of field equations representing particle-like objects was not avoided at all by the community during that epoch. For instance, we can observe this from the fact that “[i]t had become clear by the end of 1923 that none of the unified theories proposed since 1918 had such nonsingular solutions” (Vizgin, 1994). Certainly, after 1921, Einstein did dedicate attention to the problem of particle-like solutions in unified field theories. In particular, in a paper published in 1923 in collaboration with J. Grommer, he proved the non-existence of regular static spherically symmetric solutions in Kaluza’s theory (Einstein & Grommer, 1923). According to the author of (Vizgin, 1994), “they tested the [Kaluza’s theory] for the existence of centrally symmetric everywhere-regular solutions and concluded that they did not exist, this being, in Einstein’s eyes, one indication of a merely formal nature of the unification”. Despite his emphatic remarks on the failure of the theory on describing regular particle-like solutions, it results clear that Einstein’s enthusiasm about Kaluza-Klein theory did not disappear immediately and, moreover, survived through the 30’s to conclude in 1943 with his paper (Einstein & Pauli, 1943). On the other hand, Pauli and Einstein, in early periods of work, had discussed the validity of Weyl’s theory based on the (non)-existence of static solutions in it\(^7\). From this, we can infer how the existence of solutions suitable to be considered as representing particles turned out to be a criterion for suggesting the abandonment of a temptative road, even when promissory in the beginning.

In these terms, it is clear that the particle problem played a central role after 1919, and this continued through the 20’s, though with oscillating protagonism. Then, the question arises as to what was Einstein’s point of view after 1930, if a concise one can be identified, regarding the description of particles as singular solutions of field equations? And even more specifically: what was the importance that such aspect had in guiding the search for a unified field theory in that period? These questions find concise answers when one considers the following assertion\(^8\):

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\(^6\)Vizgin refers here to the epoch of the first attempts of finding a “nongeometrized” unified field theory.

\(^7\)see, for instance, the letters addressed to Michele Besso, dated on December 12, 1919 (Einstein & Besso, 1994); see also (Pauli, 1994).

\(^8\)Let us remark the use of the word “principle” in the following paragraph.
“Every field theory, in our opinion, must therefore adhere to the fundamental principle that singularities of the field are to be excluded”, as it can be read⁹ in (Einstein & Rosen, 1935).

The period 1935-1945. Furthermore, even in those papers written in collaboration with L. Infeld and B. Hoffmann in 1938 and 1939 devoted to the study of the problem of motion in general relativity (we always refer to the field equations free of sources, i.e. $T_{\mu\nu} = 0$), where “matter is represented as point singularities of the field”, Einstein had stated that “this may be due to our simplifying assumption that matter is represented by singularities, and it is possible that it would not be the case if we could represent matter in terms of a [unified] field theory from which singularities were excluded” (Einstein, Infeld & Hoffmann, 1938; Einstein & Infeld, 1939). A point that deserves to be mentioned in this context is that the treatment of the problem of motion proposed by Einstein, Infeld and Hoffmann is often referred as an example of “dualistic theory” (see for instance (Sen, 1968)) because of its property of considering particles and fields as separated entities. Rigorously speaking, it is indeed the case, but the conviction of the authors¹⁰ about the fact that this would correspond just to a “provisional” description is frequently omitted in the revisions.

Moreover, related to that conviction, even more emphasis is noted in the fact of admitting that the objection about the appearance of singular solutions in general relativity would be valid only if we consider this theory as a final theory of the whole field. Instead¹¹, the field corresponding to a particle should be considered not as a pure gravitational field, but as a solution of a unified field theory. In these terms, a necessary requirement to eventually consider a field theory as a satisfactory one should be its property of representing the particles as solutions free of singularities in the whole space. This is, in fact, the principal link existing between the problem of the description of the matter and the one of finding a unified field theory; this is the case, at least, in what respect to the Princeton years.

Certainly, Einstein preferred to admit explicitly the temporary inability to understand the structure of matter rather than to consider the image of particles as singularities of the gravitational field, which, according to his point of view, represented just an apparent solution to the problem of describing the matter (Einstein, 1953a).

In 1941 Einstein published (Einstein, 1941a) proving the non-existence of static spherically symmetric solutions in general relativity. With Pauli, in 1943, he published a paper in which the proof was extended to the case of Kaluza-Klein five-dimensional cylindrical extension of four-dimensional spherically symmetric geometries. Previously, through the 30’s, Einstein had been searching for such solutions in Kaluza-Klein theory with the intention to find a geometry suitable to be interpreted as the unified field of a particle-like object. These investigations were mainly done in collaboration with Bergmann and Bargmann. Besides, the quest for particles continued beyond the moment when his enthusiasm about Kaluza’s theory vanished; in fact, by 1949, when his investigations on unified field theories were focused on the non-symmetric theories, he persisted in the search for the coveted non-singular solutions.

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⁹Einstein commented this work (entitled “the particle problem in the general theory of relativity”) in a letter addressed to Michel Besso, dated on February 16 of 1936, (Einstein & Besso, 1994). In this letter, Einstein mentioned his hope of having achieved a first step towards a satisfactory theory of matter.

¹⁰See also the letter to Michele Besso, dated on August 16 of 1949, (Einstein & Besso, 1994).

¹¹according to Einstein’s point of view reflected in his autobiographical notes (Einstein, 1949a).
As a corollary. Regarding the question about whether the appearance of singularities should be admitted, we can state: it was Einstein’s opinion “that singularities must be excluded in a final theory of fields. It [did] not seem reasonable [to the author of general relativity] to introduce into a continuum theory points (or lines, etc\textsuperscript{12}) for which the field equations do not hold. Moreover, the introduction of singularities is equivalent to postulating boundary conditions (which are arbitrary from the point of view of the field equations\textsuperscript{13}) on surfaces which closely surround the singularities” (Einstein, 1953a).

To emphasize the importance that the description of particles had within the framework of the classical field theory, we cite (Tonnelat, 1955), where the author mentions, among the motivations for generalizing the general relativity, the fact that this theory “separates quite radically the gravitational field from [...] the sources of field which conserves a phenomenological interpretation even when one talks about uncharged particles”. It is a remarkable fact that such objection is mentioned in equal footing as the objection about the separation of gravity from the electromagnetic field (Tonnelat, 1955).

Moreover, the Einstein’s study on Kaluza-Klein theory during 1937-1943 represents an example that shows the degree of compromise that the search for singular solutions had within the framework of the search for a satisfactory field theory. This is reviewed in detail in reference (van Dongen, 2002) and we do not treat this extensively here in order to avoid redundances. Instead, we simply cite an eloquent paragraph belonging to reference (Einstein & Bergmann, 1938) in which Einstein and Bergmann presented their “generalization of Kaluza’s theory”; namely: “Many fruitless efforts to find a field representation of matter free from singularities based on this theory have convinced us, however, that such a solution does not exist”. This manifestly shows the relevance of the regular particle-like solutions within the research they did. Furthermore, if one does not understand the search of such solutions as a crucial criterion in the line of investigation, then the paragraph cited above (located at the end of section 2 of (Einstein & Bergmann, 1938)) seems to be completely disconnected from the line of arguments of the work to which it does belong. Moreover, notice that the explicit reference to “lines [...] for which the field equations do not hold” in (Einstein, 1953a) is consequent with the consideration of cylindrical five-dimensional extension of four-dimensional particle-like objects that Einstein, Bergmann and Bargmann have studied in the context of Kaluza-Klein theory (cf. (Einstein & Pauli, 1943)).

On the quantum nature of matter. Before concluding this section, we have to mention that, besides the search of regular solutions, a crucial role in the unification programme was also played by the attempts for presenting a description of the quantum behavior of nature by means of the classical field theory. This project, acquiring several forms during different periods of investigation, basically relayed on the Einstein’s conviction about that the non-linearity of the field equations corresponding to the final theory should lead to additional conditions on the exact solutions. Then, by means of the “overdetermination of the system of equations”, these conditions would result to be such that mathematical restrictions on the “manifold of solutions” would eventually coincide with the quantization rules.

\textsuperscript{12}clearly referring to his investigations on the existence of regular solutions with axial symmetry in five-dimensional Kaluza-Klein theory.

\textsuperscript{13}the emphasis on the arbitrariness is recurrent. He refers to a private communication with L. Silberstein where this aspect was explicitly pointed out and a particular example presented (Einstein & Rosen, 1935).
To understand Einstein’s position about the feasibility of conceiving the atomic (and quantum) structure of matter in terms of a classical field theory, we cite: “I believe that at the present time nobody knows anything reliable about. This is because we cannot judge in what manner and how strongly the exclusion of singularities reduces the manifold of solutions. We do not possess any method at all to derive systematically solutions that are free of singularities. Approximation methods [referred to the approximations employed in (Einstein, Infeld & Hoffmann, 1938)] are of no avail since one never knows whether or not there exists to a particular approximate solution an exact solution free of singularities” (Einstein, 1953a). Pauli emphasized the fact that Einstein had kept the faith in such a programme until his death in 1955, (Pauli, 1994). Einstein’s position about this aspect by 1950 is particularly reflected in (Einstein, 1953b) and, though in a more indirect way, in (Einstein & Strauss, 1945; 1946a) as well. It is not the aim of this brief note to discuss the details of this particular point; the overdetermination in the system of field equations as a hope of describing the quantum behavior of nature and, in a more general context, its relation with the whole programme of unification is extensively developed with erudition in (Vizgin, 1994).

Then, let us review the Einstein’s paper in 1941.

3. Einstein’s paper in 1941

Einstein’s article, which we briefly describe below, begins with the proof of a theorem which can be called Einstein’s “theorem of infinitely close solutions”. This follows from the simple observation that if an arbitrary everywhere regular metric $g_{\mu\nu}$ and its infinitesimal deformation $g_{\mu\nu} + \delta g_{\mu\nu}$ are considered, then the associate deformation in the Ricci tensor (denoted by $\delta R_{\mu\nu}$) satisfies

$$\delta R_{\mu\nu} = -\nabla_{\rho}\delta \Gamma^\rho_{\mu\nu} + \nabla_{\nu}\delta \Gamma^\rho_{\mu\nu} + (\Gamma^\sigma_{\sigma\nu} - \Gamma^\sigma_{\nu\sigma})\delta \Gamma^\sigma_{\mu\rho}$$

Then, by assuming the symmetry of the affine connection, $\Gamma^\sigma_{\sigma\nu} = \Gamma^\sigma_{\nu\sigma}$, and defining

$$U^\rho_{\mu\nu} = -\delta \Gamma^\rho_{\mu\nu} + \frac{1}{2}(\delta \Gamma^\sigma_{\mu\nu} \delta^\rho_{\nu\sigma} + \delta \Gamma^\sigma_{\nu\sigma} \delta^\rho_{\mu\nu})$$

one finds, with Einstein, that the following identity holds

$$\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} = \partial_{\rho}(\sqrt{-g}g^{\mu\nu}U^\rho_{\mu\nu})$$

This implies that, for the cases where both the original and the varied metric satisfy the Einstein equation, we get $\delta R_{\mu\nu} = 0$, i.e.

$$\partial_{\rho}(\sqrt{-g}g^{\mu\nu}( -2\delta \Gamma^\rho_{\mu\nu} + \delta \Gamma^\sigma_{\mu\sigma} \delta^\rho_{\nu\sigma} + \delta \Gamma^\sigma_{\nu\sigma} \delta^\rho_{\mu\nu})) = 0$$

Then, we can expand around Minkowski spacetime as follows

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

and consider that the Newtonian conditions are to be imposed at the infinity, which are realized by means of

$$h_{\mu\nu} = -\frac{2m}{r}\delta^\mu_{\nu}$$

Hence, we find that

$$\sqrt{-g}g^{\mu\nu}( -2\delta \Gamma^\rho_{\mu\nu} + \delta \Gamma^\sigma_{\mu\sigma} \delta^\rho_{\nu\sigma} + \delta \Gamma^\sigma_{\nu\sigma} \delta^\rho_{\mu\nu})) = -4m\delta^\rho (r^{-1})$$
These equations “hold for the infinitely close asymptotic equations. But equations (4) hold rigorously in the whole domain”, where we do not consider singularities. By integrating out the above equations over a domain defined by an hyper-surface \( V \times \mathbb{R} \), where the factor \( \mathbb{R} \) corresponds to the time direction and \( V \) is a three-volume bounded by a sphere \( S^2 = \partial V \), we find that the Gauss theorem implies

\[
\partial_\rho \int_{V \times \mathbb{R}} \sqrt{-g} g^{\mu\nu} U_\rho \, d^4 x = 2\delta m \int_{V \times \mathbb{R}} \nabla^2 r^{-1} \, d^4 x = 0
\]

since the surface integral must vanish according to (4). Here, we can consider an sphere \( S^2 \) large enough in such a way that the Newtonian limit (6) results a good approximation. Hence, we get

\[
\int_{V \times \mathbb{R}} \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \, d^4 x = 8\pi\delta m \int_{\mathbb{R}} dt = 0
\]

and then

\[
\delta m = 0
\]

This implies that, under the assumption that the solutions of the field equations are regular everywhere, one is allowed to conclude that “two infinitely close solutions without singularities have necessarily the same total mass \( m \)”.

Besides, such a result (10) confronts with the simple observation of the fact that if the metric \( g_{\mu\nu}(x^\rho) \) satisfies the field equations \( R_{\mu\nu} = 0 \), then the same system is satisfied by the configuration \( g_{\mu\nu}(\lambda x^\rho) \) for any \( \lambda \in \mathbb{R} \). Then, by identifying \( \lambda^{-1} = 1 + \delta m/m \) in the spherically symmetric solution we get a contradiction; contradiction which, of course, “disappears if we abandon the hypothesis on the non-existence of singularities.”

After Einstein’s paper. In this way, Einstein demonstrated the non-existence of gravitational fields with a non-vanishing total mass free from singularities in the general theory of relativity (Einstein, 1941a). This proof was generalized by Einstein and Pauli (Einstein & Pauli, 1943) and by Papapetrou (Papapetrou, 1948b) to the cases of other (unified) field theories.

Let us make a remark here: as it was signaled in (van Dongen, 2002), Einstein’s assertion about the scaling map \( x^\rho \to \lambda x^\rho \) as an application closed among a space of non-singular solutions is not general enough. Actually, this is the key point for explaining the existence of Kaluza-Klein non-singular particles despite the Einstein-Pauli generalization of Einstein’s proof, which again follows from this class of scaling arguments.

On the other hand, few years after the publication of (Einstein, 1941a), Papapetrou (Papapetrou, 1948b) presented an alternative proof of Einstein’s theorem which, while simpler, can be generalized without major difficulty to the case of the non-symmetric version of the gravitational theory.

Papapetrou’s proof follows from Tolman’s energy theorem (see for instance (Tolman, 1987)) which states that the energy concentrated in a three-volume \( V \) of a given static space-time is given by the formula

\[
\mathcal{U} = \int_V d^3 x \sqrt{-g}(T_1^1 + T_2^2 + T_3^3 - T_0^0)
\]
where $T^\mu_\nu$ is the stress-tensor. Then, this allows to reobtain Einstein’s conclusion in a rather simple way by noticing that the vanishing of $T^\mu_\nu$ in the interior of $V$ implies that the mass $m$ of the asymptotically flat spherically symmetric solution also vanishes.

This observation leads to extend the Einstein’s no-go theorem to the case of Einstein-Straus theory, in which the tensor $T^\mu_\nu$ turns out to be replaced by an analogue $^*T^\mu_\nu$, naturally defined by

\begin{equation}
^*T^\mu_\nu = \frac{1}{8\pi} \left( ^*R^\mu_\nu - \frac{1}{2}g^\mu_\nu g^\rho_\gamma ^*R^\rho_\gamma \right)
\end{equation}

where $^*R^\mu_\nu$ is the well known Einstein-Straus non-symmetric generalization of the Ricci tensor (Schrödinger, 1950). Then, by imposing the condition of vanishing divergence for the anti-symmetric tensor $\sqrt{-g} (g^\mu_\nu - g^\nu_\mu)$, he is able to show that, in the weak-field approximation, the symmetric part of the metric tensor $g^\mu_\nu$ satisfies the same equations that in the case of general relativity. Then, it is claimed that, even in the non-symmetric case, Einstein’s proof holds straightforwardly.

Through his deduction, Papapetrou discussed the ambiguity concerning the definition of the “correct conservation laws”, arguing that in the static case such arbitrariness results diluted. He makes use of certain results due to Schrödinger, with whom he has had conversations on these aspects.

4. Survey of subsequent investigations

Having discussed the importance that the particle problem have had within the framework of the general relativity and the efforts towards its generalization, we find interesting to review here the further investigations on the existence of regular spherically symmetric static solutions in classical field theories developed in the first half of the 20th century (since 1935). It is worth to mention that we focus our attention on the historical context and, consequently, the aim of this section is to review the first attempts to construct a classical field theory for which regular solutions were admitted.

The demonstration described in the previous section relies on basic assumptions (the non-existence of singularities and scaling properties) and on general aspects of classical field theory. Thus, it turns this result into a strong argument which can be, for instance, generalized to other theories. Hence, the question arises as to how could one avoid this generality by finding a theory for which such a proof does not turn out to be a conclusive one.

For simplicity, as it is evident from Papapetrou’s proof, the first attempt would be to consider a non-unified field theory such that the variational principle in Einstein’s argument would involve an additional term. Of course, as it is well known, coupling gravity with other field (source) does not mean that singularities are necessarily excluded; however, one could wonder whether an appropriate electromagnetic part would lead to regular geometries representing particles. Let us begin with a case of particular importance.

A non-unified field theory: the Hoffmann-Infeld particle and the Born-Infeld electrodynamics. Since the formulation of Born-Infeld electrodynamics in 1934 (Born & Infeld, 1933; 1934), the prospects of finding non-singular solutions representing particles in the field theory free of sources have been renewed. The celebrated property of Born-Infeld theory of leading to static spherically symmetric electric field well defined in the whole space motivated, then, the question
as to whether such a quality would be inherited when gravitating sources were considered\(^{14}\). It was B. Hoffmann who gave the answer by presenting in 1935 a spherically symmetric solution of Einstein theory\(^{15}\) coupled to Born-Infeld particle-like charged source (Hoffmann, 1935). In this case, as well as in the case of the search for solutions of the pure gravitational field theory, the conviction that “gravitational potentials that contain infinities are to be rejected” turned out to be one of the principal motivations.

Indeed, Hoffmann showed, “[b]y postulating that infinite relativistic gravitational potential are to be rejected in the Born-Infeld theory”, that “difficulties in the usual relativistic treatment of gravitational mass are avoided”. His solution, in terms of the Schwarzschild-like ansatz

\[ ds^2 = -e^{\psi(r)}dt^2 + e^{-\phi(r)}dr^2 + r^2d\Omega^2 \]

(13)

takes the following form\(^ {16}\)

\[ e^{\psi(r)} = e^{\phi(r)} = 1 - \frac{2m}{r} - \frac{8\pi}{r} \int_0^r dr \; H_e(r) \; , \; H_e(r) = (r^4 + e^2)^{1/2} - r^2 \]

(14)

where \(m\) and \(e\) are integration constants. This solution, as it can be verified, allows to find the Reissner-Nordström solution in certain limit (i.e. \(e \to 0\)).

Besides, the metric (13)-(14) is regular at \(r = 0\) in the case \(m = 0\). Then, the Newtonian contribution to the gravitational potential is simply given by the term of the order \(\sim r^{-1}\) in the large radius expansion of the integral in (14). Hoffmann pointed out that the infinity at the origin is then avoided by setting \(m = 0\), “not because such an infinity does not exist in the most general mathematical solution, but simply because one decides that such an infinity is objectionable on physical grounds”. This leads to an identification between “gravitating” and “electromagnetic” charges and no uncharged particle would exist. Hoffman particle (often called “BIon” in the modern literature) represents a non-localized source though presenting a Newtonian-like asymptotic behavior for large distance \(r\). By the end of his paper, Hoffmann also manifests the conviction about the fact that the vacuum field equations (i.e. free of sources) are to be preferred as a final description.

Einstein and Rosen\(^{17}\) “kindly” pointed out that Hoffmann solution, even when representing a finite gravitational field at the origin, is still singular because a conical singularity does exist there. Subsequently, a modification of Born-Infeld theory was presented by Hoffmann and Infeld in reference (Hoffmann & Infeld, 1937). Being conscious about the difficulties that the consideration of a conical singularity would imply, they were able to find a particle-like solution of the new version of Born-Infeld electrodynamics which presents a similar form, though considering \(H_e(r) = r^2 \log(1 + e^2 r^{-4})\) instead the one in (14). In fact, unlike (14), the Hoffmann-infeld solution satisfies that \(\lim_{r \to 0} e^{\psi(r)} = 1\) and, then, no conical singularity does exist. However,

\(^{14}\)Notice that an attempts of finding a regular charged gravitating solution in the context of Maxwell theory has to be excluded due to a theorem proven by A. Lichnerowicz, showing that no regular solutions of this kind do exist for Maxwell-Einstein theory. Lichnerowicz’s proof was latter generalized to the case of Kaluza-Klein theory.

\(^{15}\)We refer in this case to the general theory of relativity.

\(^{16}\)Here, the fundamental constants \(c, G\) and \(b\) are all taken as unity as in the original paper.

\(^{17}\)Hoffmann and Infeld mention Einstein-Rosen observation about the conical singularity of Hoffmann solution in (Hoffmann & Infeld, 1937), see also (Rosen, 1939). As far as we know, such objection was due to a private communication and it does not correspond to a written contribution. For our purpose here, we can consider this comment on conical singularity as a *Fuld Hall conversation.*
notice that in both Hoffmann and Hoffmann-Infeld particles the curvature is still divergent at \( r = 0 \).

These results teach us that the non-linearity of Born-Infeld electrodynamics, even though it does not represent a sufficient condition for a field theory to admit regular solutions, turns out to play a crucial role in their existence. Then, we feel tempted to ask whether an enhancement of the non-linearity of field equations could lead to overcome difficulties towards a novel solution to the particle problem. For instance, we can ask: could such an enhancement of the non-linearity allow one to heal the pathology of the conical singularity of Hoffmann’s solution and also mitigate the divergences of the curvature of the former geometries? We will return to this point shortly when discussing the higher order curvature corrections to general relativity and other models of non-linear electrodynamics. But, first, we devote a few words on another quoted proposal for unified field theories. The reason for this is that, with Einstein, the consideration of non-unified field theories (as Born-Infeld model coupled to gravity) does not seem to represent a fully satisfactory formulation.

**Papapetrou’s particle and the non-symmetric unified field theory.** The advent of the unified field theories based on the non-symmetric version of Riemannian geometry gave rise to a different proposal about from where the solution to the particle problem could come. One of the most celebrated non-symmetric unified field theories is the one proposed by Einstein in 1945 (Einstein, 1945), which was presented in a more precise way in the article (Einstein & Straus, 1946b), written in collaboration with E. Straus (see also (Einstein & Kaufman, 1953; 1954; 1955)). This theory, often called Einstein-Straus theory, is described with expertness by E. Schrödinger in his book (Schrödinger, 1950). Within this context, A. Papapetrou (Papapetrou, 1948a) and M. Wyman (Wyman, 1950) have studied the static spherically symmetric solutions representing the analogous to the Reissner-Nordstrom geometry.

Papapetrou’s solution represents a charged\(^{18}\) particle, whose gravitational field takes the form \(^{13}\) with

\[
\psi(r) = 1 - \frac{2m}{r} + \frac{e^2}{4\pi r^4} - \frac{me^2}{2\pi r^5}, \quad \phi(r) = 1 - \frac{2m}{r}.
\]

This solution is singular at the origin like in the case of general relativity and, then, it does not share the prosperity of Hoffmann geometry. We already mentioned that Papapetrou generalized the proof of the non-existence of regular solutions to the case of Einstein-Straus theory and he concluded that the Einstein’s theorem of the infinitely close solutions “is also valid in the generalized theory of gravitation”. Consequently, the divergence of (15) should not be a surprise.

Besides, it is remarkable that Einstein expressed his failure in finding a solution free of singularities in the non-symmetric field theory. The reference to this is contained in a letter addressed to Michele Besso\(^{19}\) in 1949. It does not seem obvious that Einstein did not know about Papapetrou’s generalization of his Beweis even though it is rather unlikely that Einstein and Papapetrou exchanged letters between 1946 and 1955.

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18 in the sense of being charged under the antisymmetric tensor \( g_{\mu\nu} - g_{\nu\mu} \), playing the role of electromagnetic strength tensor.

19 dated on August 16 of 1949, (Einstein & Besso, 1994).
On the other hand, a Born-Infeld inspired version of Einstein-Straus model can be formulated from the Schrödinger version of the non-symmetric unified field theory, which combines the Eddington-Einstein pure affine theory and the non-symmetric structure (Schrödinger, 1943; 1948; 1950). In fact, it was discussed whether such a formulation can lead to field equations which certainly admit regular solutions of the form (14), see (Sen, 1968; Hlavaty, 1957; Tonnelat, 1955) and references therein for details.

Another criticism to the Einstein-Straus theory, besides the singularity of Papapetrou’s geometry, is the ambiguity in the formulation, which is such that, when trying to identify a metric tensor representing gravity among the whole bestiary of symmetric tensors, one still finds the freedom to redefine the fields in order to turn the solution (15) into an exact Schwarzschild metric (Vastone, 1962). Thus, the theory, such as it was originally formulated, is not completely determined.

**Einstein-Pauli theorem and Kaluza-Klein solitons.** For completeness, let us review the observation presented by van Dongen (van Dongen, 2002), which relies in pointing out a sophism existing in Einstein’s observation of scaling invariance we referred in section 3 (see (Einstein, 1941a) and (Einstein & Pauli, 1943)). Basically, the argumentation based on affirming that a regular solution $g_{\mu\nu}(\lambda x^\rho)$ is obtained (for generic $\lambda$) starting from a given solution with $\lambda = 1$ does not hold in general. This is the case of the geometries discussed in (van Dongen, 2002). Einstein-Pauli generalization of Einstein’s proof does not apply for them since no general dilatation $x^\mu \rightarrow \lambda x^\mu$ is permitted among non-singular solutions; this is due to the periodicity conditions for excluding conical deficiency in the instantonic part of Kaluza-Klein instantons. This states that these solitons represent counterexamples which violate the “non-existence theorems”.

**Boulware-Deser-Wiltshire particle and higher order gravitational theories.** Now, let us return to the enhancement of the non-linearity of field equations as an alternative solution to the particle problem. However, in contrast with the modifications alla Born-Infeld where a non-linear field theory is proposed as source of general relativity, we will refer here to the enhancement of the non-linearity understood as higher order modifications of the gravitational part itself. It can be verified that, actually, such a type of field theory can yield non-divergent metrics even for the case of uncharged particle-like objects. As an example, let us consider a model which allows to manifestly show this property: the Gauss-Bonnet extended gravity. This extended gravity, while correcting Einstein’s theory by means of additional higher order terms, represents a relatively natural extension of general relativity (Lanczos, 1938).

Talking about pure gravitational particles, the proposal for a way of violating Einstein’s “non-existence theorem” is to elevate the order of gravitational action. A concise example of this kind of theories is the Lovelock theory of gravitation, which is obtained by dropping the requirement that the field equations have to be linear in the second derivative of the metric tensor. Lovelock (Lovelock, 1971) has constructed the most general entity (non-linear analogue to the Einstein tensor) satisfying to be a symmetric and conserved tensor depending on the metric and its derivatives up to the second order but without more restrictions. The tensor obtained in this procedure is non linear in the Riemann tensor and differs from the Einstein tensor only if the space-time has more than 4 dimensions.
Gauss-Bonnet gravity can be thought as the gravitational analogue of Born-Infeld electrodynamics and corresponds to short-distance effects correcting the Einstein’s theory of gravity at scales comparable with certain length scale \( l \), which would be a fundamental constant of the new gravitational theory. This theory has been tested for admitting static solutions with spherical symmetry and, actually, those have been found during the last decades. For instance, we can mention here the Wiltshire geometry (Wiltshire, 1988), which represents a particle-like solution of Lovelock theory coupled to Born-Infeld electrodynamics. This solution was originally obtained by D. Boulware and S. Deser in (Boulware & Deser, 1986) for the uncharged case. The interesting aspect of such a geometry is that, even for the case of uncharged particles, the metric remains finite at the origin, representing a solution of a pure gravitational theory. As in the case of Born-Infeld theory, the solution develops a conical singularity at the origin; however, it was shown in (Aiello, Ferraro & Giribet, 2004; 2005) that this can be expunged by means of an appropriate choice of the parameters. Moreover, the solution turns out to be geodesically incomplete and its curvature still diverges at \( r = 0 \). Certainly, in 1986 J.T. Wheeler proved a theorem showing that no massive spherically symmetric static solutions with everywhere finite curvature does exist in Lovelock theory of gravity in \( D \) dimensions [Wheeler, 1986a; 1986b].

The asymptotic behavior of the solution for large scales (e.g. large black holes) is well approximated by the de Sitter-Schwarzschild black hole solution while the short distance corrections start to dominate at the scale \( l \), where the physics substantially changes.

Different limits of the Wiltshire’s solution were discussed in the literature. These limits were studied in terms of the fundamental scale \( l \) and the Born-Infeld coupling constant \( b \). It can be consistently proven that these limits correspond to the expected geometries. Hence, Wiltshire’s solution presents a regular metric generalizing the quoted Hoffmann geometry for Einstein-Born-Infeld theory we discussed above and, besides, its metric remains finite (even for the uncharged case). This is well explained by the fact that the non-linear corrections involved in this case refer to the gravitational part instead to the electrodynamic part.

**Addendum: on regular solutions.** We have reviewed here some representatives of the catalogue of non-singular particle-like solutions that were appearing in the ulterior investigations (i.e. after Einstein’s paper). Furthermore, we should say that a renewed study of regular solutions has been started in the last decades within the context of gauge theories and, in particular, in the framework of the supersymmetric models inspired in the low energy limit of string theory. For instance, certain D-branes solutions in type IIB string theory and M-theory can be actually thought as gravitational solitons; however, we will not discuss this aspect here. The geometry of D3 and M5 brane solutions, monopoles, coloured and hairy (e.g. dilatonic) black holes are extensively discussed in the literature. Several quoted solutions in field theory are closely related to the search for regular solutions of the gravitational field: this is the case of the Bartnik-McKinnon \( SU(2) \) Einstein-Yang-Mills (EYM) solution, the gravitating sphalerons, the gravitating glueball-monopole which can be regarded as a massive radial excitation of t'Hooft-Polyakov monopole, the particle-like solutions with positive \( \Lambda \) in EYM theory, etc. We emphasize that the purpose of this fourth section was reviewing the first efforts devoted to find a classical field theory admitting static regular particle-like solutions (i.e. from the historical point of view) instead of presenting a catalogue of regular solutions. In particular, just as an example, we can mention that, recently, a family of charged static black hole solutions
with finite curvature was studied within the framework of the general relativity coupled to non-linear electrodynamics (Ayón-Beato & García, 1998; 1999a; 1999b; 2000; Borde, 1997). These solutions are based on the usually called Bardeen model and represent geometries which present a change of the topology of the causal structure and the finite curvature at the origin is allowed because of the non-existence of a non-compact Cauchy surface (cf. (Penrose, 1965)). Furthermore, concerning the Kaluza-Klein models, it deserves to be mentioned that, currently, the idea of dimensional extensions of spherically symmetric solutions in four dimensions acquires a renewed value in terms of the brane-world scenarios (Randall & Sundrum, 1999) and the description of black holes in this framework (Chamblin, Hawking & Reall, 2000). The discussion of singularities in this new context turns out to be an extremely interesting subject. For instance, it can be shown that, under certain conditions, divergences in the Kretschmann scalar of some four-dimensional spherical solutions are avoided. However, this also exceeds the aim and scope of this note.

5. Concluding remarks

The aim of this note was showing that the search for static non-singular spherically symmetric solutions suitable to be interpreted as describing particles in the framework of classical field theories of gravitation, represented a guide in all stages of the unification programme; having played a central role in Einstein’s investigations after 1923.

J. van Dongen pointed out how this aspect appeared within the context of Einstein’s researches on Kaluza-Klein theory. Here, we emphasize that this actually was a general issue which was present during the whole period of research during the unification programme.

Regarding the attempts for finding a satisfactory final description of the structure of matter in terms of the classical unified field theories, it became clear by the middle of the past century that such a programme had to be dismissed. Even those, who continued actively working on generalizations of Einstein’s theory by 1950, considered that the picture of reality in terms of classical (field) concepts had no chance to lead to a fruitful description. Schrödinger cristalizes this idea when concluding his book (Schrödinger, 1950) with a brief discussion on his point of view about the importance of the existence of particle-like solutions in unified non-symmetric field theory; showing, on the other hand, that this was still an open question by 1950.

The Einstein’s paper in 1941 sketches the lines which were guiding the investigations of those who were searching a satisfactory theory of the classical fields in the first half of the 20th century. It is possible to argue, based on the correspondence that documents the publishing affair of (Einstein, 1941a; 1941d), that the publication of the paper in 1941 was due to a Fubini’s request, compelling Einstein to present such a contribution to the Revista. In fact, as we already mentioned, the Beweis in 1941 acquires relevance, not as a result for general relativity itself, but because of its property of being suitable of generalizations to the cases of certain unified field theories, as in (Einstein & Pauli, 1943) and (Papapetrou, 1948b). Einstein himself qualified his demonstration in 1941 as “ein hübscher Beweis stationärer Gravitationsfelder” (Einstein, 1941c).

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