Quantum entanglement as an interpretation of bosonic character in composite two-particle systems

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(Dated: November 8, 2004)

We consider a composite particle formed by two fermions or two bosons. We discover that composite behavior is deeply related to the quantum entanglement between the constituent particles. By analyzing the properties of creation and annihilation operators, we show that bosonic character emerges if the constituent particles become strongly entangled. Such a connection is demonstrated explicitly in a class of two-particle wave functions.

PACS numbers: 03.67.-a, 03.65.Ta, 03.67.Mn

In Nature many particles are composite particles formed by two or more fundamental fermions. Atoms, molecules and nuclei are familiar examples of composite particles. As long as physical processes do not reveal the internal structure, a composite particle as a whole can be treated approximately as a point-like boson (fermion) if the number of constituent fermions is even (odd). Intuitively, the more compact the composite particle is, the better it behaves as a pure boson or fermion. This raises interesting questions how the degree of compositeness is quantified, and what physical effects would appear if the internal structure becomes apparent. In the case of particles formed by two fermions, several authors have addressed the compositeness effects of atoms in Bose-Einstein condensates [1, 2] and electron-hole pairs in semiconductor [2, 3]. In particular, Combescot and Tanguy have indicated the deviation of purely bosonic behavior, based on the properties of the creation operator associated with composite particles [3].

The issue of compositeness also appears in particles made of bosons. An important example of this kind is biphoton states generated by optical spontaneous parametric down conversion. Such highly correlated photons have been employed in fundamental experiments, e.g., to test the violation of Bell’s inequality [4], and to realize Einstein-Podolsky-Rosen paradox [5]. Recent experiments have also explored the ‘binding’ effects revealed from the de Broglie wavelength of biphotons [6, 7]. Interestingly, although the constituent photons are non-interacting and they can be spatially separated, the composite nature of biphotons can still be detected by suitable interference schemes. This suggests that the concept of compositeness is not limited to mechanically bounded particles in position space.

In this paper we examine the composite representation of two-particle systems from the view point of quantum information. We show that quantum entanglement provides an understanding of the origin of composite behavior. As we shall see below, the degree of entanglement between constituent particles determines how close a composite particle behave as a pure boson. This implies an interesting picture that constituent particles are somehow bounded by quantum entanglement. Mechanical binding forces are not essential, they serve only as physical means to enforce quantum correlations. Since the constituent particles can be correlated in many different ways, the composite representation is not limited to position space or momentum space. In this paper we will indicate the fundamental role of entanglement, based on the properties of creation and annihilation operators associated with composite particles. In contrast to previous studies that concern only fermionic constituent particles, we provide a general analysis that applies to bosonic constituents as well.

To begin with, we consider a composite particle $C$ formed by two distinguishable particles $A$ and $B$. Both $A$ and $B$ are either fermions or bosons. Let $\Psi(x_A, x_B)$ be the wave function of the two-particle system, we can always express the wave function in the Schmidt decomposition form:

$$\Psi(x_A, x_B) = \sum_{n=0}^{\infty} \sqrt{\lambda_n} \phi_n^{(A)}(x_A) \phi_n^{(B)}(x_B)$$  \hspace{1cm} (1)

where the Schmidt modes $\phi_n^{(A)}$ ($\phi_n^{(B)}$) form a complete and orthonormal set for particle $A$ ($B$). Specifically, $\phi_n^{(A)}$ and $\lambda_n$ are defined by the eigenvectors and eigenvalues of the reduced density matrix of particle $A$, and $\phi_n^{(B)}$ can be obtained similarly from the reduced density matrix of particle $B$. Eq. (1) reveals the quantum correlation by showing the pairing structure explicitly. If the particle $A$ appears in the mode $\phi_n^{(A)}$, then with certainty the particle $B$ must be in the mode $\phi_n^{(B)}$. The distribution of $\lambda_n$ provides a measure of entanglement. This is usually discussed in terms of the entanglement entropy $E = - \sum_n \lambda_n \log_2 \lambda_n$. However, a more transparent measure of entanglement is to count the ‘average’ number of Schmidt modes actively involved. The Schmidt number $K$ provides this information [8, 9]:

$$K \equiv 1/\sum_{n=0}^{\infty} \lambda_n^2.$$  \hspace{1cm} (2)
The larger the value of $K$ is, the higher the entanglement. In fact, $K$ equals the inverse of the purity of single-particle density matrix, and it also equals the linear entropy apart from a constant. A disentangled (product) state corresponds to $K = 1$, i.e., only one term in the Schmidt decomposition. If a Schmidt decomposition contains $M$ terms of equal weight (i.e., $\lambda_n = 1/M$ for $n = 1, 2, ..., M$), then we have $K = M$ which is exactly the number of mode pairs involved.

In the second quantized representation, Eq. (1) corresponds to a state generated by a creation operator $c^\dagger$ acting on the vacuum. Such a creation operator is defined by,

$$c^\dagger = \sum_{n=0}^{\infty} \sqrt{\chi_n} a_n^\dagger b_n^\dagger$$

where $a_n^\dagger (b_n^\dagger)$ is the creation operator of the particle $A$ ($B$) in the Schmidt mode $\phi_n^{(A)} (\phi_n^{(B)})$. The commutation relation between $c$ and $c^\dagger$ is given by,

$$[c, c^\dagger] = 1 + s \Delta,$$

where $s = +1$ if $A$ and $B$ are bosons, $s = -1$ if $A$ and $B$ are fermions. The operator $\Delta$ in Eq. (4) is defined by,

$$\Delta = \sum_{n=0}^{\infty} \lambda_n (a_n^\dagger a_n + b_n^\dagger b_n)$$

which has non-zero matrix elements, depending on the states involved. Therefore $c$ and $c^\dagger$ are not strictly bosonic operators.

To examine the properties of $c$ and $c^\dagger$, we consider a system containing two or more $C$ particles. Let us define the $N-$particle states by,

$$|N\rangle \equiv \chi_N^{-1/2} \frac{c^{1N}}{\sqrt{N!}} |0\rangle$$

where $\chi_N$ is a normalization constant such that $\langle N|N \rangle = 1$. In order to test how good the operator $c$ behaves as a bosonic annihilation operator, we need to determine how the operator acts on $|N\rangle$. This is given by a general equation:

$$c |N\rangle = \alpha_N \sqrt{N} |N-1\rangle + |\varepsilon_N\rangle,$$

where $\alpha_N$ is a constant, and the correction term $|\varepsilon_N\rangle$ is orthogonal to $|N-1\rangle$. Such a correction term is necessary because the set of $|N\rangle$ states (with all $N$) is only a subset of the entire Hilbert space associated with the constituent particles. A non-ideal bosonic operator would inevitably cause transitions into the states not described by (6).

From Eq. (7), the annihilation operator $c$ is bosonic if the following two conditions are satisfied:

$$\alpha_N \rightarrow 1$$

$$\langle \varepsilon_N|\varepsilon_N \rangle \rightarrow 0.$$  

After some calculations [10], for $N > 1$, we have

$$\alpha_N = \sqrt{\frac{\chi_N}{\chi_{N-1}}}$$

$$\langle \varepsilon_N|\varepsilon_N \rangle = 1 - N \frac{\chi_N^{(A)}}{\chi_{N-1}} + (N-1) \frac{\chi_{N+1}}{\chi_N}.$$  

Therefore the conditions (8) and (9) are controlled by the ratio of normalization constants. An ideal composite boson emerges in the limit $\chi_{N+1}/\chi_{N} \rightarrow 1$.

For convenience, let us write

$$\chi_N = \left\{ \begin{array}{ll}
\chi_N^F & \text{A, B are fermions} \\
\chi_N^B & \text{A, B are bosons.}
\end{array} \right.$$  

By carefully counting the states allowed for fermions and bosons, it can be shown that $\chi_N^B$ and $\chi_N^F$ are given by [11],

$$\chi_N^B = N! \sum_{p_N \geq p_{N-1} \geq \ldots \geq p_2 \geq p_1} \lambda_{p_1} \lambda_{p_2} \ldots \lambda_{p_N}$$

$$\chi_N^F = N! \sum_{p_N > p_{N-1} > \ldots > p_2 > p_1} \lambda_{p_1} \lambda_{p_2} \ldots \lambda_{p_N}.$$
The calculation of these summations can be quite complicated in general. In the case particles \( A \) and \( B \) are fermions, \( \chi_N^F \) can be analyzed by methods discussed in Ref. 3. However, exact analytical expressions of \( \chi_N \) for both fermions and bosons in closed forms are rare.

In this paper we consider a realistic class of two-particle wave functions that allows exact closed form expressions of \( \chi_N \). Such a class of two-particle wave functions is specified by the Schmidt eigenvalues,

\[
\lambda_n = (1 - z)z^n \quad n = 0, 1, 2\ldots
\]

where the parameter \( z \) is defined in the range \( 0 < z < 1 \). This parameter determines how rapid \( \lambda_n \) decreases with \( n \). A representative example of this class of wave functions is (double) gaussian given by:

\[
\Psi(x_A, x_B) = \mathcal{N}e^{-(x_A + x_B)^2/\sigma^2}e^{(x_A - x_B)^2/\sigma_r^2}.
\]

Here \( \mathcal{N} \) is a normalization constant, and \( \sigma \) and \( \sigma_r \) are widths along the \( x_A + x_B \) and \( x_A - x_B \) directions respectively. The Schmidt decomposition of Eq. (16) gives \( \lambda_n \)'s in the form of Eq. (15) with

\[
z = \left( \frac{\sigma_r - \sigma_c}{\sigma_r + \sigma_c} \right)^2.
\]

The corresponding Schmidt modes in this example are simply the eigenfunctions of a harmonic oscillator.

To evaluate \( \chi_N^B \) and \( \chi_N^F \), we let: \( p_1 = q_N, p_2 = q_N + q_{N-1}, p_3 = q_N + q_{N-1} + q_{N-2}, \ldots, p_N = q_N + q_{N-1} + \ldots + q_1 \), then Eq. (13) and (14) become,

\[
\chi_N^B = N!(1 - z)^N \sum_{q_1=0}^{\infty} \sum_{q_2=0}^{\infty} \ldots \sum_{q_{N-1}=0}^{\infty} \sum_{q_N=0}^{\infty} z^{q_1+2q_2+3q_3+\ldots+Nq_N},
\]

\[
\chi_N^F = N!(1 - z)^N \sum_{q_1=1}^{\infty} \sum_{q_2=1}^{\infty} \ldots \sum_{q_{N-1}=1}^{\infty} \sum_{q_N=0}^{\infty} z^{q_1+2q_2+3q_3+\ldots+Nq_N}.
\]

These summations can now be carried out easily,

\[
\chi_N^B = \frac{N!(1 - z)^N}{(1 - z)(1 - z^2)\ldots(1 - z^N)}
\]

\[
\chi_N^F = \frac{N!z^{N(N-1)/2}(1 - z)^N}{(1 - z)(1 - z^2)\ldots(1 - z^N)}.
\]

Hence the normalization ratios are given by,

\[
\frac{\chi_{N+1}^B}{\chi_N^B} = \frac{(N + 1)(1 - z)}{(1 - z^{N+1})},
\]

\[
\frac{\chi_{N+1}^F}{\chi_N^F} = \frac{z^{N(N + 1)(1 - z)}}{(1 - z^{N+1})}.
\]

From Eq. (7) and (10), the normalization ratio determines the modification of Bose enhancement factor. The results Eq. (22) and (23) indicate that \( \chi_{N+1}^B/\chi_N^B > 1 \) and \( \chi_{N+1}^F/\chi_N^F < 1 \). The difference between the two types of constituents can be understood, because bosons tend to stay together in the same state, while fermions do the opposite due to the exclusion principle. However, quantum statistics of constituent particles becomes less important as \( z \) approaches one, where the composite particle become a pure boson. In the example of double gaussian wave function (16), the \( z \to 1 \) limit correspond to the cases \( \sigma_r \gg \sigma_c \) or \( \sigma_c \gg \sigma_r \). We remark that that the particle number \( N \) is a key factor in Eq. (22) and (23). A larger number of particle number would require the value of \( z \) to be closer to one in order to maintain the normalization ratio.

Now we can make an explicit connection with quantum entanglement. For the Schmidt eigenvalues given by (15), the Schmidt number \( K \) defined in Eq. (2) takes the form:

\[
K = \frac{1 - z^2}{(1 - z)^2}
\]

which is a monotonic increasing function in the range \( 0 < z < 1 \). By expressing \( z \) in terms of \( K \), both \( \chi_{N+1}^B/\chi_N^B \) and \( \chi_{N+1}^F/\chi_N^F \) can be directly related to the degree of quantum entanglement. As \( K \) increases, we find that both
In deriving Eq. (13), we write

\[ \chi_{N+1}/\chi_N \approx 1 + sN/K \]  

(25)

where \( s \) defined in Eq. (4). Since the value of \( K \) corresponds to an effective number of Schmidt modes, bosonic particle description is valid when the effective number of Schmidt modes is much greater than the total number of composite particles.

The discussion above indicates that composite behavior strictly depends on the degree of quantum entanglement, at least for the class of two-particle wave functions with Schmidt eigenvalues specified by Eq. (15). We note that Eq. (15) corresponds to a wide variety of wave functions, because Schmidt modes can be chosen from any discrete, complete and orthogonal set of basis functions. Indeed, Schmidt modes do not play any role in determining \( \chi_N \). It is the distribution of \( \lambda_n \) that controls the compositeness of the particle. As long as the Schmidt eigenvalues are specified by (15), different forms of Schmidt modes correspond to the same degree of compositeness. This also shares the same feature with quantum entanglement.

To summarize, our work here examine the foundation of composite representation of two-particle systems. Within the class of two-particle wave functions, we show that the origin of composite behavior is ultimately related to quantum entanglement between constituent particles. Therefore composite representation can be applied to strongly entangled particles, which are not limited to mechanically bounded systems. Finally, we remark that the wave functions satisfying Eq. (15) are not completely general, but owing to flexibility of the parameter \( z \), these functions may be used to approximate wave functions in actual situations. This suggests that the relation between composite behavior and the degree of quantum entanglement can be quite general. However, a full analysis would require the study of Eq. (13) and (14) for arbitrary distribution of Schmidt eigenvalues, which is a topic open for future investigations.

Acknowledgments

The author thanks Prof. M.-C. Chu for discussions. This work is supported in part by the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. 400504 and Project No. 423701).

[10] The derivation of Eq. (11) requires the expectation value \( \langle N|\Delta|N \rangle \), which can be obtained by using the algebraic method in Ref. [3].
[11] In deriving Eq. (13), we write

\[ c^N |0\rangle = \sum_{k_N \geq k_{N-1} \geq \ldots \geq k_1} \sqrt{\lambda_{k_1} \lambda_{k_2} \ldots \lambda_{k_N}} F(k_1, k_2, \ldots, k_N) |k_1, k_2, \ldots, k_N\rangle \]

where \( |k_j\rangle \) denotes the occupation with an A particle and a B particle in the Schmidt mode \( k_j \), and \( F(k_1, k_2, \ldots, k_N) \) is the weight factor for the state \( |k_1, k_2, \ldots, k_N\rangle \). If these \( k \)'s have \( d \) same terms (e.g. \( k_1 = k_2 = \ldots = k_d \)), and all others terms are distinct, then the weigh factor should be \( N! / d! \times d! = N! \). The \( d! \) in the nominator comes from the fact that \( c^I \) contains operators \( a_{k_1}^d b_{k_1}^d \). This argument applies to general sequence with any combinations of degenerate terms.