Attractions of Affine Quantum Gravity*

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Abstract
All attempts to quantize gravity face several difficult problems. Among these problems are: (i) metric positivity (positivity of the spatial distance between distinct points), (ii) the presence of anomalies (partial second-class nature of the quantum constraints), and (iii) perturbative nonrenormalizability (the need for infinitely many distinct counterterms). In this report, a relatively nontechnical discussion is presented about how the program of affine quantum gravity proposes to deal with these problems.

Introduction and Survey
The program of affine quantum gravity differs from that of string theory or loop quantum gravity: specifically, it differs in the insistence on a spatial metric tensor that is strictly positive definite; in the simultaneous and uniform treatment of both first- and second-class operator constraints; in dealing with nonperturbative renormalizability; and in maintaining a close connection with the motivating classical (Einstein) gravity theory. A suitable realization of these principles is most readily presented within a formalism that is, for the most part, generally unfamiliar to many readers. The purpose of this paper is to provide a relatively simple introduction to several concepts used to study quantum gravity from this new perspective.

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Reproducing Kernel Hilbert Spaces

This important idea can be readily explained. Let $|l⟩ \in \mathcal{H}$, for all $l \in \mathcal{L}$, denote a set of states (chosen to be normalized, but that is not a requirement) that span the separable Hilbert space $\mathcal{H}$ of interest. In addition we assume that (when it is finite dimensional) the space $\mathcal{L}$ is locally equivalent to a Euclidean space, and that the states $|l⟩$ are continuously labeled by the (multi-dimensional) labels $l$. We will refer to the set of states $\{ |l⟩ \}$ as coherent states.

Since the coherent states span $\mathcal{H}$, it follows that two elements of a dense set of states may be given by

$$ |\psi⟩ = \sum_{j=1}^{J} \alpha_j |l_j⟩, \quad J < \infty,$$

$$ |\phi⟩ = \sum_{k=1}^{K} \beta_k |l_(k)⟩, \quad K < \infty. $$

As functional representatives of these abstract vectors we introduce

$$ \psi(l) \equiv \langle l | \psi⟩ = \sum_{j=1}^{J} \alpha_j \langle l | l_j⟩, $$

$$ \phi(l) \equiv \langle l | \phi⟩ = \sum_{k=1}^{K} \beta_k \langle l | l_(k)⟩. $$

Finally, as the inner product for these two elements we choose

$$ (\psi, \phi) \equiv \sum_{j,k=1}^{J,K} \alpha_j^* \beta_k \langle l_j | l_(k)⟩ = \langle \psi | \phi⟩. $$

This pre-Hilbert space is completed by adding the limit points of Cauchy sequences. The result is a functional representation composed entirely of continuous functions representing the separable Hilbert space $\mathcal{H}$.

Such spaces are called reproducing kernel Hilbert spaces because if $J = 1$ and $\alpha_1 = 1$, then it follows that $\psi(l) = \langle l | l_1⟩$ and so

$$ (\psi, \phi) = \sum_{k=1}^{K} \beta_k \langle l_1 | l_(k)⟩ = \phi(l_1), $$

a result which “reproduces” the vector $\phi(l)$. Thus the coherent state overlap $⟨l|l′⟩$ serves as the “reproducing kernel” for this space.

Observe that all properties of this representation are determined by the jointly continuous coherent state overlap function $⟨l|l′⟩$. Indeed, any continuous function of two (sets of) variables $K(l; l′)$ serves to define a reproducing kernel Hilbert space provided $K$ satisfies the condition

$$ \sum_{j,k=1}^{J,J} \alpha_j^* \alpha_k K(l_j; l_k) \geq 0 $$

for all possible complex choices of $\{ \alpha_j \}$ and finite $J$. 

**Metric Positivity**

Distinct points in a space-like 3-dimensional manifold have a positive separation distance. For a small coordinate separation $dx^a \neq 0$, that distance, as usual, is given by $ds^2 = g_{ab}(x)dx^a dx^b > 0$. We require that the associated quantum operator $\hat{g}_{ab}(x)$ also satisfy metric positivity such that $\hat{g}_{ab}(x)dx^a dx^b > 0$ in the sense of operators for all nonvanishing $dx^a$. Moreover, we insist that $\hat{g}_{ab}(x)$ becomes self adjoint when smeared with a suitable real test function. In canonical quantization one chooses the canonical (ADM) momentum $\pi^{ab}(x)$ as the field to promote to an operator, $\hat{\pi}^{ab}(x)$. However, since the momentum acts to translate the metric, such a choice is inconsistent with the preservation of metric positivity. Instead, it is appropriate to choose the mixed valence momentum field $\pi^a_c(x) \equiv \pi^{ab}(x)g_{bc}(x)$ to promote to an operator $\hat{\pi}^a_c(x)$. This choice is dictated by the relation

$$e^i \int \gamma^a_b(y) \hat{\pi}^b_c(y) d^3y \hat{g}_{cd}(x) e^{-i \int \gamma^a_b(y) \hat{\pi}^b_c(y) d^3y} = (e^{\gamma(x)/2})^e \hat{g}_{ef}(x) (e^{\gamma(x)/2})^f,$$

a relation that manifestly preserves metric positivity.

The full set of kinematical commutation relations is given by [1]

$$[\hat{\pi}^a_c(x), \hat{\pi}^b_d(y)] = \frac{i}{2} [\delta^a_d \hat{\pi}^b_c(x) - \delta^a_c \hat{\pi}^b_d(x)] \delta(x,y),$$

$$[\hat{g}_{ab}(x), \hat{\pi}^c_d(y)] = \frac{i}{2} [\delta^c_d \hat{g}_{ab}(x) + \delta^c_b \hat{g}_{ad}(x)] \delta(x,y),$$

$$[\hat{g}_{ab}(x), \hat{g}_{cd}(y)] = 0.$$

These are the so-called affine commutation relations appropriate to the affine fields $\hat{\pi}^a_c(x)$ and $\hat{g}_{ab}(x)$, both of which may be taken as self adjoint when smeared with real test functions. These commutation relations provide a realization of the group $\text{IGL}(3, \mathbb{R})$, and as such they are more in the spirit of a current algebra than traditional canonical commutation relations.

It is important to add that by choosing $\hat{\pi}^a_c(x)$ as the partner field to go with $\hat{g}_{ab}(x)$, it follows that the momentum field $\hat{\pi}^{ab}(x)$ does not make an operator when smeared but only a form.

**Quantization of Constraints**

There are several schemes in common usage to quantize canonical systems with constraints. Traditionally, these schemes treat first- and second-class constraints differently. Gravity is not a traditional gauge theory since the set
of classical constraints form an open first class system, which means that the Poisson brackets among the constraints have the form of a Lie algebra except that the structure constants are actually structure functions depending on the canonical variables. On quantization, these structure functions become operators that do not commute with the constraints, and as a consequence, the quantum constraints are partially second class in character. As noted above this usually entails a separate procedure for their analysis.

However, the recently introduced projection operator method \cite{2} to incorporate quantum constraints treats first- and second-class constraints in an identical fashion and thereby it seems ideal to apply to gravity. Here, we content ourselves with a sketch of how this procedure is applied to simple systems.

We start by assuming that \( \phi_\alpha(p,q) = 0, \alpha = 1,\ldots,A, \) represent a set of real classical constraints for some multi-dimensional system. We choose some quantization procedure and identify \( \{ \Phi_\alpha(P,Q) \} \) as a set of self-adjoint operators representing the constraints. Ideally, following Dirac, we would identify the physical Hilbert space \( H_{\text{phys}} \subset H \) as composed of vectors \( |\psi_{\text{phys}}\rangle \) that have the property that

\[
\Phi_\alpha(P,Q) |\psi_{\text{phys}}\rangle = 0
\]

for all \( \alpha \). Consistency of this procedure requires that (i) \( \langle \psi_{\text{phys}} | \psi_{\text{phys}} \rangle < \infty \), and (ii) \( [\Phi_\alpha(P,Q),\Phi_\beta(P,Q)] |\psi_{\text{phys}}\rangle = 0 \). Unfortunately, for certain constrained systems, either one or both of these consistency conditions is violated. In that case it is useful to propose another scheme.

One alternative procedure, known as the projection operator method, involves a projection operator

\[
\mathbb{E} = \mathbb{E}(\Sigma_\alpha \Phi^2_\alpha(P,Q) \leq \delta(h)^2)
\]

an expression which means that

\[
0 \leq \mathbb{E} \Sigma_\alpha \Phi^2_\alpha(P,Q) \mathbb{E} \leq \delta(h)^2 I.
\]

In these expressions, \( \delta(h) \) denotes a small, positive cutoff, generally dependent on \( h \), that can be reduced to a suitable level. In this approach the (regularized) physical Hilbert space is taken as \( \mathcal{H}_{\text{phys}} = \mathbb{E}\mathcal{H} \).

It is pedagogically useful to illustrate this procedure with three simple examples:
(1) Let $\Phi_\alpha = J_1, J_2, J_3$ denote the generators of $SU(2)$, and the desired physical Hilbert space satisfies $J_k|\psi_{phys}\rangle = 0$ for $k = 1, 2, 3$. We can secure the physical Hilbert space of interest by choosing

$$E(\Sigma_k J_k^2 \leq \frac{1}{2} \hbar^2) .$$

This example represents a first-class constrained system.

(2) Let $\Phi_\alpha = P, Q$, a pair of canonical operators. In this case we choose

$$E(P^2 + Q^2 \leq \hbar) ,$$

which projects onto states $|\psi_{phys}\rangle$ that satisfy $(Q + iP)|\psi_{phys}\rangle = 0$. This example represents a second-class constrained system.

(3) Let $\Phi_\alpha = Q$, a single operator with zero in the continuous spectrum and for which $Q|\psi_{phys}\rangle = 0$ has no normalizable solution. Here we choose

$$E(Q^2 \leq \delta^2) ,$$

with no $\hbar$ dependence necessary. As $\delta \to 0$, this projection operator passes strongly (hence weakly) to the zero operator. To overcome this fact, we rescale the projection operator and take a suitable limit as $\delta$ goes to zero. As one example, we introduce coherent states

$$|p, q\rangle \equiv e^{-iqP}e^{ipQ}|0\rangle ,$$

and we consider

$$\langle \langle p'', q''|p', q'\rangle \equiv \lim_{\delta \to 0} \frac{\langle p'', q''|E(Q^2 \leq \delta^2)|p', q'\rangle}{\langle 0|E(Q^2 \leq \delta^2)|0\rangle} .$$

The resultant expression forms a suitable reduced reproducing kernel which can be used to characterize the physical Hilbert space as a reproducing kernel Hilbert space.

If $P$ and $Q$ form an irreducible pair, and for the sake of illustration we choose $|0\rangle$ as a normalized solution of $(Q + iP)|0\rangle = 0$, i.e., $|0\rangle$ is the oscillator ground state, then

$$\langle \langle p'', q''|p', q'\rangle = e^{-\frac{1}{2}(q''^2 + q'^2)} ,$$

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a reproducing kernel which characterizes a one-dimensional Hilbert space. Different choices of the fiducial vector $|0\rangle$ may lead to different functional representatives, but they nevertheless still describe one-dimensional Hilbert spaces.

It is noteworthy that path integral expressions exist that directly generate matrix elements of any desired projection operator. For example, staying with elementary examples, coherent state path integrals that generate expressions such as $\langle p'', q'' | E | p', q' \rangle$ may formally be written as conventional phase-space path integrals save for one change, namely, the choice of the integration measure for the Lagrange multipliers [2].

How these general ideas may be applied to quantum gravity can be found in [1].

Perturbative Nonrenormalizability

One of the most challenging aspects of conventional approaches to quantum gravity is its perturbative nonrenormalizability. Divergences can be regularized by the introduction of cutoffs, as usual, and then counterterms developed on the basis of perturbation theory can be identified and included in the formalism. For renormalizable theories there are only a finite number of distinct types of counterterms, while for nonrenormalizable theories—such as gravity—an infinite set of qualitatively distinct counterterms is mandated by perturbation theory. It is no wonder that the morass created by renormalized perturbation theory has driven many workers to alternative approaches such as string theory. On the other hand, perhaps we are deceiving ourselves; could it be that perturbatively suggested counterterms to nonrenormalizable models are in fact irrelevant? This heretical viewpoint is indeed suggested by the hard-core picture of nonrenormalizable interactions which we now outline [3].

To present the essential ideas as simply as possible let us initially examine certain singular potentials in quantum mechanics. In particular, consider the Euclidean-space path integral for a free particle in the presence of an additional potential $V(x) \geq 0$. In symbols, let us study

$$W_\lambda \equiv \mathcal{N} \int_{x(0)=x'}^{x(T)=x''} e^{-\frac{1}{2} \int \dot{x}(t)^2 dt - \lambda \int V(x(t)) dt} \mathcal{D}x.$$
As $\lambda \to 0^+$, it appears self evident that $W_\lambda$ passes to the expression

$$W_0 \equiv \mathcal{N} \int_{x(0)=x'}^x e^{\frac{1}{2}\int x^2(t) \, dt} \, Dx = \frac{1}{\sqrt{2\pi T}} e^{-(x''-x')^2/2T}$$

appropriate to a free particle. Whatever the analytic dependence of $W_\lambda - W_0$ for small $\lambda$ (e.g., $O(\lambda)$, $O(\lambda^{1/3})$, $O(e^{-1/\lambda})$, etc.), it is tacitly assumed that as $\lambda \to 0^+$, $W_\lambda \to W_0$, i.e., that $W_\lambda$ is \textit{continuously connected} to $W_0$. However, this limiting behavior is \textit{not} always true.

Consider the example $V(x) = x^{-4}$. In this case the singularity at $x = 0$ is so strong that the contribution from all paths that reach or cross the origin is \textit{completely suppressed} since $\int x(t)^{-4} \, dt = \infty$ for such paths, no matter how small $\lambda > 0$ is chosen. As a consequence, as $\lambda \to 0^+$ for $V(x) = x^{-4}$, it follows that

$$\lim_{\lambda \to 0^+} W_\lambda = W'_0 \equiv \frac{\theta(x''x')}{\sqrt{2\pi T}} \left[ e^{-(x''-x')^2/2T} - e^{-(x''+x')^2/2T} \right].$$

Stated otherwise, when $V(x) = x^{-4}$, $W_\lambda$ is \textit{decidedly not} continuously connected to the free theory $W_0$, but is instead continuously connected to an alternative theory – called a pseudofree theory – that accounts for the \textit{hard-core} effects of the interaction. The interacting theory may well possess a perturbation expansion about the pseudofree theory (to which it is continuously connected), but the interacting theory will \textit{not} possess any perturbation expansion about the free theory (to which it is not even continuously connected).

Let us next pass to scalar field theory and the Euclidean-space functional integral

$$S_\lambda(h) \equiv \mathcal{N} \int \exp\{ \int h \phi \, d^n x - \frac{1}{2} \int (\nabla \phi)^2 + m^2 \phi^2 \, d^n x - \lambda \int \phi^4 \, d^n x \} \, D\phi$$

appropriate to the $\phi^4_n$ model in $n$ spacetime dimensions. We recall for such expressions that there is a Sobolev-type inequality to the effect that

$$\left\{ \int \phi(x)^4 \, d^n x \right\}^{1/2} \leq K \int [((\nabla \phi(x))^2 + m^2 \phi(x)^2] \, d^n x$$

holds for \textit{finite} $K$ (e.g., $K = 4/3$) whenever $n \leq 4$, but which \textit{fails} to hold (i.e., $K = \infty$) whenever $n \geq 5$. Thus for nonrenormalizable interactions
φ₄ⁿ, for which \( n \geq 5 \), it follows that there are fields \( \phi \) for which the free action is finite while the interaction action is infinite. Just as in the elementary example, there is no reason to believe that counterterms suggested by a regularized perturbation analysis (the underlying premise of which is to maintain a continuous connection with the free theory!) should have any relevance in defining the pseudofree theory \( S'_0(h) \).

It is noteworthy that proposals have been advanced to define \( S'_0(h) \) and thereby to develop a meaningful and nontrivial theory of nonrenormalizable scalar fields [4]. Monte Carlo studies of such proposals are currently under way.

Lastly we observe that gravity is also a theory for which the free action (limited to quadratic terms) does not dominate the interaction action (remaining terms), and consequently gravity would seem to be a candidate theory to be understood on the basis of a hard-core interaction, which, when regularized, leads to its perturbatively nonrenormalizable behavior. As plausible as this scenario seems, it will involve a considerable effort to establish it convincingly.

References

