1 Introduction

The conceptual problems of quantum mechanics still attract much attention. Among the most famous and illustrative of these problems are the Schrödinger cat paradox[1] and the problem of wave function reduction[2]. The ruling Copenhagen interpretation of quantum mechanics does not address them to the satisfaction of all. This has led to the development of variant interpretations of quantum mechanics. In this paper, we explain how the interpretation of quantum mechanics due to Landé deals with these problems in a natural and satisfying way.

We briefly outline the cat paradox and the collapse of the wave function. Both of them arise from the superposability of quantum states. Let the eigenfunctions $\psi_1(x)$ and $\psi_2(x)$ be solutions of the Schrödinger equation corresponding to the energy eigenvalues $E_1$ and $E_2$. The functions $\psi_1(x)$ and $\psi_2(x)$ are states of the system; so also is their superposition

$$\Psi(x) = a_1 \psi_1(x) + a_2 \psi_2(x) \quad (1)$$

where $a_1$ and $a_2$ are complex constants. If the energy is measured while the system is in the state $\Psi(x)$, either $E_1$ or $E_2$ or is obtained; the respective probabilities of obtaining these eigenvalues are $|a_1|^2$ and $|a_2|^2$. If in such a measurement the value $E_1$ is obtained, then another measurement of the energy will yield $E_1$ with certainty, because the first measurement of the energy threw the system into the state $\psi_1(x)$. The fact that a measurement of the energy when the system is in the state $\Psi(x)$ can give either $E_1$ or $E_2$ or shows that in that state, the system is to be considered as being in a superposition of the states $\psi_1(x)$ and $\psi_2(x)$. The act of measuring the energy actualizes one of the eigenvalues of the energy. In this example, the process whereby the system state changes from the superposition $\Psi(x)$ to $\psi_1(x)$ or $\psi_2(x)$ constitutes collapse or reduction of the wave function.

Superposition states are a general feature of quantum theory and can be formed with the eigenfunctions of any observable, not just the energy. A hypothetical observable in which the eigenvalues are the states "dead" and "alive" of a cat illustrates very strikingly the implications of superposition states. With this observable, the state

$$\Psi = a_{\text{dead}} \psi_{\text{dead}} + a_{\text{alive}} \psi_{\text{alive}} \quad (2)$$

1
is possible. Since this is a superposition state, the cat is neither dead nor alive until an experiment is performed that reduces the system wave function to that of the state ”dead” or ”alive”. The conclusion that a macroscopic entity like a cat can be in a superposition of two states is the essence of the cat paradox.

The conceptual problems arising from the collapse of the wave function and from the cat paradox are well known. Next we show how the Landé interpretation of quantum mechanics disposes of them.

2 Background Theory

The Landé interpretation of quantum mechanics was outlined in a series of books and papers[3-6]. Here we highlight those aspects of it needed to discuss the cat paradox and wave function collapse.

According to the Landé approach, nature is inherently indeterministic and this fact must be ready-built into quantum theory[4]. In treating measurements, we must therefore be reconciled to the fact that all we can hope to do with theory is to predict the probabilities of various possible outcomes of an experiment. If a quantum system contains several observables, then we can hope to use theory to determine the probabilities of obtaining given values of a particular observable after having measured and obtained specific values of another.

Let a quantum system possess the three observables $A$, $B$ and $C$ with respective eigenvalue spectra $A_1, A_2, ..., A_N, B_1, B_2, ..., B_N$ and $C_1, C_2, ..., C_N$. A measurement of $B$ from a state defined by the eigenvalue $A_i$ of the observable $A$ can yield any of the eigenvalues $B_j$ of $B$ with probabilities determined by the probability amplitudes $\chi(A_i, B_j)$. Similarly, a measurement of $C$ is described by the probability amplitudes $\psi(A_i, C_j)$. On the other hand, a measurement of $C$ from the state of $B$ defined by the eigenvalue $B_i$ yields any of the values $C_j$ with probabilities given by the probability amplitudes $\phi(B_i, C_j)$. The probabilities for measurements in the opposite direction are given by the same probability amplitudes. Thus, for example,

$$\psi(A_i, C_j) = \psi^*(C_j, A_i)$$ (3)

This means that the matrix formed from the probability amplitudes is Hermitian.
As would be expected of quantities pertaining to one system, the three sets of probability amplitudes are not independent of each other. The relation that connects them is[4]

\[ \psi(A_i, C_n) = \sum_{j=1}^{N} \chi(A_i, B_j) \phi(B_j, C_n) \] (4)

with identical relations for the other probability amplitude. This is evidently the law of probability addition, a fundamental axiom of quantum mechanics. A careful interpretation of this relation is all we need to clear up the cat paradox and the problem of wave function collapse.

The probability addition law Eq. (4) is evidently the ultimate origin of the superposition state. In order to obtain the standard superposition state from this expression, we need only to label the quantities incompletely. Firstly, since in the treatment of solutions of the Schrödinger equation, \(C\) is the position and is continuous, we make the assignment \(C_n = x\). We omit the labelling of the initial state, so that the left-hand side becomes \(\Psi(x)\).

Finally, because the initial state is not properly identified, the structure of the probability amplitudes \(\chi(A_i, B_j)\) is obscured and we can treat them as mere constants \(b_j\). Hence, the expansion becomes

\[ \Psi(x) = \sum_{j=1}^{N} b_j \phi_j(x) \] (5)

A second crucial feature of the Landé approach is the idea that every probability amplitude connects a well-defined initial state and a well-defined final state[4]. Wave functions are just probability amplitudes and this applies to them as well. From this, we deduce that solutions of eigenvalue equations connect initial states defined by the eigenvalue and final states defined by the variable in terms of which the differential operator is defined[7,8]. To use the Schrödinger equation as an example, its eigenfunctions are probability amplitudes corresponding to the energy eigenvalues as defining the initial states and the position as characterising the final state.

3 Analysis of the Paradoxes

Having understood that a superposition state is just an under-labelled form of the probability-addition law Eq. (4), we can proceed to analyse the cat
paradox and the process of wave function collapse.

As explained above, the origin of the cat paradox is the superposability of quantum states. Because of the suppression of the initial state index $A_i$, the role of defining the state passes to the index $j$. But since many values of $j$ appear in the expansion, we are forced to think of the system as being in a superposition of states of the observable $B$. We are then left with no choice but to derive the cat paradox and wave function reduction from the expansion and to grapple with all the conceptual problems these conclusions engender.

In truth however, the state represented by an expansion is not described by one of the values of the expansion observable. The state is actually given by the label $A_i$. Therefore, a "superposition state" is not one in which the system is in some kind of limbo. It is one in which the system is in an eigenstate of the observable $A$. Our conceptual problems arise mainly from our failure to label the probability amplitude $\Psi$ with the initial-state parameter. In reality, the state $\Psi(x)$ is the probability amplitude $\Psi(A_i, x)$. In this quantity, the main parameters of interest are the initial-state eigenvalue $A_i$ and the final-state eigenvalue $x$. If we desire to express this probability amplitude as an expansion in terms of the probability amplitudes $\chi(A_i, B_j)$ and $\phi(B_j, x)$, this is for convenience only and does not mean that the system is in a superposition of the states of the observable $B$.

It is true that the ability to be able to express a probability amplitude in terms of two other sets of probability amplitudes is a very powerful tool indeed. We may sometimes not have an obvious means of finding the probability amplitude $\Psi(A_i, x)$, while we do have the sets of probability amplitudes $\chi(A_i, B_j)$ and $\phi(B_j, x)$. In that case the expansion enables us to find $\Psi(A_i, x)$ in terms of the other sets of probability amplitudes. However, any other two sets of probability amplitudes could be used for this purpose, provided that one set corresponds to the initial-state eigenvalue being $A_i$ and the other set corresponds to the final-state eigenvalue being $x$. Thus, we could equally use the two sets $\chi'(A_i, D_j)$ and $\phi'(D_j, x)$, where $D$ is another observable of the system. But whether the third observable is $B$ or $D$, the state of the system is determined only by the eigenvalue $A_i$.

Once we have shifted the definition of the state to the parameter $A_i$, we are able to assign to the index $j$ its true role. It is the state label for the probability amplitudes $\phi_j(x) = \phi(B_j, x)$ which describe measurements of the observable $C$ from states of the observable $B$. But it is not the state label
for the states described by the probability amplitude $\Psi(x) = \Psi(A_i, x)$. The appearance of the label $j$ is entirely artificial since it arises only from our decision to expand the probability amplitude $\Psi(A_i, x)$ in terms of the states of the observable $B$. We could have chosen another observable, say $D$, of the system for this purpose. Also, this expansion is not essential, and is justified only when used as a tool for determining the expression for $\Psi(A_i, x)$ in the event that there is no direct means of obtaining it.

The question arises as to the identity of the observable $A$, and indeed as to whether a quantum system is always in a well-defined state. The answer to this is that before any kind of measurement can be performed on a system, the system has to exist. This means that it contains some property which can be measured by which this existence is ascertained. The eigenvalues of this quantity in principle furnish the values $A_i$. Thus indeed a quantum system can always be thought to be in a well-defined state when we first encounter it. This state could be thought of as having being brought about by a measurement. Since the parameter $A$ always exists, it should never be omitted from the labelling of the probability amplitudes $\Psi(x)$, which should in reality always be written as $\Psi(A_i, x)$.

At this point, it is necessary to recapitulate the interpretation of eigenfunctions. In the context of quantum theory, an eigenfunction is a probability amplitude that connects an initial state corresponding to the eigenvalue and a final state corresponding to the variable represented by the variable in terms of which the differential operator is cast. Thus, the solutions of the time-independent Schrödinger equation in coordinate space connect initial states corresponding to the energy eigenvalues and final states belonging to position eigenvalues. The solutions of the momentum eigenvalue equation connect initial states belonging to the linear momentum and final states corresponding to the position.

4 Resolution of the Paradoxes

We are now in a position to dispose of the problem of the collapse of the wave function of the cat paradox. These are seen to result merely from assigning a false significance to the index $j$ that appears in the expansion of the primary probability amplitude $\Psi(A_i, x)$.

The problem of wave function collapse vanishes as soon as we realise
that a system in the state $\Psi(A_i, x)$ is not in a superposition of states of the observable $B$. The system is in the state corresponding to the eigenvalue $A_i$. The observable $B$ is not of necessary or fundamental importance to the description of the probability amplitude $\Psi(A_i, x)$. In cases where the probability amplitude $\Psi(A_i, x)$ is ready known, it is not necessary to associate it with the observable $B$.

If on the other hand we ask what the probabilities are of obtaining the different eigenvalues of $B$ upon measurement when the system state is initially $A_i$, we must appeal to the probability amplitudes $\chi(A_i, B_j)$ for the answers. In answering that question, the probability amplitude $\Psi(A_i, x)$ does not figure, unless we do not have a ready-made expression for $\chi(A_i, B_j)$ and we seek to obtain it by means of the expansion Eq. (4). Since measurements from the states of the observable $A$ to the observable $B$ do not involve the "superposition" state $\Psi(A_i, x)$ but are instead described by the probability amplitude $\chi(A_i, B_j)$, the issue of wave function collapse simply does not play a role in the measurement. The measurement of $B$ throws the system from an eigenstate of $A$ to an eigenstate of $B$. That is all there is to it. No reduction of the wave function occurs.

We remark that the cat paradox and wave function collapse are much more pervasive and ubiquitous in quantum theory than one would at first imagine. Even the seemingly-innocuous probability amplitude $\chi(A_i, B_j)$ could easily engender the cat paradox and become susceptible to wave function reduction if we decided that its simple form is not satisfactory and we should express the probability amplitude as an expansion. Using the third observable of the system we obtain

$$\chi(A_i, B_j) = \sum_n \Psi(A_i, C_n) \xi(C_n, B_j)$$

(6)

where $\xi(C_n, B_j)$ are the probability amplitudes for measurements from states of the observable $C$ to states of the observable $B$. In view of property Eq. (3) of the probability amplitudes, we have

$$\xi(C_n, B_j) = \phi^*(C_n, B_j)$$

(7)

We can in fact express this as

$$\chi(A_i, B_j) = \sum_n \Psi(A_i, C_n) \phi^*(C_n, B_j)$$

(8)
We see that we again have a superposition state if we choose to think that the label \( n \) is the state label. Hence, we have a Schrödinger cat state. Moreover, if we choose to analyse the problem of measuring the observable \( C \) by means of this probability amplitude instead of by the appropriate probability amplitudes \( \Psi(A_i, C_n) \), we are faced with wave function collapse, since in that case any value \( C_n \) can result from the measurement.

We can go further with this argument. Since any probability amplitude in quantum mechanics can be expressed as an expansion, we conclude that in fact all states are superposition states. In particular, even probability amplitudes representing repeat measurements are superposition states!

A simple example suffices to demonstrate this. Suppose that the energy of a system is measured and the value \( E_k \) is obtained. The measurement has left the system in the state \( \psi_k(x) \). If we measure the energy again, the same value is obtained with certainty and so the probability amplitude for this is

\[
\zeta(E_k, E_k) = e^{i\alpha}
\]

where \( \alpha \) is a real constant. But this can be expressed as

\[
\zeta(E_k, E_k) = \sum_i \eta(E_k, B_j) \varphi(B_j, E_k)
\]

We note in passing that since \( \eta(E_k, B_j) = \varphi^*(B_j, E_k) \), Eq. (10) is essentially the closure relation.

The expansion Eq. (10) tells us that even the probability amplitude \( e^{i\alpha} \) is a superposition state which therefore contains wave function reduction and the cat paradox! If it is argued that the expansion of this trivial probability amplitude in the form Eq. (10) is gratuitous, then the reply is that in principle the same applies to the expansion of the probability amplitude \( \Psi(A_i, x) \). Where there is an expression for the probability amplitude \( \Psi(A_i, x) \), there is no need for an expansion, and hence there is no necessary derivation of either the cat paradox or wave function collapse from the analysis.

Given the foregoing, we can see that the Schrödinger cat paradox is artificial. The moment we stop thinking of the expansion of a probability amplitude in terms of the states of a third observable as being a limbo superposition state, the cat paradox is banished. The expansion Eq. (11) should really be written as

\[
\Psi(A_i, C_n) = \chi(A_i, B_{\text{dead}}) \psi(B_{\text{dead}}, C_n) + \chi(A_i, B_{\text{alive}}) \psi(B_{\text{alive}}, C_n)
\]

\[11\]
to emphasize that the actual state is described by $A_i$, and that the observable $B$ whose eigenvalues are $B_{\text{dead}}$ and $B_{\text{alive}}$ appears arbitrarily; the expansion of the primary probability amplitude $\Psi(A_i, C_n)$ is optional. In fact, if this system has another observable $D$ whose eigenvalues are $D_{\text{fat}}$ and $D_{\text{thin}}$, this very same probability amplitude can be written as

$$\Psi(A_i, C_n) = \xi(A_i, D_{\text{fat}})\mu(D_{\text{fat}}, C_n) + \xi(A_i, D_{\text{thin}})\mu(D_{\text{thin}}, C_n)$$  \hspace{1cm} (12)$$

In both cases, the system state is given by $A_i$. If we turn our attention to the issue of whether the value of the observable $B$ will turn out to be $B_{\text{dead}}$ or $B_{\text{alive}}$ upon measurement, we should base our deliberations on the probability amplitudes $\chi(A_i, B_{\text{dead}})$ and $\chi(A_i, B_{\text{alive}})$, and not on the probability amplitude $\Psi(A_i, C_n)$.

To amplify on the issue of gratuitous expansion of a probability amplitude, consider the case of probability amplitudes connecting initial states of given energy to final states of specified position. These probability amplitudes are just the solutions of the time-independent Schrödinger equation. Since a great deal of quantum mechanics consists in solving this equation, these functions are known for many cases. We normally think of the eigenfunctions of the Schrödinger equation as expressing pure states. But if we decide to expand one of them using a third observable of the system, we end up with a "superposition state". A standard such expansion is

$$\psi_{E_n}(x) = \int e^{ikx}\Phi_{E_n}(p)dp$$  \hspace{1cm} (13)$$

where $\Phi_{E_n}(p)$ are the momentum space eigenfunctions for the system. If we apply the reasoning that leads to the cat paradox, we must say that the state of the system is determined by the quantum numbers for the momentum. Thus, we have the cat paradox and the problem of wave function reduction arising if we address the problem of measuring the momentum of the system with reference to this state rather than to the correct probability amplitude $\Phi_{E_n}(p)$. But of course in this case, it is accepted that the state of the system is described by the particular value of $E$ which obtains. In fact, we can clarify this expression by first setting $\psi_{E_n}(x) = \psi(E_n, x)$ so as to show that this is the probability amplitude for obtaining specified values of the position when the system state is initially characterised by the energy.
$E_n$. We next write $\Phi_{E_n}(p) = \Phi(E_n, p)$, because this is the probability amplitude for obtaining specified values of the momentum when the system state is initially characterised by the energy $E_n$. We then write $e^{ikx} = \chi(p, x)$; this is the probability amplitude for obtaining specified values of the position when the system is initially in the state of momentum $p$. We then have

$$\psi(E_n, x) = \int \Phi(E_n, p)\chi(p, x)dp$$  \hspace{1cm} (14)

Here we see that the system state is characterised only by $E_n$, and not by the label $p$ which occurs only because of the optional decision to use the eigenfunctions of $p$ to expand the function $\psi(E_n, x)$ (whose form is already known). There is neither wave function reduction nor cat paradox proceeding from this state. Where the form of the probability amplitude $\psi(E_n, x)$ is known, the expansion Eq. (14) is not needed, but of course, this is not to deny the important role that this expansion plays in the formal development of quantum theory.

Incidentally, we may use this discussion to reinforce the interpretation of eigenfunctions. The connection between the states involved and the parameters and variables appearing in an eigenvalue equation of which the eigenfunctions are solutions is well illustrated when we remind ourselves of the eigenvalue equations which the functions under discussion satisfy.

The function $\psi(E_n, x)$ is the solution of the time-independent Schrödinger equation

$$H(x)\psi(E_n, x) = E_n\psi(E_n, x)$$  \hspace{1cm} (15)

The function $\Phi(E_n, p)$ is the solution of the time-independent Schrödinger equation

$$H(p)\Phi(E_n, p) = E_n\Phi(E_n, p)$$  \hspace{1cm} (16)

while the function $e^{ikx} = \chi(p, x)$ is a solution of the eigenvalue equation

$$-i\hbar \frac{\partial}{\partial x}\chi(p, x) = p\chi(p, x)$$  \hspace{1cm} (17)

In each case, the eigenvalue corresponds to the initial state, while the variable in terms of which the differential operator is cast defines the final state.
5 Discussion

In this paper, we have argued that the cat paradox and the problem of wave function reduction are merely artifices of a misunderstanding of quantum theory. We have explained how they can be resolved, but our analysis has produced several questions that deserve to be touched upon.

One of the most profound consequences of the cat paradox is the argument that certain quantities have no objective existence before they are measured. According to the present approach, the probability amplitudes of quantum theory do not warrant this conclusion. We may best interpret them as giving probabilities for obtaining values of one observable upon measurement after a previous measurement had thrown the system in question into an eigenstate of one of its observables. The question whether or not the quantities being measured pre-exist is not of over-riding importance. But one thing is certain. The fact that for a given system only certain quantities can be measured shows that something or other inheres in the system and makes possible only the measurement of those quantities. Something therefore pre-exists that relates directly to those measurable properties of the system. Thus, for example, the fact that no spin projection can be measured for a particle of spin zero shows that something pre-exists that precludes the possibility of obtaining non-zero values of the spin projection.

We emphasize that, in our opinion, the cat paradox and the problem of wave function reduction are primarily a consequence of confusion and of mislabelling of quantum quantities. One reason for this, already pointed out by Landé [4], is the pre-eminent role given in quantum theory to solutions of the Schrödinger equation at the expense of other probability amplitudes. If solutions of the Schrödinger equation in coordinate (momentum) space are interpreted as probability amplitudes for obtaining eigenvalues of position (momentum) when the system is in a given energy eigenstate, then it is clear that for that system other probability amplitudes can be defined. If we do not appreciate this fact sufficiently, we are tempted to address all problems of measurement to the solutions of the Schrödinger equation, when we should sometimes seek for the answers elsewhere.

The solutions of the Schrödinger equation are only one possible set among those that can be used to define energy states. Any other probability amplitudes which have energy states for their initial states, but variables other than the position for the final observables, are also energy states. We acknowledge
that, in general, the solutions of the Schrödinger equation in coordinate space tend to be the most useful since in many cases we find ourselves interested in working out expectation values and other properties of quantities which are most conveniently expressed in terms of the coordinates. But this should not detract from the fact that other probability amplitudes corresponding to the energy as defining the initial eigenvalue also constitute energy states of a system.

We conclude the paper by remarking that this approach to quantum mechanics, which derives much of its utility from the principle that all probability amplitudes should be characterised by a well-defined initial state and a well-defined final state, is very powerful. For example, it has enabled us to make useful generalizations of angular momentum theory[9-15]. It very likely could be used to clarify more of the conceptual problems of quantum mechanics.

6 References