New Issues in the Inflationary Scenario

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We briefly review the arising of an inflationary phase in the Universe evolution in order to discuss an inhomogeneous cosmological solution in presence of a real self interacting scalar field minimally coupled to gravity in the region of a slow rolling potential plateau. During the inhomogeneous de Sitter phase the scalar field dominant term is a function of the spatial coordinates only. We apply this generic solution to the Coleman-Weinberg potential and to the Lemaitre-Tolman metric. This framework specialized nearby the FLRW model allows a classical origin for the inhomogeneous perturbations spectrum.

Keywords: inhomogeneous inflation, perturbation spectrum

I. GENERAL STATEMENTS

The homogeneous and isotropic Universe so far observed at sufficient large scales (of the order of 100 Mpc) inevitably breaks down when considering galaxies, clusters, etc. The early Universe, as testified by the extreme uniformity of the Cosmic Microwave Background Radiation (CMBR), exhibits a level of isotropy and homogeneity of order of $10^{-4}$; this is well tested up to $10^{-3} - 10^{-2}$ seconds of its life by the very good agreement between the abundances of light elements as predicted by the Standard Cosmological Model (SCM) and the one observed.

The theoretical framework of the Einstein equations applied to cosmology when following the evolution towards the initial singularity shows instability for density perturbations, eventually giving space to cosmological models differing from the SCM. Furthermore, reliable indications support the idea that the Universe evolved through an inflationary scenario and since that age it reached isotropy and homogeneity on the horizon scale at least.

Despite of these evidences favourable to the large scale homogeneity, many relevant features suggest that in the very early stage of evolution it had to be described by much more general inhomogeneous solutions of the Einstein equations.

With respect to this we remark the following two issues:

(i) the backward instability of the density perturbations to an isotropic and homogeneous Universe allows us to infer that, when approaching the initial singularity, the dynamics evolved as more general and complex models; thus the appearance of an oscillatory regime is expected in view of its general nature, i.e. because its perturbations correspond simply to redefine the spatial gradients involved in the Cauchy problem.

(ii) Since the Universe dynamics during the Planckian era underwent a quantum regime, then no symmetry restrictions can be imposed on the cosmological model; in fact, the wave functional of the Universe can provide information on the casual scale at most, and therefore the requirement for a global symmetry to hold would imply a large scale correlation of different horizons. Thus, the quantum evolution of the Universe is appropriately described only in terms of a generic inhomogeneous cosmological model.

A peculiar feature of the inflationary scenario consists of the violent expansion the Universe underwent during the de Sitter phase; indeed via such a mechanism the inflationary model provides a satisfactory explanation of the so-called horizons and flatness paradoxes by stretching the inhomogeneities at a very large scale. When referred to a (homogeneous and isotropic) Friedmann–Lemaitre–Robertson–Walker (FLRW) model, the de Sitter phase of the inflationary scenario rules out the small inhomogeneous perturbations so strongly, that it makes them unable to become seeds for the later structures formation. This picture emerges sharply within the inflationary paradigm and it is at the ground level of the statements according to which the cosmological perturbations arise from the scalar field quantum fluctuations.

Though this argument is well settled down and results very attractive even because the predicted quantum spectrum of inhomogeneities takes the Harrison-Zeldovich form, nevertheless the question whether it is possible, in a more general context, that classical inhomogeneities survive up to a level relevant for the origin of the actual Universe
Indeed here we show the behaviour of an inhomogeneous cosmological model which undergoes a de Sitter phase and show how such a general scheme allows the scalar field to retain, at the end of the exponential expansion, a generic inhomogeneous term to leading order (for connected topic see [14]). Thus our analysis provides relevant information either with respect to the morphology of an inhomogeneous inflationary model, either stating that the scalar field is characterized by an arbitrary spatial function which plays the role of its leading order. The paper is structured as follows: in Section II we review the basic equations of the SCM and the arising of the typical inflationary behaviour of the metric scale factor. In Section III we introduce the formalism of, in terms of the invariant function $\zeta$, to treat density perturbations in an inflationary scenario. In Section IV we will show the dynamics of an inhomogeneous cosmological model coupled with a real self-interacting scalar field. The solution concerns the phase when the scalar field slow rolls on a potential plateau and the Universe evolution is dominated by the effective cosmological constant related to the energy level over the true vacuum state of the theory. In Section V.A we specify such a framework to the Coleman-Weinberg expression for the potential, while in Section V.B we consider also the explicit spherically symmetric Tolman-Bondi metric. Then we apply in Section V.C the density perturbations estimate to the FLR W Universe and in Section VI we draw some conclusions. In the overall work, we can neglect the contribution of the ultra-relativistic matter which is relevant only for higher order terms and becomes more and more negligible as the exponential expansion develops (for a discussion of an inflationary scenario with relevant ultra-relativistic matter and different outcoming behaviour, see [1, 10]).

II. FROM THE SCM TO INFLATION

A. The Friedmann Equation

The FLRW metric is the most general spatially homogeneous and isotropic one which, in terms of the comoving coordinates $(t, r, \theta, \phi)$, has a line element reading as

$$ds^2 = dt^2 - R^2(t) \left( \frac{dr^2}{1 - \kappa r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right)$$

(1)

where the scale factor $R(t)$ is a generic function of time only and, for an appropriate rescaling of the coordinates, the factor $\kappa = 0, \pm 1$ distinguishes the sign of constant spatial curvature. The FLRW dynamics is reduced to the time dependence of the scale factor $R(t)$, solving the Einstein equations with a diagonal stress-energy tensor $T_{\mu \nu}$ for all the fields present (matter, radiation, etc.); for a perfect fluid it is characterized by a space-independent energy density $\rho(t)$ and pressure $p(t)$ as given by $T_{\mu \nu} = \text{diag}(\rho, -p, -p, -p)$; in this case the $0-0$ component of the Einstein equations (the Friedmann one) and the $i-i$ components read as

$$\frac{\dot{R}^2}{R^2} + \frac{\kappa}{R^2} = \frac{8\pi G}{3} \rho, \quad 2\frac{\dot{R}}{R} + \frac{\ddot{R}}{R} + \frac{\kappa}{R^2} = -8\pi G p,$$

(2)

respectively. The difference between them leads to

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p),$$

(3)

which can be solved for $\ddot{R}(t)$ once provided an equation of state, i.e. a relation between $\rho$ and $p$.

When the Universe was radiation-dominated, as in the early period, the radiation component provided the greatest contribution to its energy density and for a photon gas we have $p = \rho/3$. The present-time Universe is, on the contrary, matter-dominated: the “particles” (i.e. the galaxies) have only long-range (gravitational) interactions and can be treated as a pressureless gas (“dust”): the equation of state is $p = 0$.

The energy tensor appearing in the Einstein field equations describes the complete local energy due to all non-gravitational fields, while gravitational energy has a non-local contribution. An unambiguous formulation for such a non-local expression is found only in the expressions used at infinity for an asymptotically flat space-time. This is due to the property of the mass-energy term to be only one component of the energy-momentum tensor which can be reduced only in a peculiar case to a four-vector expression which can not be summed in a natural way.

Bearing in mind such difficulties, the conservation law $T^{\mu \nu} = 0$ leads to $d(\rho R^3) = -pd(R^3)$, i.e. the first law of
thermodynamics in an expanding Universe which, via the equation of state, leads to a differential relation for $\rho(R)$. A solution $\rho(R)$ for it in the Friedmann equation (2) corresponding to $\kappa = 0$ provides the following behaviours

\[ \rho \propto R^{-4}, \quad R \propto t^{1/2}; \quad \text{Matter: } \rho \propto R^{-3}, \quad R \propto t^{2/3} \]  

where as long as the Universe is not curvature-dominated, i.e. for sufficiently small values of $R$, the choice of $\kappa = 0$ is not relevant. Let us define the Hubble parameter $H \equiv \dot{R}/R$ and the critical density $\rho_c \equiv 3H^2/8\pi G$ in order to rewrite (2) as

\[ \frac{\kappa}{H^2 R^2} = \frac{\rho}{3H^2/8\pi G} - 1 \equiv \Omega - 1, \]  

where $\Omega$ is the ratio of the density to the critical one $\Omega \equiv \rho/\rho_c$; since $H^2 R^2$ is always positive, the relation between the sign of $\kappa$ and the sign of $(\Omega - 1)$ reads

\[ \begin{align*}
\text{Closed: } & \kappa = +1 \Rightarrow \Omega > 1 ; \\
\text{Flat: } & \kappa = 0 \Rightarrow \Omega = 1 ; \\
\text{Open: } & \kappa = -1 \Rightarrow \Omega < 1
\end{align*} \]  

B. Shortcomings of the SM: Horizon and Flatness Paradoxes

Despite the simplicity of the Friedmann solution some paradoxes occur when taking into account the problem of initial conditions. The observed Universe has to match very specific physical conditions in the very early epoch, but [18] showed that the set of initial data that can evolve to a Universe similar to the present one is of zero measure and the standard model tells nothing about initial conditions.

Flatness.

The value of $\Omega$ at present time $\Omega_0$ is related to the radius of curvature and to the density by Eq. (5). The observational data restrict $\Omega_0$ to be of order of few units and consequently $R_{\text{curv}} \sim H_0^{-1}$ and $\rho_0 \sim \rho_c$. By eliminating $H^2$ via the two equations in (5) one finds the dependence of $\Omega$ on time as approaching unity for decreasing time as

\[ |\Omega(10^{-43} \text{ sec}) - 1| \lesssim O(10^{-60}), \quad |\Omega(1 \text{ sec}) - 1| \lesssim O(10^{-16}), \]  

and for the radius of curvature this would imply

\[ R_{\text{curv}}(10^{-43} \text{ sec}) \gtrsim 10^{30} H^{-1}, \quad R_{\text{curv}}(1 \text{ sec}) \gtrsim 10^8 H^{-1}. \]  

These estimates imply that the FLRW model was characterized at the beginning by very special initial data in order to evolve in the Universe we observe today as, for example, a primeval deviation from the critical density when the temperature was $T = 10^{17}$ GeV (at the Grand Unification epoch) given by

\[ \left| \frac{\rho - \rho_c}{\rho} \right|_{T=10^{17}\text{GeV}} < 10^{-55}. \]  

A flat Universe today requires $\Omega$ at ancient time close to unity up to a part in $10^{55}$. A little displacement from flatness at the beginning – for example $10^{-30}$ – would produce an actual Universe either very open or very closed, so that $\Omega = 1$ is a very unstable condition: this is the flatness problem. The natural time scale for cosmology is the Planck time ($\sim 10^{-43}$ sec): in a time of this order a typical closed Universe would reach the maximum size while an open one would become curvature dominated. The actual Universe has survived $10^{60}$ Planck times without neither recollapsing nor becoming curvature dominated.

Horizon.

The other important problem arising from the SCM regards the explanation of the smoothness of the CMBR: the entire observable Universe could not have been in causal contact at remote time and eventually smoothed out, as a consequence of particle horizon. In fact, a light signal (as a sample of the fastest possible interaction) emitted at $t = 0$ travelled during a time $t$ the physical distance

\[ l(t) = R(t) \int_0^t R(t')^{-1} dt' = 2t \]  

(10)
in a radiation-dominated Universe with $R \propto t^{1/2}$, measuring the physical horizon size, i.e., the linear size of the greatest region causally connected at time $t$. The distance [11] has to be compared with the radius $L(t)$ of the region which will evolve in our currently observed part of the Universe. Conservation of entropy for $s \propto T^3$ gives

$$L(t) = (s_0/s(t))^{1/3} L_0,$$

where $s_0$ is the present entropy density and $L_0 \sim H^{-1} \simeq 10^{10}$ years is the radius of the observed Universe. An estimate of the ratio of the volumes provides, as $T \sim 10^{17}$ GeV,

$$l^3/L^3_{T=10^{17}\text{GeV}} \sim 10^{-83}.\tag{12}$$

The actual observable Universe is composed of several regions which have \textit{not} been in causal contact for the most part of their evolution, preventing an explanation about the present days Universe smoothness. In particular, the spectrum of the CMBR is uniform up to $10^{-4}$. Moreover, we have at the time of recombination, i.e., when the photons of the CMBR last scattered, the ratio $l^3/L^3 \sim 10^5$: the present Hubble volume consists of about $10^5$ causally disconnected regions at recombination and no process could have smoothed out the temperature differences between these regions without violating causality. The particle horizon at recombination subtends an angle of only $0.8^\circ$ in the sky today, while the CMBR is uniform across the sky.

### C. The Inflationary Paradigm

The basic ideas of the theory of inflation rely firstly on the original work [4], i.e. the old inflation, which provides a phase in the Universe evolution of exponential expansion; then the formulation of new inflation by [15] introduced the slow-rolling phase in inflationary dynamics; finally, many models have sprung from the original theory (for a criticism, see [16]).

In [1] is described a scenario capable of avoiding the horizon and flatness problems: both paradoxes would disappear dropping the assumption of adiabaticity. In such a case, the entropy per comoving volume $s$ would be related as $s_0 = Zs_{\text{early}},$ where $s_0$ and $s_{\text{early}}$ refer to the values at present and at very early times, for example at $T = T_0 = 10^{17}$ GeV, and $Z$ is some large factor. The estimate of the value of $|\rho - \rho_0|/\rho$ is then multiplied by a factor $Z^2$ and would be of the order of unity if $Z > 3 \cdot 10^{27}$, getting rid of the flatness problem. The right-hand side of (11) is multiplied by $Z^{-1}$: for any given temperature, the length scale of the early Universe is smaller by a factor $Z$ than previously evaluated, and for $Z$ sufficiently large the initial region which has evolved in our observed one would have been smaller than the horizon size at that time; for $Z > 5 \cdot 10^{27}$ the horizon problem disappears.

Making some \textit{ad hoc} assumptions, the model accounts for the horizon and flatness paradoxes while a suitable theory needs a physical process capable of such a large entropy production. A simple solution relies on the assumption that at very early times the energy density of the Universe was dominated by a scalar field $\phi(\vec{x}, t),$ i.e. $\rho = \rho_\phi + \rho_{\text{rad}} + \rho_{\text{mat}} + \ldots$ with $\rho_\phi \gg \rho_{\text{rad}}, \rho_{\text{mat}},$ etc and hence $\rho \simeq \rho_\phi.$

The quantum field theory Lagrangian density in this case and the corresponding stress-energy tensor read as

$$\mathcal{L} = \partial^\mu \phi \partial_\mu \phi/2 - V(\phi), \quad T^\mu_\nu = \partial^\mu \phi \partial_\nu \phi - \mathcal{L} \delta^\mu_\nu,\tag{13}$$

respectively, and hence for a perfect fluid leads to

$$\rho_\phi = \dot{\phi}^2/2 + V(\phi) + R^{-2}/2\nabla^2 \phi, \quad p_\phi = \dot{\phi}^2/2 - V(\phi) - R^{-2}/6\nabla^2 \phi.\tag{14}$$

Since spatial homogeneity implies a slow variation of $\phi$ with position, the spatial gradients are negligible and the ratio $\omega = p/\rho$ reads

$$\frac{p_\phi}{\rho_\phi} \simeq \frac{\dot{\phi}^2/2 + V(\phi)}{\dot{\phi}^2/2 - V(\phi)}.$$

For a field at a minimum of the potential $\dot{\phi} = 0$ and [15] becomes an equation of state as $p_\phi = -\rho_\phi,$ giving rise to a phase of exponential growth of $R \propto e^{Ht},$ where the Hubble parameter $H$ remains constant: this is the inflationary or de Sitter phase.

A very different evolution arises for a field in a thermal bath, in which case the coupling can be summarized by adding a term $-(1/2)\lambda T^2 \dot{\phi}^2$ to the Lagrangian. The potential $V(\phi)$ is replaced by the \textit{finite-temperature} effective potential

$$V_T(\phi) = V(\phi) + \lambda T^2 \phi^2/2.\tag{16}$$
This model can recover the Standard Cosmology via a phase transition of the scalar field between a metastable state (false minimum) and the true vacuum; the reheating due to oscillations around this state are damped by particle decay and, when the corresponding products thermalize, the Universe is reheated, and inflation comes to an end. This process (we will not discuss here the details) nevertheless leaves some problems open as, schematically: (i) inflation never ends, due to smallness of the tunnelling transition rate between the two minima; (ii) the phase transition is never completed; (iii) the discontinuous process of bubble nucleation (exponential expansion of vacuum phases) via quantum tunnelling should produce a lot of inhomogeneities which aren’t actually observed.

D. New Inflation: the Slow Rolling Model

In 1982, both [15] proposed a variant of Guth’s model, now referred to as new inflation or slow-rolling inflation, in order to avoid the shortcomings of the old inflation. Their original idea considered a different mechanism of symmetry breaking, the so-called Coleman-Weinberg (CW) one, based on the gauge boson potential with a finite-temperature effective mass $m_T \equiv \sqrt{-m^2 + XT^2}$ reading as

$$V_T(\phi) = \frac{B\sigma^4}{2} + B\phi^4 \left[ \ln \left( \frac{\phi^2}{\sigma^2} \right) - \frac{1}{2} \right] + \frac{1}{2} m_T^2 \phi^2,$$

where $B \approx 10^{-3}$ is connected to the fundamental constants of the theory, while $\sigma \approx 2 \cdot 10^{15}$ GeV gives the energy associated to the symmetry breaking process.

The quantity $m_T^2$ can be used to parametrize the potential [14]:

1. when $m_T^2 > 0$, the point $\phi = 0$ is a minimum of the potential, while when $m_T^2 < 0$ it is a maximum;

2. when $m_T^2 < 4\sigma^2/e \approx 1.5\sigma^2$, a second minimum develops for some $\bar{\phi} > 0$; initially this minimum is higher than the one at 0, but when $m_T$ becomes lower than a certain value $m_T^* (0 < m_T^* < 1.5\sigma^2)$ it will eventually become the global minimum of the potential.

If at some initial time the $\phi$-field is trapped in the minimum at $\phi = 0$, the true minimum can disappear as the temperature lowers: as $m_T$ approaches 0, the potential barrier becomes low and can be overcome by thermal (weakly first order process) tunnelling, i.e. due to classical fluctuations of the $\phi$ field around its minimum; the barrier can disappear completely when $m_T = 0$ (second order process). The phase transition doesn’t proceed via a quantum tunnelling – a very discontinuous and a strongly first order process. The transition occurs rather smoothly, avoiding the formation of undesired inhomogeneities.

When the $\phi$-field has passed the barrier (if any), it begins to evolve towards its true minimum. The model [17] has the feature that if the coefficient of the logarithmic term is sufficiently high, the potential is very flat around 0 and the field $\phi$ “slow rolls” in the true vacuum state, rather than falling abruptly: during this phase the inflation takes place, lasting enough to produce the required supercooling. When the field reaches the minimum, it begins to oscillate around it thus originating the reheating.

The problems of Guth’s originary model are skipped moving the inflationary phase after the field has escaped the false vacuum state, by adding the slow-rolling phase.

Virtually all models of inflation are based upon this principle.

III. EVOLUTION OF DENSITY PERTURBATIONS

During the de Sitter phase the Hubble radius $H^{-1}$ is roughly constant while it increases during the FLRW phase. The scale factor $R$ undergoes an $e$-folding in either case so that microphysics (and then interaction between different close points) can operate only at scale less then $O(H^{-1})$. Hence, during the late inflationary epoch the evolution of the perturbations is essentially scale independent, since nothing can alter the amplitude of a real physical perturbation when its scale is larger then $H^{-1}$.

The perturbed metric tensor gives place to the equations [15]

$$[k^2 + 12\pi G(\rho_0 + p_0)R^2]a = -k^2(1 + 3\epsilon^2)h, \quad (\dot{h} - Ha) + 3H(\dot{h} - Ha) = 0,$$

where $h$ and $a$ denote two functions measuring the metric perturbations independent of the choice of the spatial coordinates, $\rho_0$ and $p_0$ have to be considered as the background energy density and pressure and $\epsilon^2 = dp_0/d\rho_0$ is
The equations (18) permit to identify, in the uniform Hubble constant gauge, the amplitude
\[ \zeta \equiv \hbar \left[ 1 + \frac{k^2}{12\pi G(\rho_0 + p_0)R^2} \right] \]  
(19)
as a nearly time-independent quantity. Hence, by the second of (18), there are two independent modes: a decaying one for which \( \dot{h} - H_a \sim \exp(-3HT) \), clearly negligible, and a constant one for which \( \dot{h} = Ha \). This second mode, with a few calculations, during inflation is expressed by the time-independence of the function (19) \( \zeta = \zeta_f = \text{const.} \) and has the form \( \zeta = \delta \rho / (\rho + p) \); its value has to match the final one \( \zeta_f \) of the FLRW epoch.

### IV. INHOMOGENEOUS INFLATIONARY MODEL

In a synchronous reference, the generic line element of a cosmological model takes the form (in units \( c = \hbar = 1 \))
\[ ds^2 = dt^2 - \gamma_{\alpha\beta}(t, x^\mu)dx^\alpha dx^\beta, \quad \alpha, \beta, \mu = 1, 2, 3 \]  
(20)
where \( \gamma_{\alpha\beta}(t, x^\mu) \) is the three-dimensional metric tensor describing the geometry of the spatial slices. The Einstein equations in the presence of a self interacting scalar field \( \{\phi(t, x^\mu), V(\phi)\} \) read explicitly
\[ \frac{1}{2} \partial_t k_\alpha^\alpha + \frac{1}{4} k_\beta^\beta k_\alpha^\alpha = \chi \left[ - (\partial_\mu \phi)^2 + V(\phi) \right] \]  
(21a)
\[ \frac{1}{2} (k_\alpha^\beta \partial_\alpha \phi - k_\beta^\alpha \partial_\beta \phi) = \chi (\partial_\alpha \phi \partial_\beta \phi) \]  
(21b)
\[ \frac{1}{2} \sqrt{\gamma} \left[ \partial^\mu (\sqrt{\gamma} k_\alpha^\mu) + P_\alpha^\beta \right] = \chi \left[ \gamma^{\beta\mu} \partial_\alpha \phi \partial_\mu \phi + V(\phi) \delta_\alpha^\beta \right], \]  
(21c)
where \( \chi = 8\pi G \), the three-dimensional Ricci tensor \( P_\alpha^\beta \) is constructed via \( \gamma_{\alpha\beta}, \gamma \equiv \det \gamma_{\alpha\beta}, k_{\alpha\beta} \equiv \partial_t \gamma_{\alpha\beta}. \) The dynamics of the scalar field \( \phi(t, x^\mu) \) is coupled to the system (21) and is described by the equation
\[ \partial_t \phi + \frac{1}{2} k_\alpha^\alpha \partial_t \phi - \gamma^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \phi + \frac{dV}{d\phi} = 0. \]  
(22)

In what follows we will consider the three fundamental statements:

(i) the three metric tensor is taken in the general factorized form \( \gamma_{\alpha\beta}(t, x^\mu) = \Gamma^2(t, x^\mu) \xi_{\alpha\beta}(x^\mu) \), where \( \xi_{\alpha\beta} \) is a generic symmetric three-tensor and therefore contains six arbitrary functions of the spatial coordinates, while \( \Gamma \) is to be determined by the dynamics;

(ii) the self interacting scalar field dynamics is described by a potential term which satisfies all the features of an inflationary one, i.e. a symmetry breaking configuration characterized by a relevant plateau region;

(iii) the inflationary solution is constructed under the assumptions
\[ 1/2 (\partial_t \phi)^2 \ll V(\phi) \]  
(23a)
\[ | \partial_t \phi | \ll | k_\alpha^\alpha \partial_t \phi |. \]  
(23b)

Our analysis, following [12], concerns the evolution of the cosmological model when the scalar field slow rolls on the plateau and the corresponding potential term is described as
\[ V(\phi) = \Lambda_0 - \lambda U(\phi), \]  
(24)
where \( \Lambda_0 \) behaves as an effective cosmological constant of the order \( 10^{15} - 10^{16} \) GeV and \( \lambda (\ll 1) \) is a coupling constant associated to the perturbation \( U(\phi) \).

Since the scalar field moves on a plateau almost flat, we infer that to lowest order of approximation \( \phi(t, x^\gamma) \sim \alpha(x^\gamma) \) (see below) and therefore the potential reduces to a space-dependent effective cosmological constant
\[ \Lambda(x^\gamma) \equiv \Lambda_0 - \lambda U(\alpha(x^\gamma)). \]  
(25)
We search for a solution of the dynamical equation (28) in the form

\[ 3 \partial_t \ln \Gamma + 3 (\partial_t \ln \Gamma)^2 = \chi \Lambda(x^\gamma), \quad (\partial_t \ln \Gamma)\delta_\alpha^\beta + 3 (\partial_t \ln \Gamma)^2 \delta_\alpha^\beta = \chi \Lambda(x^\gamma) \delta_\alpha^\beta, \]

respectively. A simultaneous solution for \( \Gamma \) of both equations (26) takes the form

\[ \Gamma(x^\gamma) = \Gamma_0(x^\gamma) \exp \left[ \sqrt{\chi \Lambda(x^\gamma)} / 3 (t - t_0) \right], \]

where \( \Gamma_0(x^\gamma) \) is an integration function while \( t_0 \) is a given initial instant of time for the inflationary scenario. Under the same assumptions and taking into account (27) for \( \Gamma \), the scalar field equation (22) rewrites as

\[ 3H(x^\gamma) \partial_t \phi - \lambda W(\phi) = 0, \quad H(x^\gamma) = \partial_t \ln \Gamma = \sqrt{\chi \Lambda(x^\gamma) / 3}, \quad W(\phi) = dU / d\phi. \]

We search for a solution of the dynamical equation (28) in the form

\[ \phi(t, x^\gamma) = \alpha(x^\gamma) + \beta(x^\gamma)(t - t_0). \]

Inserting expression (29) in (28) and considering it to the lowest order, it is possible to express \( \beta \) in terms of \( \alpha \) as

\[ \beta = \frac{\lambda W(\alpha)}{\sqrt{3 \chi \Lambda_0 - \lambda U(\alpha)}}. \]

Of course the validity of solution (30) takes place in the limit

\[ t - t_0 \ll \left| \frac{\alpha}{\beta} \right| = \left| \frac{\alpha}{W(\alpha)} \sqrt{3 \chi \Lambda_0 - \frac{U(\alpha)}{\lambda}} \right| \]

where the ratio \( \Lambda_0 / \lambda^2 \) takes in general very large values.

The \( 0 - \alpha \) component (24), in view of (26) and (29) through (30), reduces to \( \partial_t (\Lambda + \lambda U) = 0 \), which for \( \Lambda(x^\gamma) \) is an identity as (26).

The spatial gradients, either of the three-metric field either of the scalar one, behave as \( \Gamma^{-2} \) and decay exponentially.

If we take into account the coordinate characteristic lengths \( L \) and \( l \) for the inhomogeneity scales regarding the functions \( \Gamma_0 \) and \( \xi_{\alpha\beta} \), i.e.

\[ \partial_\gamma \Gamma_0 \sim \Gamma_0 / L, \quad \partial_\gamma \xi_{\alpha\beta} \sim \xi_{\alpha\beta} / l, \]

respectively, negligibility of the spatial gradients at the beginning leads to the inequalities for the physical quantities

\[ \Gamma_0 l = l_{\text{phys}} \gg H^{-1}, \quad \Gamma_0 L = L_{\text{phys}} \gg H^{-1}. \]

These conditions state that all the inhomogeneities have to be much greater then the physical horizon \( H^{-1} \).

Negligibility of the spatial gradients at the beginning of inflation is required (as well known) by the existence of the de Sitter phase itself; however, spatial gradients having a passive dynamical role allow to deal with a fully inhomogeneous solution: space point dynamically decouple to leading order.

The condition (24) is naturally satisfied since states that the dominant contribution in \( \Lambda(x^\gamma) \) is provided by \( \Lambda_0 \), i.e. \( \lambda U(\alpha) \ll \Lambda_0 \); the same can be said for (26).

The only important restriction on the spatial function \( \alpha(x^\gamma) \) is \( |\alpha| \ll |U^{-1} (\Lambda_0 / \lambda)| \).

A satisfactory exponential expansion able to overcome the SCM shortcomings, i.e. to freeze the density fluctuations between the beginning and the end of the inflation, requires that in each space point the condition for the e-folding \( H(t_f - t_i) \sim O(10^2) \) holds, where \( t_i \) and \( t_f \) denote the instants when the de Sitter phase starts and ends, respectively. We may take \( t_i \equiv t_0 \) and hence for \( t_f \) we have, by (28) and (30),

\[ H(t_f - t_i) \ll (\Lambda_0 / \lambda) \left| \alpha / W(\alpha) \right|. \]

Since \( \Lambda_0 / \lambda \) is a very large quantity, no serious restrictions appear for the e-folding of the model.

The obtained solution possesses a very general feature: in fact, once satisfied all the dynamical equations, still eight arbitrary spatial functions remain, i.e. six for \( \xi_{\alpha\beta}(x^\gamma) \), and then \( \Gamma_0(x^\gamma), \alpha(x^\gamma) \). However, taking into account the possibility to choose an arbitrary gauge via the set of the spatial coordinates, we have to kill three degrees of freedom; hence five physically arbitrary functions finally remain: four corresponding to gravity degrees of freedom and one related to the scalar field.

This picture corresponds exactly to the allowance of specifying a generic Cauchy problem for the gravitational field, on a spatial non-singular hypersurface, nevertheless one degree of freedom of the scalar field is lost against the full generality.
V. APPLICATIONS

A. Coleman–Weinberg Model

Let us specify this solution in the case of the vanishing mass zero-temperature CW potential \[ V(\phi) = \frac{B\sigma^4}{2} + B\phi^4 \left[ \ln \left( \frac{\phi^2}{\sigma^2} \right) - \frac{1}{2} \right] . \] (35)

In the region \(|\phi| \ll |\sigma|\) the potential (35) approaches a plateau profile similar to (24) and acquires the form

\[ V(\phi) \simeq \frac{B\sigma^4}{2} - \lambda\phi^4/4, \quad \lambda \simeq 80B \simeq 0.1 . \] (36)

This is reducible to (24) by setting \(\Lambda_0 = \frac{B\sigma^4}{2}, U(\phi) = \frac{\phi^4}{4}\) and \(W(\phi) = \frac{\phi^3}{3}\). Hence the relations (30) rewrites as

\[ \beta = \lambda \alpha^3/3H \]

while the inequality for \(\alpha\) is equivalent to fulfil the initial assumption \(\Lambda_0 \gg \lambda U(\alpha) \sim \lambda \alpha^4/4\), that is

\[ |\alpha| \ll \sqrt[4]{\Lambda_0/\lambda} \sim \sigma . \] (37)

B. Lemaitre–Tolman Spherically Symmetric Metric

It is interesting to consider the framework developed so far when in presence of a spherically symmetric line element written as

\[ ds^2 = dt^2 - e^{2\pi}dr^2 - e^{2\psi}d\Omega \] (38)

where \(\pi\) and \(\psi\) are functions of \(t\) and \(r\) only. The corresponding Einstein equations read as

\[ (\dot{\psi})^2 + 2\dot{\pi}\dot{\psi} + e^{-2\psi} = e^{-2\pi} [2\psi'' + 3(\psi')^2 - 2\pi'\psi'] = \frac{\chi}{2} \left[ (\partial_t \phi)^2 + (\partial_r \phi)^2 e^{-2\pi} + V(\phi) \right] \] (39a)

\[ 2\dot{\psi} + 3(\dot{\psi})^2 + e^{-2\psi} = e^{-2\pi} = -\frac{\chi}{2} \left[ (\partial_t \phi)^2 + (\partial_r \phi)^2 e^{-2\pi} - V(\phi) \right] \] (39b)

\[ 2\dot{\psi}' + 2\psi'\dot{\psi} - 2\dot{\pi}\psi' = \chi \dot{\phi} \phi' \] (39c)

while the scalar field \(\phi(t, x^\gamma)\) is described by the equation

\[ \partial_t \phi + \left[ \dot{\pi} + 2\dot{\psi} \right] \partial_r \phi - \left[ \phi'' + \phi' (\pi' + 2\psi') \right] e^{-2\pi} + \frac{dV}{d\phi} = 0 . \] (40)

Once defined \(e^{\psi} = rm(r,t)\), without entering in the details of the derivation [21], we can find again for large values of \(m\)

\[ \sqrt{\frac{2\chi\Lambda}{3}m} = e^{\sqrt{\chi\Lambda/6} (t-t_0)} , \] (41)

which admits linear solution for the scalar field similar to (30), satisfying the set of Einstein equations under the same assumptions of the previous discussion.

C. Towards FLRW Universe

The conditions [33] state that the validity of the inhomogeneous inflationary scenario discussed so far requires the inhomogeneous scales to be out of the horizon when inflation starts. The situation is different when treating the small perturbations to the FLRW case; in fact, the negligibility of the spatial curvature corresponds to require the radius of curvature of the Universe to be much greater than the physical horizon, the inhomogeneous terms being small in amplitude. To this end, let us consider the three-metric

\[ \gamma_{\alpha\beta} = \Gamma(t, \varphi')^2 \left[ h_{\alpha\beta} + (t - t_0)\delta_{\alpha\beta} (\varphi'') \right] , \] (42)
where $h_{\alpha\beta}$ denotes the FLRW spatial part of the three metric ($\{\varphi^\mu\}$ are the three usual angular coordinates) and $\delta\theta_{\alpha\beta}$ denote a small inhomogeneous perturbation. The Einstein equations (24) coupled to the scalar field dynamics (22) on the plateau admit, leading to order in the inhomogeneities, the solution

$$
\Gamma = \Gamma_0 e^{H(t-t_0)} , \quad H = H_0 - \frac{\delta\theta}{6} , \quad H_0 = \sigma^2 \sqrt{\frac{\chi B}{6}} ,
$$

(43a)

$$
\phi = a_0 \left[ 1 + \frac{\lambda \alpha_0^2}{3H_0} (t-t_0) \right] + \frac{\delta\theta}{3\chi} \left[ 1 + \frac{\lambda \alpha_0^2}{H_0} (t-t_0) \right] , \quad \delta\theta_{\alpha\beta} = \frac{\delta\theta}{3} h_{\alpha\beta} ,
$$

(43b)

where $t_0$ and $\Gamma_0$ are constants. The solution holds and provides the correct $e$-folding of order $O(10^2)$ when the inequalities

$$
t - t_0 \ll 3H_0/\lambda\alpha_0^2 , \quad \alpha_0 \ll O \left( 10^{-1} \sqrt{\lambda_0/\lambda} \right) ,
$$

(44a)

take place, the first one ensuring that the dominant term of the scalar field remains the time-independent one during the de Sitter phase, while the second one accounts for the $e$-folding; the spatial gradients in the Einstein field equations behave as (see, for instance, the RHS of Eq. (21c)) $\Gamma^{-2} \partial_\alpha \delta\theta_{\alpha\beta} \sim \Gamma^{-2} \delta^2 / l^2$ which have to be compared to the potential term $V \sim \lambda \alpha^2$ and therefore are negligible, while the case for the scalar field is similar. Hence we get

$$
\Gamma_0 l = l_{\text{phys}} \gg H_0^{-1} \delta , \quad l_{\text{phys}} \gg \delta / \sqrt{\lambda\alpha_0^3} ,
$$

(44c)

respectively, where $\delta \ll H_0/100$ and $l$ in $l_{\text{phys}}$ denote the characteristic amplitude and length, of the arbitrary function $\delta\theta$, trace of the tensor $\partial_\alpha \delta\theta_{\alpha\beta}$. When inflation starts the inhomogeneous scales can be inside the physical horizon $H_0^{-1}$.

The physical implications on the density perturbation spectrum of such a nearly homogeneous model rely on the dominant behaviour of the potential term over the energy density $\rho_{\phi}$ associated to the scalar field during the de Sitter phase and therefore

$$
\Delta \equiv \left| \frac{\delta\rho}{\rho_{\phi}} \right| \sim \left| \frac{d \ln V}{d \phi} \right| \delta\phi \sim \left| \frac{\lambda}{\Lambda_0} W(\alpha_0) \delta\alpha \right| ,
$$

(45)

where $\delta\alpha = \delta\theta / (3\chi)$ for our scalar field solution (43). In particular, in the CW case, Equation (45) reduces to

$$
\Delta_{\text{CW}} \approx \frac{50}{\sigma^4 \alpha_0^3} \frac{\delta\theta}{\chi} ,
$$

(46)

However, to compute the physically relevant perturbations after the scales re-entry in the horizon, let us evaluate the gauge invariant quantity $\zeta$ which has now the form (19)

$$
\zeta = \frac{\delta\rho}{\rho + p} \approx \frac{\delta\theta}{W(\alpha_0)} \frac{\Lambda_0}{\lambda} ,
$$

(47)

when the perturbations leave the horizon and $\rho + p = (\partial_\phi^2)^2$ (see Eq. 14); in the CW case it reads as

$$
\zeta_{\text{CW}} = \frac{\sigma^4}{160} \frac{\delta\theta}{\alpha_0^3} .
$$

(48)

Since $\zeta$ remains constant during the super-horizon evolution of the perturbations, then at the re-entry to the causal scale in the matter-dominated era we get $\zeta_{\text{MD}} \sim \delta\rho / \rho \sim \zeta_{\text{CW}}$.

By restoring physical units and assuming $\alpha_0 \lesssim 10^{-4} \pi / \sqrt{\lambda c}$ in agreement with (19), then it is required $\delta\alpha / \alpha_0 \lesssim 10^{-2}$ in order to obtain perturbations $\delta\rho / \rho \sim 10^{-4}$ at the horizon re-entry during the matter-dominated age.

Hence the expression (45) explains how the perturbation spectrum after the de Sitter phase can still arise from classical inhomogeneous terms. Indeed, the function $\delta\theta(\varphi^\mu)$ is an arbitrary one and can be chosen for it a Harrison–Zeldovich spectrum by assigning its Fourier transform as

$$
|\delta\alpha(k)|^2 \propto \text{const.} / k^3 ;
$$

(49)

such a spectrum has to hold for $k \ll \Gamma_0 / (H_0^{-1} \delta)$.

Thus, the pre-inflationary inhomogeneities of the scalar field remain almost of the same amplitude during the de Sitter phase as a consequence of the linear form of the scalar field solution (22). We get that the Harrison–Zeldovich spectrum can be a pre-inflationary picture of the density perturbations and it survives to the de Sitter phase, becoming a classical seed for structure formation. The existence of such a classical spectrum is not related with the quantum fluctuations of the scalar field, whose effect is an independent contribution to the classical one.
VI. CONCLUDING REMARKS

The merit of our analysis relies on having provided a dynamical framework within which classical inhomogeneous perturbations to a real scalar field minimally coupled with gravity can survive even after that the de Sitter expansion of the Universe stretched the geometry; the key feature underlying this result consists (i) of constructing an inhomogeneous model for which the leading order of the scalar field is provided by a spatial function and then (ii) of showing how the very general case contains as a limit a model close to the FLRW one. Moreover such results have been extended to the spherically symmetric Lemaitre–Tolman metric.

It is relevant to remark that the metric tensor (42) seems of the same form as the one considered in [10]; however in the present paper the function \( \eta(t) \) appearing in the previous work is linear in time and does not decay exponentially. The different behaviour relies on the negligibility of the matter with respect to the scalar field which is at the ground of the present analysis. We are here assuming the dynamics of \( \eta(t) \) to be driven by the scalar field alone, instead of by the ultra-relativistic matter. This situation corresponds to an initial conditions for which the scalar field dominates over the ultra-relativistic matter when inflation starts and this is the reason for the resulting different issues of the two analyses.

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