Symmetries of Non-relativistic Field Theories on the Non-Commutative Plane

P. A. Horváth (a), L. Martina (b), P. C. Stichel (c)

(a) Lab. Math. Phys. Théor., Université de Tours, P. Grandmont, F-37 200 TOURS (France)
(b) Dip. Fisica and Sez. INFN, Università di Lecce, v. Arnesano, CP. 193, I-73 100 LECCE (Italy).
(c) An der Krebskuhle 21, D-33 619 BIELEFELD (Germany)

June 4, 2005

Abstract

New developments on non-relativistic field theoretical models on the non commutative plane are reviewed. It is shown in particular that Galilean invariance strongly restricts the admissible interactions. Moreover, if a scalar field is coupled to a Chern-Simons gauge field, a geometrical phase emerges for vortex-like solutions, transformed by Galilei boosts.

hep-th/0411139

Non commutative solitons - finite action solutions of the classical equations of motion of noncommutative field theories - have attracted great interest in the last few years, mainly in connection with strings and brane dynamics [1]. However, at very low energy (i.e. in condensed matter physics), the analysis of the Fractional Quantum Hall Effect (FQHE) [2], has suggested that the phenomenology can be expressed in terms of quasi-particles, related to states of strongly correlated electrons in the Lowest Landau Level. These quasi-particles are imbedded into an effective gauge connection of entirely quantum mechanical nature, related to Berry’s phase [3].

As a consequence, the quasi-particles (anyons) have fractional statistics. In a field theoretical approach, the topological origin of this effect is encoded into a non-dynamical vector potential \( \vec{A} \) which reproduces an Abelian Chern-Simons term in the action, minimally interacting with the massive field \( \psi \) of a Landau-Ginzburg theory. The original quasiparticles re-emerge as vortex-like solutions, which carry fractional electric charge and unit magnetic flux.

On the other hand, in the limit of vanishing mass, \( m_e \rightarrow 0 \), the classical Lagrangian for a system of interacting charged particles (electrons) in the plane becomes first order in time derivatives, providing us with hamiltonian equations of motion for non commuting variables [4]. Analogous Lagrangian (and also hamiltonian [5]) equations can be obtained for the mean values of the position and momentum for wave packets in magnetic Bloch bands [5]. A Mead-Berry connection [7], depending on the quasi-momentum, appears also in this case. It again encodes a geometric phase into the semiclassical description of the microscopic system.

Similar, simplified particle models in the plane were introduced on a purely axiomatic mechanical setting in [8], resorting to an acceleration-dependent Lagrangian, and in [9], where a formalism à la Souriau was used. More interestingly, fine-tuning the magnetic field with respect to a new parameter \( k \) specific to the particle provides us with equations of motion, which reduce to the Hall equations for a charged particle in perpendicular electric and a magnetic fields.

Three phenomena arise: i) the dimensional reduction of the phase space, ii) the non commutativity of the reduced ”configurational” variables, iii) the parameter $k$ generates a second central extension of the Galilei group in the plane. This can be seen by computing the Poisson brackets of the boost generators

$$\{G_1, G_2\} = k.$$  \hfill (1)

Some time ago it has been proved in fact [10] that the Galilei group in 2+1 dimensions admits a 2-dimensional central extension, in contrast with the usual one, associated with the particle mass. However, its physical meaning remained obscure for a long time and the result was considered a mere mathematical curiosity. Thus, in order to reproduce the properties of the Laughlin quasiparticles, it is natural to consider a Chern-Simons field theory on a non commutative plane (NC-plane) as a better description of the FQHE. This idea was stressed in [11]. The NC-plane is represented as the $C^*$-algebra of the bounded operators generated by the Heisenberg algebra

$$[\hat{x}_i, \hat{x}_j]= -i \epsilon_{ij} \theta, \quad (i, j = 1, 2)$$  \hfill (2)

where $\theta$ is a characteristic scalar parameter, playing the same role as $\hbar$ in the phase space, and $\hat{\epsilon} = (\epsilon_{ij})$ represents the antisymmetric tensor in two dimensions.

There exists a one-to-one mapping between the space $S$ of the Schwartzian functions $\psi$ on $\mathbb{R}^2$ and the $C^*$-algebra. It is defined by the Weyl quantization formula

$$\hat{\psi} = \int \psi(\vec{x}) \hat{\Delta}(\vec{x}) d^2(\vec{x}), \quad \text{where} \hat{\Delta}(\vec{x}) = \frac{1}{(2\pi)^2} \int e^{i \vec{k} \cdot (\vec{x} - \vec{y})} d^2 \vec{k}$$

is the point-like quantizer operator. The inverse is given by the Wigner de-quantization formula

$$\psi(\vec{x}) = \text{Tr} \left( \hat{\psi} \hat{\Delta}(\vec{x}) \right), \quad \text{where the translation invariant trace map} \quad \text{Tr} \left( \hat{\psi} \right) = \int \psi(\vec{x}) d^2 \vec{x} \quad \text{can be introduced.}$$

Thus, one is lead to a new associative non abelian algebra in the space $S$ in terms of the Moyal $\star$ product

$$\psi \star \varphi(\vec{x}) = \text{Tr} \left( \hat{\psi} \hat{\varphi} \hat{\Delta}(\vec{x}) \right).$$  \hfill (3)

This result allows us to rephrase any field theory, defined by an action

$$S \left[ \hat{\psi}_\alpha \right] = \int dt \text{Tr} \left[ \mathcal{L} \left( \hat{\psi}_\alpha, \hat{\partial}_i \hat{\psi}_\alpha, \ldots \right) \right] = \int dt d^2 \vec{x} \mathcal{L} (\psi_\alpha, \partial_i \psi_\alpha, \ldots),$$  \hfill (4)

for the operators $C^*$, in terms of a nonlocal Lagrange density, involving the classical fields $\psi_\alpha$, their derivatives and their $\star$-products.

The noncommutative version (see [11] [12] [13]) of the non relativistic scalar field theory $(m = 1, \epsilon = 1)$ coupled to the Chern-Simons gauge field is given by the Lagrange density

$$\mathcal{L} = i \bar{\psi} \star D_i \psi - \frac{i}{2} \bar{D} \psi \star D \psi + \kappa \left( \frac{1}{2} \epsilon_{ij} \partial_i A_j \star A_j + A_0 \star F_{12} \right) - V (\psi, \bar{\psi}).$$  \hfill (5)

In (5) one has introduced the $\star$-covariant derivative and the $\star$ field - strength tensor

$$D_\mu \psi = \partial_\mu \psi - i A_\mu \star \psi, \quad D_\mu \bar{\psi} = \partial_\mu \bar{\psi} + i \bar{\psi} \star (A_\mu)$$  \hfill (6)

$$F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i (A_\mu \star A_\nu - A_\nu \star A_\mu)$$  \hfill (7)

respectively. According to (6) the matter field $\psi$ is in the fundamental representation of the gauge group $U(1)_\star$, i.e. the locally gauged fields are given by

$$\bar{\psi} = e^{i \lambda(\vec{x})} \star \psi, \quad \bar{A}_\mu = e^{i \lambda(\vec{x})} \star (A_\mu + i \partial_\mu) \star e^{-i \lambda(\vec{x})}, \quad \bar{F}_{\mu \nu} = e^{i \lambda(\vec{x})} \star F_{\mu \nu} \star e^{-i \lambda(\vec{x})}.$$
A remarkable feature of the $U(1)_s$ gauge theory \cite{14} is the quantization of the coupling constant \cite{14} $\kappa = \frac{n}{2\pi}$, $n \in \mathbb{Z}$, corresponding to the quantized filling factor in the FQHE \cite{11} \cite{13}. Vortex-like solutions of such a model were discussed in \cite{12} \cite{13}.

As shown in \cite{10}, the Lagrangian can be expressed as a trace over a Hilbert space, namely

$$L = i\pi\kappa \text{Tr} \left[ K^\dagger D_t K - KD_t K^\dagger \right] - 2\pi\kappa \text{Tr} [A_0] + 2\pi \theta \text{Tr} \left[ \hat{\psi}^\dagger D_t \hat{\psi} - \frac{1}{2\theta} \left( D\hat{\psi} \left( D\hat{\psi} \right)^\dagger + \bar{D}\hat{\psi} \left( \bar{D}\hat{\psi} \right)^\dagger \right) + V \left( \hat{\psi}, \hat{\psi}^\dagger \right) \right]$$

where we have redefined the gauge field operators as

$$K = \frac{1}{\sqrt{2\theta}} \left( \hat{\alpha}_1 - i\hat{\alpha}_2 - i\theta \left( \hat{A}_1 \hat{A}_2 \right) \right) = \hat{\alpha} - i\sqrt{2\theta} \hat{A}_+$$

and the corresponding adjoint $K^\dagger = \hat{\alpha}^\dagger + i\sqrt{2\theta} \hat{A}_-$. The operator $\hat{\alpha} = \frac{1}{\sqrt{2\theta}} (\hat{\alpha}_1 - i\hat{\alpha}_2)$ and its adjoint $\hat{\alpha}^\dagger$ satisfy the canonical commutation relation $[\hat{\alpha}, \hat{\alpha}^\dagger] = 1$, furthermore in terms of complex variables $(z, \bar{z})$ one has the representation $[\hat{\alpha}, \cdot] = \sqrt{2\theta} \partial_z$, $[\hat{\alpha}^\dagger, \cdot] = -\sqrt{2\theta} \partial_{\bar{z}}$. The covariant derivatives act as

$$D_t \hat{\psi} = \partial_t \hat{\psi} - i\hat{A}_0 \hat{\psi}, \quad D_t O = \partial_t O - i \left[ \hat{A}_0, O \right],$$

$$D\hat{\psi} = \sqrt{ \frac{\theta}{2} } (D_1 - iD_2) \hat{\psi} = K\hat{\psi} - \hat{\psi} \hat{c}, \quad \bar{D}\hat{\psi} = \sqrt{ \frac{\theta}{2} } (D_1 + iD_2) \hat{\psi} = \hat{\psi} \hat{c}^\dagger - K^\dagger \hat{\psi}$$

for any vector operator $O$.

Explicit, and at quantum level renormalizable, solutions can be found \cite{15} for the (ungauged) fourth-order self-interacting Landau - Ginzburg model

$$\mathcal{L} = L_0 - V^* = \left( i\bar{\psi} \star \partial_t \psi + \bar{\psi} \star \frac{\Delta \psi}{2} \right) - \frac{\lambda}{2} \bar{\psi} \star \bar{\psi} \star \psi \star \psi.$$  

In particular, two-particle bound states were found. They are characterized by a dipolar length proportional to the transverse total momentum, and the parameter $\theta$. The latter signals the breaking of galilean symmetry. This behaviour is analogous to what found in certain string models. The question of galilean invariance in NC-theories is addressed in \cite{16} \cite{17}. Below we review and amplify the results, reported in \cite{19} \cite{20}, on boost - invariant solutions in the NC-plane.

In our attempt to perform a systematic symmetry analysis of field theories on the NC-plane, we start with observing that for the free version of \cite{13} $L_0$ is quasi - invariant and acquires divergence-like terms (possibly proportional to $\theta$) with respect to an "exotic" version of the 10-dimensional Schrödinger symmetry algebra. Note that because of the bilinear form of the Lagrangian and the integral property

$$\int f \star g (\bar{x}) d^2 \bar{x} = \int f (\bar{x}) g (\bar{x}) d^2 \bar{x},$$

the action concides with the usual one in the commutative plane. The Euclidean subgroup is implemented by the inner automorphisms

$$\psi_{\text{trasl}} = \psi (\bar{x} - \bar{t}, t) = e^{-i \frac{\theta}{\theta'} \bar{x}} \star \psi (\bar{x}, t) \star e^{i \frac{\theta}{\theta'} \bar{x}}$$

$$\psi_{\text{rot}} = \psi \left( R_{\varphi}^{-1} \bar{x}, t \right),$$

with $R_{\varphi}$ being the rotation in the plane.
where \( \tilde{h} \) and \( \varphi \) represent parameters of the translations \( \vec{x} \rightarrow \vec{x} + \tilde{h} \) and of the rotations \( \vec{x} \rightarrow R_{\varphi} \vec{x} \), respectively. These relations express i) the nonlocality of the theory related to the scale of the momentum \( \tilde{\rho} \), ii) the emergence of the space translations as particular gauge transformations.

This allowed us to express covariant derivatives in terms of the gauge field in [1] using "covariant fields" (9) (see [1]).

The usual one-parameter centrally extended Galilei transformation is replaced by "exotic" two-parameter ones with (for \( m = 1 \)) infinitesimal and finite expressions

\[
\delta^\ast \psi = (ib \cdot \vec{x}) \ast \psi - t\tilde{b} \cdot \nabla \psi = (ib \cdot \vec{x})\psi - (\theta/2)\tilde{b} \times \nabla \psi - t\tilde{b} \cdot \nabla \psi
\]  

(15)

and

\[
\psi^+_b(\vec{x}, t) = e^{-i(\frac{\vec{g}_t}{\theta} \cdot \vec{x} t)} e^{\vec{g}_t \cdot \vec{x}} \ast \psi \left( \vec{x} - \tilde{b} t, t \right),
\]

(16)

respectively, where \( e^{\vec{b} \cdot \vec{x}} \) is the exponential w.r.t. the \( \ast \) product. Analogously, a \( \theta \) - deformed expansion symmetry is allowed. In infinitesimal form, it is given as [16]

\[
\delta^\ast \vec{x} = \eta t \vec{x}, \quad \delta^\ast t = \eta \vec{t}, \quad \delta^\ast \psi = -\eta \left[ \left( -\frac{i}{2} \vec{x}^2 + t \right) \psi + t \vec{x} \cdot \nabla \psi + \vec{t} \partial_t \psi \right] - \eta \left[ \frac{\theta}{2} \vec{x} \times \nabla \psi + \frac{\theta^2}{4} \partial_t \psi \right],
\]

(17)

and the usual dilations [18]

\[
\delta_{\Delta} \vec{x} = \Delta \vec{x}, \quad \delta_{\Delta} t = 2\Delta t, \quad \delta_{\Delta} \psi = -\Delta \left[ \psi + \vec{x} \cdot \nabla \psi + 2t \partial_t \psi \right].
\]

(18)

where \( \eta \) and \( \Delta > 0 \) are real parameters.

The Noether theorem still holds and the associated conserved quantities may get new terms,

\[
M = \int d^2 x |\psi|^2, \quad H_0 = \int d^2 x \frac{1}{2} |\tilde{\nabla} \psi|^2, \quad P_i = -i \int d^2 x \tilde{\nabla} \partial_i \psi, \quad J = -i \int d^2 \vec{x} \epsilon_{ij} \tilde{\nabla} \partial_j \psi,
\]

(19)

\[
G_i = - \int d^2 \vec{x} x_i |\psi|^2 + t P_i + \frac{\theta}{2} \epsilon_{ij} P_j
\]

(20)

\[
D = -2t H_0 + \frac{1}{2i} \int d^2 \vec{x} x_i (\tilde{\nabla} \partial_i \psi - (\partial_i \tilde{\nabla}) \psi), \quad K = t^2 H_0 + t D - \frac{1}{2} \int d^2 \vec{x} \vec{x}^2 |\psi|^2 + \frac{\theta}{2} J - \frac{\theta^2}{4} H_0.
\]

(21)

Since we can use the usual Poisson brackets \( \{ \psi(\vec{x}, t), \tilde{\psi}(\vec{x}', t') \} = -i \delta(\vec{x} - \vec{x}') \), the symmetry algebra, expressed in terms of the above generators, contains additional \( \theta \) - dependent terms, as in [1]. One establishes the relation

\[
k = \theta M.
\]

(22)

The other modified brackets are

\[
\{ K, G_i \} = \theta \epsilon_{ij} G_j, \quad \{ D, K \} = -2K + \theta J - \theta^2 H_0, \quad \{ D, G_i \} = -G_i + \theta \epsilon_{ij} P_j.
\]

(23)

All other commutation relations among the symmetry generators remain the same as in the commutative case [18].

Turning to the interacting theory, one can observe that all the self-interactions can be written in terms of the ”chiral” densities \( \rho_+ = \tilde{\psi} \ast \psi \), and \( \rho_- = \psi \ast \tilde{\psi} \).

Now, if the usual implementation of the Galilei is applied, the densities transform infinitesimally as \( \delta^\ast \rho_{\pm} = \pm \frac{\theta}{2} \vec{x} \times \nabla \rho_{\pm} - t\tilde{b} \cdot \nabla \rho_{\pm} \). Analogously, resorting to the ”exotic” boost (10), they change according to

\[
\delta^\ast \rho_+ = -t\tilde{b} \cdot \nabla \rho_+, \quad \delta^\ast \rho_- = -t\tilde{b} \cdot \nabla \rho_- - \theta \tilde{b} \times \nabla \rho_-.
\]
Consequently, the variation of a generic potential $V$ becomes exact w.r.t. such a transformation if it only depends on one type of $\rho_\pm$:

$$\delta_\theta \dot{V}_+ = -\theta \cdot \vec{b} \cdot \vec{D} \dot{V}_+, \quad \delta_\theta \dot{V}_- = -\theta \cdot \vec{b} \cdot \vec{D} \dot{V}_- - \theta \cdot \vec{b} \cdot \vec{D} \dot{V}_-.$$

(24)

In conclusion, any “pure” expression $V_\pm = V(\rho_\pm)$ provides us with a theory which is Galilei-invariant both in the the conventional and the *-implementation”. Chiral potentials of this kind were considered by several authors [21].

Finally, one can verify by direct computation that any potential made of products of $\rho_\pm$ necessarily breaks the conformal invariance. This is a consequence of the non local character of the theory and it could be related to the so-called UV/IR mixing [1].

Analyzing now the symmetry properties of the gauged model (5) with the pure fourth order self-interaction $V = \lambda (\psi \star \dot{\psi})^2$ [22], we see that the equations of motion (in Moyal and operatorial form, respectively)

$$iD_t \psi + \frac{1}{2} \vec{D}^2 \psi + \left(2\lambda - \frac{1}{2\kappa} \right) \psi \star \dot{\psi} \star \psi = 0, \quad \delta_\theta \psi = \frac{\theta}{\kappa} \cdot \vec{b} \cdot \vec{D} \psi,$$

$$\kappa E_i - \varepsilon_{ikj} j_{-k} = 0, \quad \delta_\theta \vec{b} \cdot \vec{x} = 0,$$

$$\kappa B + \rho_\pm = 0,$$

(25)

with $B = \epsilon_{ij} F_{ij}$, $E_i = F_{i0}$ and $j_\pm = \frac{1}{2\kappa} \left( \vec{D} \psi \star \dot{\psi} \star \psi (\vec{D} \psi) \right)$, possess an explicit “chirality” in the Hall and the Gauss law, respectively. Both the usual and the $\delta^* \left[ 16 \right]$ implementations break the invariance of the Gauss law under a Galilean boost. However, Galilean symmetry is restored if we consider the antifundamental representation

$$\delta_\star \psi = \psi \star (i\vec{b} \cdot \vec{x}) - \theta \cdot \vec{b} \cdot \vec{D} \psi = (i\vec{b} \cdot \vec{x}) \psi + \frac{\theta}{2} \vec{b} \times \vec{D} \psi - \theta \cdot \vec{b} \cdot \vec{D} \psi,$$

(26)

which is obtained as a right action of the $U(1)_\rho$ group, or more simply, changing the sign of $\theta$ in [16]. At the same time, the gauge field $A_\mu$ transforms as usual by

$$\delta A_i = -\theta \vec{b} \cdot \vec{D} A_i, \quad \delta A_0 = -\vec{b} \cdot \vec{A} - \theta \vec{b} \cdot \vec{D} A_0.$$

(27)

Finally, it is straightforward to prove that ”pure” chiral self-interactions $V_\pm$ are also invariant under the $\delta_\star$ transformations. Thus, the required galilean symmetry strongly selects the type of interaction, as we considered in [25].

On the other hand, any potential breaks the conformal invariance.

A remarkable consequence is that one can build up the new conserved Noether charges associated to the boost charge

$$G^r = t\vec{P} - \int \vec{x} \rho_+ d^2\vec{x}. \quad \text{(28)}$$

Their Poisson brackets differ from [1] only for the sign of $\theta$.

One could actually define the family of conserved quantities $G^{(\alpha)}_i = G^r_i + \bar{\theta} \epsilon_{ij} P_j$, parametrized by a real $\alpha$. This leads to new transformation rules

$$\delta^{(\alpha)} \psi = \bar{\theta} \cdot \{ \psi, \bar{G}^{(\alpha)} \}, \quad \delta^{(\alpha)} A_i = \vec{b} \cdot \{ A_i, \bar{G}^{(\alpha)} \},$$

depending on $\alpha$. The Poisson brackets for the boost charges are

$$\{ G^{(\alpha)}_i, G^{(\alpha)}_j \} = \epsilon_{ij} (\alpha - \theta) \int |\psi|^2 d^2x \quad \text{(29)}$$

So that, for $\alpha = 0$ we recover the *-implementation [1]. For $\alpha = \theta$ instead, we act on the matter field as in the commutative case, because of vanishing of the second central charge. But the
gauge potential changes non-conventionally. However, following [22], the gauge fields in the non-commutative ($\theta \neq 0$) domain must be related to the commutative ($\theta = 0$) case by a differential relation in $\theta$, precisely by
\[
\frac{\partial}{\partial \theta} A_i(\theta) = -\frac{1}{4} \epsilon_{kli} \left( A_k \ast (\partial_l A_i + F_{li}) + (\partial_l A_i + F_{li}) \ast A_k \right).
\] (30)
This equation is manifestly form-invariant w. r. t. boosts, provided $\alpha$ does not depend on $\theta$.
On the other hand, the boost transformation for the gauge fields on the ordinary plane ($\theta = 0$) holds only for $\alpha = 0$. In conclusion, the boost generator (28) is the only admissible one, because it is continuous for $\theta \to 0$. Hence, the non-trivial second charge is dynamically defined by (29) for $\alpha = 0$.
However, the proposed infinitesimal transformations (26 - 27) are not gauge covariant. This fact may provide difficulties in deriving correct gauge invariant conserved quantities and invariant configurations under specific symmetry subgroups, as it is well known in dealing with gauge field theories. But, resorting to this analogy, in correspondence to an infinitesimal linear coordinates transformation $\delta f^{\mu} = -f^\mu$ of the NC-plane, in [23] it was proposed to associate an infinitesimal gauge-covariant transformation on a vectorfield $\delta \hat{A}_\mu$ by
\[
\hat{\delta}_{\tilde{b}} A_\mu = \epsilon_{ij} b_j B, \quad \hat{\delta}_{\tilde{b}} A_0 = -\tilde{b} \cdot \tilde{E},
\] (31)
Of course, these transformations can be expressed in terms of $\rho_-$ and $\bar{\jmath}_-$, accordingly to [25]. It is remarkable that their algebra closes up to a gauge transformations. Specifically, noticing that
\[
\hat{\delta}_{\tilde{b}} B = -\tilde{b} \cdot \tilde{D} B, \quad \hat{\delta}_{\tilde{b}} E_i = -\tilde{b} \cdot \tilde{D} E_i - \epsilon_{ij} b_j B,
\] (32)
one can find
\[
\begin{align*}
\left[ \hat{\delta}_{\tilde{b}}, \hat{\delta}_{\tilde{b}} \right] \psi &= \tilde{b} \times \tilde{b} \left( \theta - t^2 B \right) \psi, \\
\left[ \hat{\delta}_{\tilde{b}}, \hat{\delta}_{\tilde{b}} \right] A_i &= -t^2 \tilde{b} \times \tilde{D} A_i B, \\
\left[ \hat{\delta}_{\tilde{b}}, \hat{\delta}_{\tilde{b}} \right] A_0 &= -\tilde{b} \times \tilde{D} \left( t^2 D_0 B + 2t B \right).
\end{align*}
\] (33)
Thus, the algebra of the covariant boosts closes up to a gauge transformation generated by $-\tilde{b} \times \tilde{b} t^2 B$. Moreover, the matter field acquires a time independent phase factor plus the gauge transformation contribution. Because of this result, the integration of such infinitesimal transformations is in general prevented.
However, the integrability can be obtained in the special case $D_\mu B = 0$. Because of the Gauss law, also the covariant time derivative of the chiral density $\rho_-$ is vanishing. Thus, since a covariant version of the continuity equation holds, the chiral current $\bar{\jmath}_-$ is covariantly solenoidal, i.e. $\tilde{D} \cdot \bar{\jmath}_- = 0$. Finally, taking into account the field-current identity, we end up with the covariant irrotational condition for the electric field $\tilde{D} \times \tilde{E} = 0$, consistently with the Bianchi identity for the connection $A_\mu$.
Although the above conditions restrict us to special field configurations, they still allow for non-trivial solutions.
Let us start with static solutions of (25) in the BPS limit $2\lambda \kappa = 1$ [12]. They are given by
\begin{align}
\hat{\psi} &= \sqrt{\frac{\kappa}{\theta}} |0\rangle \langle \zeta_0|, \quad K = \zeta_0 |0\rangle \langle 0| + S_1 \hat{c} S_1^\dagger, \quad A_0 = -\frac{1}{2\kappa} |0\rangle \langle 0|, \quad (34)
\end{align}

where $S_1 = \sum_{i=0}^{\infty} |i\rangle \langle i+1|$. Here $|0\rangle$ is the Fock vacuum $\hat{c} |0\rangle = 0$ and $|\zeta_0\rangle$ is a coherent state centered at the position $\zeta_0$, i.e.

$$|\zeta_0\rangle = e^{-\frac{1}{2}|\zeta_0|^2} e^{\zeta_0 \hat{c}^\dagger} |0\rangle.$$  (35)

The Gauss law give us $B = -\frac{1}{2\theta} |0\rangle \langle 0|$, i.e. we have a field configuration from the class $D_\mu B = 0$ and $\vec{E} = 0$.

Now, time dependent solutions are obtained by acting on the static solutions (34) by a finite gauge covariant boost (with velocity $v$). In fact, substituting the static solution into the infinitesimal transformations (31) and (32), one obtains

$$\hat{\psi} = \sqrt{\frac{\kappa}{\theta}} |0\rangle \langle \phi(t)|,$$  (36)

where

$$|\phi(t)\rangle = e^{(t-i\theta)\hat{c}^\dagger - \bar{v}(t+i\theta) + t(\bar{v}\zeta_0 - v\zeta_0)} |\zeta_0\rangle.$$  (37)

Manipulations of this expression, resorting to the Baker - Campbell - Hausdorff formula, lead to

$$|\phi(t)\rangle = e^{\imath \alpha(t)} |\zeta_0 + v(t - i\theta)\rangle,$$  (38)

which represents a coherent state, uniformly moving, with velocity $v$. The time dependent phase factor $\alpha(t)$ has the expression

$$\imath \alpha(t) = -\frac{\theta}{2} (\bar{v}\zeta_0 + v\bar{\zeta}_0) + \frac{t}{2} (\bar{v}\zeta_0 - v\bar{\zeta}_0).$$  (39)

For the gauge component $K$ one obtains the expressions

$$K(t) = (\zeta_0 + vt) |0\rangle \langle 0| + S_1 \hat{c} S_1^\dagger,$$  (40)

while $A_0$ remains static. Note that $B$ is left invariant, i.e. it still belongs to the class $D_\mu B = 0$.

Solutions of the kind (36) were found in [19]. From (32) we see that the CS-electric field takes the value $E_i = \frac{1}{\theta} \varepsilon_{ij} v_j |0\rangle \langle 0|$. Notice that this field configuration is similar as in the Hall motion. In particular, the relation $|\frac{E_i}{B\theta}| = v$ holds, while the particle is moving in the direction orthogonal to $E$.

Finally, in the BPS limit $2\lambda \kappa = 1$, one obtains a simple expression for energy of the particle interacting with the Chern - Simons field and with itself. In fact, one has $\mathcal{E} = -2\pi \text{Tr} \left[ \hat{\psi}^\dagger \hat{D} \hat{D} \hat{\psi} \right]$. For the solution (36), one sees that $\mathcal{E} = 2\pi \kappa |v|^2 = \frac{1}{2} \theta M |v|^2$, which confirms the continuous spectrum of the vortex energy. More interesting it is its expression as kinetic energy for a particle possessing the second central charge $\{22\}$ as mass and moving at the selected Hall velocity, found by the ratio $|E|/|B|$.

In conclusion, we have discussed the symmetries of a field theory on the non commutative plane. The first result is that we can recover galilean invariant theories, but only "chiral" interactions are admissible. However, the boost generators no longer commute, but they close up by the introduction of the second central extension parameter $\kappa$, which is dynamically determined by the particle mass and the non commutativity parameter of the plane. Finally, classes of uniformly moving solutions can be built by resorting to freely moving coherent states. However,
the associated CS-magnetic and electric field are not trivial at all, even if they result to be static. More general type of moving solutions can be found [20] and then boosted. In particular, one can show that two vortex solutions can be boosted, without changing their relative positions. This is in contrast with the dipolar solutions found by [15].

This work was partially supported by the Murst (grant SINTESI 2002) and by the INFN (Iniz. Spec. Le41). Two of us (PAH and LM) acknowledge the organizers for their hospitality at the International Workshop Nonlinear Physics. Gallipoli’2004.

References


