Vacuum induced Stark shifts for quantum logic using a collective system in a high quality dispersive cavity

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(Dated: November 17, 2004)

A collective system of atoms in a high quality cavity can be described by a nonlinear interaction which arises due to the Lamb shift of the energy levels due to the cavity vacuum [Agarwal et al., Phys. Rev. A \textbf{56}, 2249 (1997)]. We show how this collective interaction can be used to perform quantum logic. In particular we produce schemes to realize cnot gates not only for two-qubit but also for three-qubit systems. We also discuss realizations of Toffoli gates. Our effective Hamiltonian is also realized in other systems such as trapped ions or magnetic molecules.

I. INTRODUCTION

The possibility of doing quantum computation with neutral atoms is becoming more realistic with the advances in techniques relating to the trapping of few atoms which could even be addressed individually\textsuperscript{1,2,3}. However a number of experiments so far have been done with flying qubits\textsuperscript{4,5} and a number of proposals exist on implementing quantum logic operations using cavity QED\textsuperscript{6,7,8,9,10,11}. We note that the realization of a cnot gate between two qubits requires some form of interaction between the qubits. There are thus realizations which depend on the interaction between the center of mass degrees and the electronic degrees of freedom as in the case of ions\textsuperscript{12,13,14}, the interaction between the photonic qubit and the atom as in case of cavity QED\textsuperscript{4}. Thus for doing logic operations with neutral atoms one would require an effective interaction between them. Note that we have to keep the distance between atoms such that selective addressing is possible for one qubit operations. On the other hand if atoms are far apart then the electrostatic interaction between them is very weak. These problems can be overcome by using a high quality dispersive cavity. It has been shown earlier that the interaction of trapped atoms with a single mode of the radiation field produces an effective interaction which can be utilized for doing quantum logic\textsuperscript{15}. Though we shall work in the framework of this physical system, it is notable that the considered Hamiltonian is a special case of the Lipkin model\textsuperscript{16}, and similar Hamiltonians can be associated with the dynamics of ion traps\textsuperscript{17} and Fe\textsuperscript{3+} ions of a large magnetic molecule\textsuperscript{18}.

The outline of this paper is as follows. We shall introduce our system and qubits in section II, present a brief summary of some key mathematical tools used during our calculations in section III, then in section IV and V we shall give specific constructions of cnot gates for $N = 2$ and $N = 3$ atoms, respectively. We discuss realizations of Toffoli gates in section VI, and section VII is dedicated to our conclusions. We note that we work strictly with qubits and do not use any additional levels for the logic operation\textsuperscript{19}. The use of additional levels may lead to additional sources of decoherence either due to local environment or due to the fields which are used to address such levels.

II. PHYSICAL SYSTEM

We consider $N$ two-level atoms trapped in a cavity, with the atomic transition frequency $\omega_0$ detuned from the cavity resonance frequency $\omega$ by some value $\Delta$, and denote the dipole coupling between an atom and the cavity by $g$. The cavity losses are characterized by a constant $\kappa$ that describes the coupling to an external reservoir.

We do not consider atomic spontaneous decay explicitly, but we note that for large enough detuning $\Delta$, the modification of the decay rate due to the Purcell effect becomes negligible. Further, in order to facilitate individual addressing of the atoms, we require that the atoms are well separated, i.e. their spatial wave functions are non-overlapping.

It was shown in Ref.\textsuperscript{12} that if the cavity is in a thermal state with mean photon number $\bar{n}$, tracing out for this cavity mode in the limit $g\sqrt{N} \ll |i\Delta + \kappa|$ results in a time evolution of the atoms that can well be approximated by a unitary process. The effective Hamiltonian corresponding to this evolution may be written most conveniently in terms of the $S_i$ collective spin-$N/2$ and $S_i^2$ total angular momentum square operators defined through

$$S_i = \frac{1}{2} \sum_{k=1}^{N} \sigma_i^{(k)},$$

(1)

where the $\sigma_i^{(k)}$ operators are the Pauli-$i$ operators ($i = x, y, z$ or $+, -$) defined on the computational basis as usual. We define the computational basis states $|0\rangle_k$ and $|1\rangle_k$ as the ground ($|g\rangle$) and excited ($|e\rangle$) states of the $k$th atom, respectively. The number of qubits is there-
fore equal to the number of atoms $N$ trapped within the cavity. The effective Hamiltonian for $N$ atoms reads

$$
H_N = \hbar \eta (S_+ S_- + 2 \bar{n} S_z) \tag{2a}
$$

$$
= \hbar \eta \left[ S^2 - S_z^2 + (2\bar{n} + 1)S_z \right], \tag{2b}
$$

where the coupling factor is $\eta = g^2 \Delta / (\kappa^2 + \Delta^2)$.

The main theme of this paper shall be to prove the universality of this interaction Hamiltonian. For simplicity we assume that 1-qubit operations can be carried out much faster than the characteristic time of the collective interaction. To prove universality we show that this way it is possible to generate all CNOT gates.

We note that the terms linear in $S_z$ in Hamiltonians \ref{2a} correspond to 1-qubit operations. Let

$$
R_{x,y,z}(\vartheta) = \exp(-i \vartheta \sigma_{x,y,z}/2) \tag{3}
$$

denote to standard SU(2) rotations. Since $S_z$ commutes with the rest of the Hamiltonian, it follows that the linear terms in Hamiltonians \ref{2a} and \ref{2b} may be effectively cancelled from the time-evolution by applying $R_z(-2\eta \bar{n} t)$ and $R_z(-\eta (2\bar{n} + 1) t)$, respectively, to every qubit. It is important that the angle of rotation depends on the actual mean photon number (i.e. temperature) of the cavity. It also depends on the time $t$ for which $H_N$ is to be applied, however, as it shall be shown, in the course of quantum logic gate operation, this $t$ is known a priori. Also, because of the mentioned commutation properties this rotation can be carried out any time within the time-window prescribed by $t$. Later in this paper we shall work with Hamiltonians \ref{2b}, without the linear terms, and assume that the linear terms are always compensated with these 1-qubit rotations.

### III. ENGINEERING TWO-QUBIT GATES

For construction of desired two-qubit gates we have used a technique introduced in Ref. \ref{27}. For conciseness we briefly summarize this technique.

We consider two two-qubit gates $M$ and $L$, with unit determinants for now. We term them equivalent if they can be transformed into each other using only one-qubit operations $O = O_1 \otimes O_2$ and $O' = O_1' \otimes O_2'$ as

$$
L = O'MO. \tag{4}
$$

Here we used the tensorial product notation $\otimes$ to distinguish operators acting on different subsystems. A very important result of \ref{27} is that this equivalence is perfectly characterized by two numbers: Let $M_B = Q^\dagger MQ$ ($L_B$ similarly) with $Q$ being the unitary transformation to a specific entangled basis related to the standard Bell-states, and $m = M_B^\dagger M_B$ (l similarly) where $T$ denotes real transpose. Then the pairs (Tr$^2 m$, Tr$m^2$) and (Tr$l^2$, Tr$l^2$) coincide if and only if $L$ and $M$ are equivalent in the sense of Eq. \ref{4}.

An interesting side-effect is that these matrices $m$ and $l$ can be used to find the one-qubit operators $O$ and $O'$ connecting two equivalent $M$ and $L$. The recipe is as follows: first find the diagonal basis of $m$ ($l$), i.e. write $m = O_M^T d_M O_M$ where $d_M$ is a diagonal matrix. Then the solution can be written as

$$
O = O_M O_L^T \tag{5a}
$$

$$
O' = O_L^T O^T M_B^\dagger. \tag{5b}
$$

These results can also be applied to matrices with non-unit determinants, and to cover that case also we associate the numbers

$$
[\text{Tr}^2 m/16 \det M, (\text{Tr}^2 m - \text{Tr} m^2)/4 \det M] \tag{6}
$$

with a matrix $M$. This pair of numbers can be viewed as invariants under one-qubit operations.

The equations \ref{5} can be used to construct $L$ using $M$ only if $L$ and $M$ are equivalent. However, the invariants \ref{6} can also be used to construct $L$ from a given matrix $A$ not belonging to the same equivalence class. The method relies on splitting the problem into two, by first searching for a matrix equivalent to our target $L$ in the form of $M = AO_f A^\dagger$, and then using \ref{5} to obtain $L$. Solving the invariant equations is generally less involved than the solution of matrix equations. Unfortunately the first step on the course is not guaranteed to work for all $L$ and $A$. For example, a CNOT gate (or any equivalent) may never be constructed from SWAP gates and 1-qubit operations. For some other $A$, the construction may be possible, but only by using $A$ more than twice, e.g. $AO_f AO_f^\dagger A$.

### IV. CASE OF $N = 2$ ATOMS

As the simplest case we consider two atoms in the cavity. The collective spin operators now describe a spin-1 system, and a number of simplifications apply to this case. For example, the Dicke states \ref{28, 29} span the complete Hilbert space of the two atoms and hence, the solution of matrix equations. Unfortunately the first step on the course is not guaranteed to work for all $L$ and $A$. For example, a CNOT gate (or any equivalent) may never be constructed from SWAP gates and 1-qubit operations. For some other $A$, the construction may be possible, but only by using $A$ more than twice, e.g. $AO_f AO_f^\dagger A$.

$$
U^{(2)}(t) = e^{-\frac{i}{\hbar} H_Z t} = 1 - e^{-i \eta t} 2t G \sin(\eta t). \tag{7}
$$

In the computational basis this corresponds to the matrix

$$
U^{(2)}(t) = e^{-i\varphi} \begin{pmatrix}
    e^{-i\varphi} & 0 & 0 & 0 \\
    0 & \cos \varphi & -i \sin \varphi & 0 \\
    0 & -i \sin \varphi & \cos \varphi & 0 \\
    0 & 0 & 0 & e^{i\varphi}
\end{pmatrix}, \tag{8}
$$

with $\varphi = \eta t$. The invariants \ref{6} of this matrix are

$$
[\cos^4 \varphi, 4 \cos^2 \varphi - 1], \tag{9}
$$
while for a CNOT gate we would require $[0, 1]$. We see that this requirement is not met by any real $\varphi$. After some algebra, however, we obtain that with $U^{(2)}_t = U^{(2)}(\frac{\varphi}{2} \eta^{-1})$, the sequence

$$\hat{U}^{(2)} = U^{(2)}_t O_f U^{(2)}_t,$$

is equivalent to a CNOT gate if $O_f = R_y(\pi/4) \otimes \mathbb{I}$. In particular, using this CNOT equivalent gate $\hat{U}^{(2)}$ a CNOT with first bit as control and second as target bit can be produced as

$$\text{CNOT} = e^{i\pi/4}O_c^{\prime} \hat{U}^{(2)} O_c,$$

the one-qubit operations of this formula being

$$O_c^{\prime} = [R_z(-\pi/2)R_z(\pi/4)] \otimes [R_z(\pi/2)R_z(\pi/4)],$$

$$O_c = [R_z(\pi/4)R_y(-\pi/2)] \otimes R_z(5\pi/4).$$

The phase factor is in principle irrelevant and is written there for didactical reasons only. The construction is depicted as a quantum circuit diagram on Fig. 1. We note here, that assuming two-qubit gates $U^{(2)}$ with equal $t$, this construction is optimal in terms of operation time for the complete CNOT gate.

V. CASE OF $N = 3$ ATOMS

To build quantum logic for three atoms using the Hamiltonian in (23) we first show how to construct CNOT gates between any two atoms. The reason for constructing first a two-qubit gate from the three-atom collective interaction, rather than directly constructing a universal three-qubit gate (e.g., Toffoli or Fredkin gate) is the lack of convenient characterization of higher qubit gates. Up to date, invariants such as (6) have been discovered only for two-qubit gates.

In this section we shall consider $H_3$ of (23) without the linear terms, and using this we first construct a two-qubit gate that connects only two atoms and leaves the third atom unchanged, i.e.

$$U^{(3)}_{23} = \mathbb{I} \otimes U_{23}.$$  

Following the scheme similar to the spin-echo technique, we search for operators fulfilling (14) in the form

$$U^{(3)}_{23}(t) = X_1 U^{(3)}(t) X_1 U^{(3)}(t') \tag{15},$$

where $U^{(3)}(t) = \exp(-\frac{i}{\hbar} t H_3)$ is the time-evolution generated by the chosen Hamiltonian, and $X_1 = R_z(\pi) \otimes \mathbb{I} \otimes \mathbb{I}$ which is essentially a NOT gate. We pose the condition (14) on (15) to find the appropriate $t$ and $t'$.

The time evolution operator $U^{(3)}(t)$ is diagonal in the Dicke-state basis. To develop further insight into the problem, we apply the theory of angular momentum addition, and separate our spin-3/2 system into a product of a spin-1/2 and a spin-1 subsystem. Transformation matrix to the product basis is given by the relevant Clebsch-Gordan coefficients.

We require that (15) acts on the spin-1/2 subsystem as the identity, and this condition translates to $t' = t$ and $\sin(3/2\eta t) = 0$. This gives three distinct solutions for $U_{23}$ for each $i = -1, 0, 1$ via $\eta t = 2/3\pi(3k + i) (k \in \mathbb{Z})$. Out of these three, $i = 0$ corresponds to the identity operator and is therefore irrelevant. The solutions for $i = -1$ and $i = 1$ are essentially the same (adjoint of one another) and they both have invariants $[1/4, 3/2]$. In the following we work out the CNOT gate explicitly for $i = 1$, because its implementation requires less time. This operation is represented by

$$U_{23} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & e^{i\pi/3} & 0 & 0 \\
0 & 0 & e^{-i\pi/3} & 0 \\
0 & 0 & 0 & e^{-i\pi/3}
\end{pmatrix} \tag{16}$$

in the computational basis. We note that (16) may be written as $\exp(-i\pi/3 \sigma_z \otimes \sigma_z)$ resembling the Heisenberg spin-spin interaction that has found many applications in Quantum Information Processing, most notably NMR Quantum Computing[30, 31, 32]. However, in this case the interaction time is fixed by the conditions on $t$ and $t'$ of (15). Nevertheless, it will be seen that it is possible to express CNOT gates using this operator and 1-qubit gates.

Having a well-defined two-qubit gate in hand we can again follow the recipe of Sec. III. After some algebra we find that using the one-qubit operators acting on the subspace of qubits 2 and 3,

$$U_f = 1 \otimes R_y(\varphi_f) \tag{17},$$

$$U_c = R_z(-\pi/2) \otimes [R_z(\pi)R_z(\varphi_c)] \tag{18},$$

$$O_c^{\prime} = [R_z(\pi/2)R_y(-\pi)]$$

$$\otimes [R_z(\varphi_c)R_y(-\pi/2)R_z(\pi/2)], \tag{19}$$

with

$$\tan(\varphi_f/2) = 1/\sqrt{2} \tag{20},$$

$$\tan(\varphi_c/2) = \sqrt{2/3} - 1 \tag{21},$$

$$\tan(\varphi_c'/2) = (1 - \sqrt{3})/\sqrt{2}. \tag{22}$$

FIG. 1: Quantum circuit diagram depicting the sequence to prepare CNOT gate from the time-evolution operator in (3).
the CNOT gate can be constructed as
\[
\text{CNOT} = e^{-i\pi/4}O_c U_{23} O_f U_{23} O_c. \tag{23}
\]

This CNOT gate acts on qubit 2 and 3 as control and target bits, respectively. However, due to the symmetry of \(H_3\), a CNOT gate acting the other way around or connecting different qubits is achievable simply by exchanging the role of qubits appropriately.

We note here that while this implementation of CNOT gates is exact, it is not necessarily optimal, and the CNOT may be realized on this system more efficiently.

VI. TOFFOLI GATES

Universality of CNOT gates implies that having them in all configurations for three qubits allows the construction of any three-qubit quantum gate, i.e. any SU(2\(^3\)) operator. As an example we consider another important building block for systematic construction of complex quantum circuits, the Toffoli gate. We also discuss a simplified version of the Toffoli gate (Fig. 2) that differs from the Toffoli gate only in one conditional phase shift whereas requiring only half the CNOT gates. In some circumstances Toffoli gates may be replaced by the simplified versions.

Using the expressions for CNOT gates in our three-atom system, it is straightforward to implement both of these important quantum gates. Simple arithmetic counting the number of applications of \(U^{(3)}(2\pi/3\eta^{-1})\) operations shows that the gate times for the Toffoli and its simplified version add up to \(16\pi/\eta\) and \(8\pi/\eta\), respectively. Following DiVincenzo’s criteria for efficient error-free quantum computation these gate times should be much shorter than the coherence time of the complete system.

VII. CONCLUSIONS

In this paper we have shown the computational universality upto three qubits of a cavity assisted interaction between two-level atoms trapped in a dissipative cavity. In addition to the collective interaction we only needed single-qubit operations to implement multi-qubit gates. This required that the atoms are separately addressable, and we also assumed that single-atom operations can be performed on much shorter time scales than the collective interaction. The formalism used is not specific to two or three atom systems and therefore allows for further generalizations to more qubits.

In comparison with earlier proposals, our scheme is not only robust to cavity decay, but also allows dealing with thermal cavity states in a straightforward manner. We believe therefore that this scheme may find useful applications in situations where good localization of atoms had been achieved but the possibility of constructing good cavities is limited.

Acknowledgments

This work was partly supported by the National Research Fund of Hungary under contract Nos. T 034484 and T 043079; the Marie Curie Programme of the European Commission; and by the Hungarian Ministry of Education under contract No. CZ-5/03. A. G. would also like to thank the Director of P.R.L., Ahmedabad for his warm welcome and for all his support, and J. Janszky for useful discussions.