Classical Solutions of SU(3) Yang-Mills Theory and Heavy Quark Phenomenology

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Abstract. It is showed that potentials derived from classical solutions of the SU(3) Yang-Mills theory can provide confining potentials that reproduce the heavy quarkonium spectrum within the same level of precision as the Cornell potential.

In order to solve the classical Yang-Mills equations of motion, usually one writes an ansatz that simplifies the Euler-Lagrange equations and, hopefully, includes the relevant dynamical degrees of freedom. In [1] it was proposed a generalized Cho-Faddeev-Niemi-Shabanov ansatz for the gluon field, where the gluon is given in terms of two vector fields, $\hat{A}_\mu$ and $Y^a_\mu$, and a covariant constant real scalar field $n^a$,

$$A^a_\mu = n^a \hat{A}_\mu + \frac{3}{2g} f_{abc} n^b \partial_\mu n^c + Y^a_\mu$$

(1)

with the constraints

$$D_\mu n^a = 0, \quad n^a Y^a_\mu = 0.$$  

(2)

In [1] it was showed that the above decomposition of the gluon field is gauge invariant but not necessarily complete. In the weak coupling limit, $g \to 0$, a finite gluon field requires either $n = 0$ or $\partial_\mu n = 0$. If $n = 0$, the gluon field is reduced to a vector field in the adjoint representation of SU(3) gauge group. For the other case, $\partial_\mu n = 0$, the gluon is written in terms of the vector fields $\hat{A}_\mu$ and $Y^a_\mu$ and includes the previous solution as a particular case. Accordingly, a field such that $n \neq 0$ or $\partial_\mu n \neq 0$ does not produce a finite gluon field in the weak coupling limit and, in this sense, can be viewed as a nonperturbative field. Among this class of fields, the simplest parametrisation for the covariant scalar field\(^1\) is

$$n^a = \delta^{a1}(-\sin \theta) + \delta^{a2}(\cos \theta).$$

Then

$$A^a_\mu = n^a \hat{A}_\mu + \delta^{a3} \frac{1}{g} \partial_\mu \theta + \delta^{a8} C_\mu,$$

(3)

where $C_\mu = Y^8_\mu$. The classical Lagrangian and equations of motion are independent of $\theta$ and are abelian like in $\hat{A}_\mu$ and $C_\mu$. Among the possible nonperturbative gluons given

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\(^1\) From the constraint equation $Dn = 0$ it follows that $n^2$ is constant. Our choice was $n^2 = 1$. A different value for the norm of $n$ is equivalent to a rescaling of $\hat{A}$.\n
by (1), the simplest configuration has $\hat{A} = C = 0$. The coupling to the fermionic fields requires only the Gell-Mann matrix $\lambda^3$, decoupling the different colour components. This suggests, naively, that such a field is able to produce either confining, non-confining or free particle solutions for the quarks.

The classical equations of motion are independent of $\theta$. However, a choice of a gauge condition, provides an equation for this field. For the Landau gauge, $\theta$ verifies a Klein-Gordon equation for a massless scalar field. Note that there is no boundary condition for $\theta$, i.e. the usual free particle solutions of the Klein-Gordon equation are not the only possible ones. Indeed, writing $\theta(t, \vec{r}) = T(t) V(\vec{r})$, then

$$\frac{T''(t)}{T(t)} = \frac{\nabla^2 V(\vec{r})}{V(\vec{r})} = \Lambda^2 > 0, \quad (4)$$

$$T(t) = a e^{\Lambda t} + b e^{-\Lambda t}, \quad (5)$$

$$V(\vec{r}) = \sum_{l,m} V_l(r) Y_{lm}(\Omega), \quad (6)$$

where $z = \Lambda r$ and $I_{l+1/2}(z)$ and $K_{l+1/2}(z)$ are modified spherical Bessel functions of the $1^{st}$ and $1^{rd}$ kind. The lowest multipole solution is

$$V_0(r) = A \frac{\sinh(\Lambda r)}{r} + B \frac{e^{-\Lambda r}}{r} \quad (8)$$

and the associated gluon field is given by

$$\vec{A}_0^3 = \Lambda \left( e^{\Lambda t} - b e^{\Lambda t} \right) V_0(r), \quad (9)$$

$$\vec{A}_0^3 = - \left( e^{\Lambda t} - b e^{\Lambda t} \right) \nabla V_0(r). \quad (10)$$

From the lowest multipole solution one can derive a potential, which maybe suitable to describe heavy quarkonium. Indeed, assuming that quarks do not exchange energy, in the nonrelativistic approximation and leading order in $1/m$, the spatial function in $\vec{A}_0^3$, $V_0(r)$, can be viewed as a nonrelativistic potential and one can try to solve the associated Schrödinger equation. For the potential (8), the wave function goes to zero faster than an exponential for large quark distances,

$$\psi(\vec{r}) = \exp \left\{ \frac{-2}{\Lambda} \sqrt{\frac{2\Lambda}{m}} \exp \left( \frac{\Lambda r}{2} \right) \right\}. \quad (11)$$

As a first try to compute the heavy quarkonium spectra, we fixed $A, B$ and $\Lambda$ minimising the square of the difference between $V_0(r) + \text{Constant}$ and the Cornell potential $V_{Cornell} = \frac{e}{r} + \sigma r \ (e = -0.25, \sqrt{\sigma} = 427 \text{ MeV}$) integrated between 0.2 fm and 1 fm.

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2 Note that, by definition, the mass scale $\Lambda$ is independent of a rescaling of the gluon field.

3 The potential is $\sim 1/r$ for short distances and goes to infinity for large quark distances.
This optimisation provides the following parameters $A = 5.4$, $B = -1.0$, $\Lambda = 281$ MeV, $Constant = -1190$ MeV; for these values $-24 MeV \leq V_{Cornell} - (V_0 + Constant) \leq 64$ MeV in the integration range considered. Then, we can compare the Schrödinger equation spectrum for the charmonium ($m_c = 1.25$ GeV) and for the bottomonium ($m_b = 4.25$ GeV) for the two potentials. The spectrum for the new potential shows an equal level spacing for both the charmonium and bottomonium spectra. If the $V_0$ charmonium spectrum is quite close to the Cornell spectrum, the bottomonium shows clear deviations; see figure 1. The differences are the result of overestimating the strength of $V_0(r)$ for smaller distances. Indeed, one can improve our potential linearising the full QCD equations around the above configuration. To lowest order, this is equivalent to add a term like $k/r$ to $V_0$. Computing $k$ perturbatively\(^4\) adjusting the $M[(1P)] - M[(1S)]$ bottomonium mass difference, gives $k = 0.2448251$. The heavy quarkonia spectra, including this correction, is given in figure 1.

In conclusion, classical configurations seem to be able to produce a spectra close to the Cornell potential. Hopefully, this is an indication that these configurations can be of help to understand strong interaction physics. Of course, there are a number of issues that need to be further investigated (definition of the potential parameters, inclusion of time dependence, decay rates). We are currently working on these topics and will provide a report soon.

REFERENCES


\(^4\) For each stationary, the shift in energy due to this term is compatible with a perturbative treatment. Corrections are clearly below 10-20%.