Anatomy of three-body decay
I. Schematic models

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Abstract

Sequential three-body decay proceeds via spatially confined quasi-stationary two-body configurations. Direct three-body decay populates the three-body continuum without intermediate steps. The relative importance of these decay modes is discussed in a schematic model employing only Coulomb or centrifugal barrier potentials. Decisive dimensionless charge, mass and energy ratios are derived. Sequential decay is usually favored for charged particles. Small charge and small mass of high energy is preferably emitted first. Without Coulomb potential the sequential decay is favored except when both resonance energy and intermediate two-body energy are large.

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1 Introduction.

Resonances and excited states decaying into two clusters have been thoroughly studied from the early days of quantum mechanics. The prominent examples are α-decay, nucleon emission and fission, see e.g. [1]. More complicated final states with many fragments also occur and in particular three-body decay has been studied on and off over many years, see e.g. [2]. Many recent investigations focused on the unusual nuclear three-body halo structures which are naturally inclined to decay into their constituent clusters [3,4]. General scaling properties for direct multi-cluster Coulomb decay were also recently derived [5].
The primary experimental decay information is the distribution of cluster energies after the decay. From these spectra the decay can be analysed as sequential decay, i.e. emission of one particle populating a subsequently decaying resonance of the two remaining particles, or direct decay into the three-body continuum. Early attempts were made to characterize nuclear three-body decay [6,7]. Here the sequential decay is two subsequent two-body decays. The direct decay is parametrized by one or two terms of an expansion in hyperspherical harmonics. This may be sufficient for short-range interactions although even then the crucial asymptotic behavior should be influenced by admixtures of higher harmonics arising via the structure of the decaying resonance. In contrast a few hyperharmonics are most likely inadequate for charged particles in the final state.

The experimental techniques improved tremendously over the last decade. Kinematically complete and accurate experimental information becomes available. The three-body decay experiments are essentially all consistent with sequential decay with emission of one particle at a time. The observables available are widths of the decaying resonances and angular distributions and energy spectra of the emitted particles, see e.g. references in [8].

In quantum mechanics all paths connecting initial and final states contribute to the decay width. However, the least action path often dominates and provides sufficient accuracy. For given masses the least action path can be determined from suitable potential energy surfaces covering both initial and final states. The width can then be calculated and the decay mechanism related to tunneling properties of the dominating potentials. In this paper we assume that an initial state is a many-body resonance decaying into three fragments. We model this as a three-body system at intermediate and large distances. The three particles in the final state are then formed before entering the barrier precisely as the preformation factor in the description of \( \alpha \)-decay. This invites to definitions (and subsequent experimental determination) of three-body spectroscopic factors describing the fraction of corresponding three-body content in the initial many-body wave function as previously attempted for \( \alpha \)-cluster states [9].

The purpose of the present paper is to investigate the mechanism for three-body decay employing only Coulomb and centrifugal barriers. This is a generalization of classical \( \alpha \)-decay calculations to analogous three-body tunneling computations. We shall in particular concentrate on differences between sequential and direct decay. Realistic computations including both short and long-range potentials are discussed in the companion paper [8].
2 Theoretical concepts

The hyperspherical adiabatic expansion method is often quantitatively fairly accurate with only the dominating potential [10]. The corresponding generalized effective radial potentials are expressed as function of the hyperradius, \( \rho \), defined by

\[
\rho^2 \equiv \frac{1}{mM} \sum_{i<k} m_i m_k r_{ik}^2 = (x_j^2 + y_j^2) , \quad r_{ik}^2 = (r_i - r_k)^2 , \tag{1}
\]

where \((i, j, k)\) is a permutation of \((1, 2, 3)\), \(r_i\) is the coordinate of particle \(i\), \(M = \sum m_i\) and \(m\) is an arbitrary normalization mass, \(x_j\) and \(y_j\) are respectively proportional to the distance between two particles and the distance between their center of mass and the third particle. The remaining coordinates are all angles [10], but they are not necessary in the present paper.

For short-range interactions the lowest hyperspherical adiabatic potentials dominate in the expansion of the wave function. The efficiency is reflected by the ability to describe the Efimov effect arising when two subsystems simultaneous have large scattering lengths [11]. For systems with both short and long-range interactions the lowest potentials also dominate at small (and intermediate) distances whereas more components usually are needed for asymptotically large distances.

We shall assume that one adiabatic potential is sufficient in analogy to the two-body problem where short-range, Coulomb and centrifugal barrier terms provide an effective one-dimensional radial potential with possible bound states and resonances. With the dominating adiabatic potential the width of a given three-body resonance can be estimated by the WKB tunneling probability multiplied by the knocking rate [1]. This is a conceptual extension from two to three-body decay which is far from being obvious but shown recently to be approximately valid [12].

In principle different paths lead from the initial resonance state located at small distances to the final free three-particle state. Large separation between all pairs of particles can for example be achieved in two steps, i.e. first by moving one particle to infinity while the other two remain at essentially the same distance from each other and second by moving the two close-lying particles apart. Let us define sequential decay as this two-step process if the second step is started after the first particle is at a distance larger than the initial size of the three-body system.

The intermediate configuration does not have to be a two-body resonance. A substantial attraction for example in s-waves arising from a close-lying virtual
s-state could be sufficient to produce the signature of such a sequential decay. This is in close analogy to the effects of final state interactions in fragmentation reactions of three-body systems where two final-state fragments remain close together resulting in a significantly narrower momentum distribution [13].

To clarify the idea we assume that the third particle is emitted with relative energy \( E_{12,3} = E - E_{12} \), where \( E \) is the total energy and \( E_{12} \) is the energy of the remaining two-body resonance. The corresponding velocity is \( v_{12,3} = (2E_{12,3}/\mu_{12,3})^{1/2} \), where \( \mu_{12,3} \) is the related reduced mass. The two-body resonance has a lifetime of \( t_{12} = \hbar/\Gamma_{12} \). The distance, \( d_{12,3} \), particle 3 moves before the 12-system decays compared to the size of the three-body system \( R_0 \) is then

\[
\frac{d_{12,3}}{R_0} = \frac{v_{12,3}t_{12}}{R_0} = \frac{1}{\Gamma_{12}} \sqrt{\frac{2\hbar^2 (E - E_{12})}{\mu_{12,3}R_0^2}},
\]

which should be larger than one to fulfill the conditions for sequential decay. Typical numbers for nuclei gives that the two-body decay width \( \Gamma_{12} \) then has to be smaller than about 1 MeV. However, this condition is not sufficient to decide the character of the decay. The picture is not useful when the initial three-body resonance wave function has no configuration similar to a two-body subsystem in a resonance. Then direct decay has an advantage. The overall conclusion is that the preference for sequential decay is indicated by a narrow two-body resonance, but this is only a necessary and not a sufficient condition.

\section{Schematic models}

Let us assume the fragments are formed at distances smaller than or equal to the inner classical turning point of the barrier defined by the dominating adiabatic potential. This is analogous to \( \alpha \)-decay with a preformation factor equal to one. If this clusterization of the final state fragments is not fully present a pre-exponential preformation factor has to be included. The terminology is identical to the description of \( \alpha \)-decay and our formulation is a direct generalization to three particles in the final state.

To discuss direct versus sequential decay we want to compare the corresponding decay probabilities. Assume that the sequential process occurs by first emitting one of the particles and subsequently the remaining two-body system decays. The boundary condition of an outgoing flux means that we should compare the direct decay width with the width for the first step sequential decay multiplied by the branching ratio for decaying into the final state by the next step. The second step may be in competition with other decay modes.
This is the classical picture where a decision of paths is chosen both initially and in the next step. Thus the sequential decay width is the width for the first step multiplied by the branching ratio for the second step, i.e. the second width divided by itself added to the total width for decay into competing channels.

It is rewarding to reconcile this procedure with the sequential decay condition of small partial two-body width $\Gamma_{12}$ arising from Eq.(2). If $\Gamma_{12}$ is large the total sequential decay width is given by $\Gamma_{12,3}$ arising from the first step. However, Eq.(2) then also indicates direct decay. For large $\Gamma_{12}$ other modes are strongly coupled to the sequential decay channel. The system does therefore not survive long enough for the third particle to move outside the radius of the initial three-body system. The other channels, directly populating continuum states, take all the probability before even the first step in the sequential decay process is completed. This means that the direct decay seems to be very likely but also sequential decay may be comparably large depending on the size of $\Gamma_{12,3}$. Thus large $\Gamma_{12}$ may favor direct decay perhaps coupling to other adiabatic channels. This cannot be described by one-channel estimates.

3.1 The WKB approximation for an effective hyperradial potential

As soon as one effective potential is responsible for the decay we can derive simple estimates by use of the WKB tunneling transmission $T$, i.e.

$$T = \frac{1}{1 + \exp(2S)} \approx \exp(-2S), \quad S = \frac{1}{\hbar} \int_{\rho_0}^{\rho_t} d\rho \sqrt{2m(V(\rho) - E)}, \quad (3)$$

where $E$ is both the total energy and the kinetic energy of the particles after separation, $\rho_0$ and $\rho_t$ are the classical turning points where $V(\rho_0) = V(\rho_t) = E$. The integration path here is along $\rho$ but the expression remains valid for any other path. This perhaps unusual expression for $T$ is really the second order WKB approximation which substantially extends the validity range [1]. For example the harmonic oscillator transmission coefficient is then exactly reproduced. The value of $T = 1/2$ obtained at the top of the barrier ($\rho_0 = \rho_t, S = 0$) is quantum mechanically correct. When $S \gg 1$ the usual exponential expression in Eq.(3) is obtained.

A rather good estimate is found with the dominating adiabatic effective hyperradial potential $U_{eff}$ inserted instead of $V$, see e.g. [12]. A number of crucial effects are collected in $U_{eff}$, e.g. the three distinguishable contributions from centrifugal barrier, Coulomb and short-range potentials, and the subtle adiabatic adjustments of structure as the distance between clusters increases.
Then intermediate structures are picked up along the decay path, e.g. a particular two-body resonance or an especially strong s-wave attraction between two particles. Furthermore, several configurations can be present and interfere to produce $U_{\text{eff}}$, e.g. different resonances in the two-body subsystems or the coherent contributions arising due to (anti)symmetry of identical particles. All these effects are accounted for since each adiabatic potential represents a very specific weighted combination of paths continuously leading from small to large distances.

Different adiabatic potentials represent different paths which separately can be estimated by the WKB approximation. Effects of several simultaneously contributing adiabatic potentials would one way or another require coupled channel calculations. Thus it is usually not sufficient to assume a given classical path and compute the expectation values of the different terms in the three-body Hamiltonian. However, as for $\alpha$-decay, it is very illuminating to exhibit the effects of the most prominent terms in the classical picture. We shall therefore explicitly investigate Coulomb and centrifugal barrier terms. We then only implicitly include the short-range interaction by choices of geometric paths, of two-body resonance energies and relative angular momenta. The same philosophy was recently applied to direct decay for the Coulomb potential [5].

3.2 Geometries

A given decay path is defined by specifying how the distance between particles increase as function of $\rho$. We assume a constant scaling although any function in principle could be used. With the use of Eq.(1) we then define the positive scaling constants $s_{ik}$ by

$$\frac{r_{ik}^2}{\rho^2} \equiv s_{ik}^2, \quad mM \equiv \sum_{i<k} m_i m_k s_{ik}^2, \quad s_{ik}^2 = \frac{mM}{\sum_{i<k} m_i m_k}, \quad (4)$$

where the upper limit are given by $s_{ik} < \sqrt{mM/(m_i m_k)}$ as seen from Eq.(4) since where each term must be smaller than the sum $mM$. The last expression for $s_{ik}$ is obtained by assuming that all $s_{ik}$ are identical.

A number of different paths are now parametrized by choices of $s_{ik}$ as illustrated in Fig.1. When for example $s_{12} \approx 0$ and $s_{13} \approx s_{23}$ we obtain the typical sequential path where particles 1 and 2 stay close until the distance to particle 3 is much larger than the initial size of the three-body system. After this emission of particle 3, also particles 1 and 2 increase their mutual distance until all particle distances are large, i.e. the final three-body decay has been completed.
Fig. 1. Different geometric configurations illustrating possible decay paths, i.e. removal of one particle while the other two remain close (left) perhaps in a two-body resonance, scaling of a linear configuration (middle), and an overall proportional scaling of all distances (right). Cyclic permutations are also allowed.

When \( s_{13} \approx s_{23} \approx s_{12}/2 \) the decay proceeds through the linear chain also depicted in Fig. 1. Direct decay to the three-body continuum is described by a simultaneous increase of all \( s_{ik} \) until all particle distances are large compared to the initial size. Then no subsystem is used as an intermediate stepping stone and no preference for two-body substructures are exploited. Cyclic permutations or renaming of the particles then cover all extreme structures.

3.3 Coulomb potential

When long-range repulsive Coulomb interactions are present they strongly influence the decay. We assume that the effect of the short-range interaction is vanishingly small in the classically forbidden barrier region. Unless the charges are very small or the angular momenta are large the (generalized) centrifugal barrier also has relatively small effect on the barrier penetrability [1]. Thus we use Eq.(3), either for direct decay with \( V \) defined as

\[
V(\rho) = \sum_{i<k} Z_i Z_k e^2 / r_{ik} = \frac{1}{\rho} \sum_{i<k} Z_i Z_k e^2 / s_{ik},
\]

or with \( \rho \) and \( m \) substituted by \( r_{13} \) or \( r_{12} \) and the related masses \( \mu_{12,3} \) and \( \mu_{12}, \) i.e.

\[
V(r_{13}) = E_{12} + \frac{Z_1 Z_3 e^2}{r_{13}} + \frac{Z_2 Z_3 e^2}{r_{23}} = E_{12} + \frac{(Z_1 + Z_2) Z_3 e^2}{r_{13}},
\]
\[
V(r_{12}) = \frac{Z_1 Z_2 e^2}{r_{12}},
\]

corresponding to the two steps of the sequential decay process when particle 3 is emitted. The constant energy \( E_{12} \) is tied up in the intermediate configuration with particles 1 and 2 close together. This two-body structure could correspond to a resonance or maybe result from an s-wave attraction keeping the particles together.
The WKB exponents are in any case proportional to a generic function like

\[
S \propto \int_{x_0}^{x_t} dx \sqrt{\frac{x_t}{x} - 1} = x_t \left( \arctan \sqrt{\frac{x_t}{x_0} - 1} - \frac{x_0}{x_t} \sqrt{\frac{x_t}{x_0} - 1} \right) \approx \frac{\pi x_t}{2}, \tag{8}
\]

where we assumed that \( x_0 \ll x_t \). This approximation implies that the exponent is fairly accurately determined, but the decay width itself is substantially more inaccurate.

Using Eqs. (3), (5), (6) and (8) we then arrive at the WKB exponents for the direct and the two-step sequential decays, i.e.

\[
S = \frac{\pi}{2} \left( \sum_{i<k} Z_i Z_k e^2 \right) \sqrt{\frac{2m}{\hbar^2 E}} = \frac{\pi}{2} \left( \sum_{i<k} Z_i Z_k e^2 \right) \sqrt{\frac{2 \sum_{i<k} m_i m_k}{2 \hbar^2 E M}}, \tag{9}
\]

\[
S_{12,3} = \frac{\pi}{2} (Z_1 + Z_2)Z_3 e^2 \sqrt{\frac{2 \mu_{12,3}}{\hbar^2 (E - E_{12})}} \equiv \frac{b_{12,3}}{\sqrt{E - E_{12}}}, \tag{10}
\]

\[
S_{12} = \frac{\pi}{2} Z_1 Z_2 e^2 \sqrt{\frac{2 \mu_{12}}{\hbar^2 E_{12}}} = \frac{b_{12}}{\sqrt{E_{12}}}, \tag{11}
\]

where the notation is self-explanatory. We used the appropriate reduced masses \((\mu_{12,3})\) for the sequential decay corresponding to particles 1 and 2, and their center of mass and the third particle, respectively.

The last expression in Eq. (9) is obtained by using identical scaling, i.e. all \( s_{ik} \) are the same as expressed in Eq. (4). Another set of scaling parameters can be obtained for the potential in Eq. (5) by minimizing the action integral \( S \) in Eq. (3) with respect to \( s_{ik} \). With the constraint on \( s_{ik} \) in the middle equation of Eq. (4) we obtain the least action result

\[
S_{\text{min}} = \frac{\pi e^2}{2} \sqrt{\frac{2}{\hbar^2 E M} \left( \sum_{i<k} (Z_i Z_k)^{2/3} (m_i m_k)^{1/3} \right)^{3/2}} \equiv \frac{b}{\sqrt{E}}. \tag{12}
\]

The corresponding optimum path is defined for scaling parameters given by \( s_{ik}^3 m_i Z_i = s_{jk}^3 m_j Z_j \). Thus the action is minimized with the ratio of distances between particles 1 – 3 and 2 – 3 given as the cubic root of the ratio of charges \( Z_1/Z_2 \) multiplied by the ratio \( m_2/m_1 \). For identical particles the path is scaling of an equal sided triangle and given by the upper limit in Eq. (4). The same scaling property was also found in [5] by use of time dependent equations of motion for the scaling parameters.

The direct to sequential branching ratio is a sum of two terms,
\[ P_{d,\text{seq}} = \frac{T}{T_{12,3}} \left( 1 + \frac{T_{\gamma}}{T_{12}} \right), \tag{13} \]

each related to the first and second steps of the process, respectively. From Eq.\((3)\) we have that \( T = \exp(-2S_{\text{min}}) \), \( T_{12,3} = \exp(-2S_{12,3}) \), \( T_{12} = \exp(-2S_{12}) \), and \( T_{\gamma} \) is proportional to the decay probability into other modes in the second step of the sequential decay process. In most cases of interest we have \( T_{\gamma} \ll T_{12,3} \) and the second term in Eq.\((13)\) can be ignored. Thus, when \( T < T_{12,3} \) the sequential decay is preferred and vice versa. This inequality is controlled by the relative size of \( S_{12,3} \) and \( S_{\text{min}} \). If we also measure \( S_{12} \) relative to \( S_{\text{min}} \) we have two crucial quantities appearing in these expressions, i.e.

\[ R_a \equiv \frac{S_{12,3}}{S_{\text{min}}} = \frac{b_{12,3}}{b} \epsilon_a, \quad R_b \equiv \frac{S_{12}}{S_{\text{min}}} = \frac{b_{12}}{b} \epsilon_b, \tag{14} \]

where the dimensionless charge-mass and energy ratios are defined by Eqs.\((10)\), \((11)\), \((12)\) and

\[ \epsilon_a \equiv \sqrt{\frac{E}{E - E_{12}}}, \quad \epsilon_b \equiv \sqrt{\frac{E}{E_{12}}}. \tag{15} \]

We first notice that \( T > T_{12,3} \) when \( R_a > 1 \). This condition is determined by only one combinations of masses, charges and energies. For an energy \( E_{12} \) approaching \( E \) the ratio \( R_a \) becomes infinitely large expressing that then direct decay is preferred independent of charges and masses. Otherwise we notice that emission in the first step of a particle with small charge, small mass and high energy favors the sequential decay mode. The system would then choose the best of the stepwise disintegrations through the different possible intermediate configurations, in all cases provided the process can compete with the direct decay.

### 3.4 Centrifugal barrier

We now consider the case without Coulomb interaction. Still we assume that the effect of the short-range interaction is vanishingly small in the classically forbidden barrier region. Then we are left with the potentials corresponding to the centrifugal barrier, i.e.

\[ V(\rho) = \frac{\hbar^2(K + 3/2)(K + 5/2)}{2m\rho^2} \approx \frac{\hbar^2(K + 2)^2}{2m\rho^2}, \tag{16} \]

\[ V(r_{13}) = E_{12} + \frac{\hbar^2\ell_{12,3}(\ell_{12,3} + 1)}{2\mu_{12,3}r_{13}^2} \approx E_{12} + \frac{\hbar^2(\ell_{12,3} + 1/2)^2}{2\mu_{12,3}r_{13}^2}, \tag{17} \]
\[ V(r_{12}) = \frac{\hbar^2 \ell_{12}(\ell_{12} + 1)}{2\mu_{12} r_{12}^2} \approx \frac{\hbar^2 (\ell_{12} + 1/2)^2}{2\mu_{12} r_{12}^2}, \]  

(18)

respectively for direct decay and the two steps of the sequential decay where particle 3 is emitted first. We used again both the appropriate reduced masses and the lowest allowed angular momentum quantum numbers \( \ell_{12,3} \) and \( \ell_{12} \) corresponding to the relative two-body motion. The direct decay has the lowest hypermomentum quantum number \( K = \ell_{12,3} + \ell_{12} \) \[10\]. More than one partial wave may contribute but in general the lowest values dominate as they correspond to the lowest barriers. We inserted the usual improved semiclassical approximation in Eqs.(17) and (18) for the expectation value of the angular momentum.

The WKB exponents are then all obtained from the same type of generic integrals, i.e.

\[ S \propto \int_{x_0}^{x_t} dx \sqrt{\frac{x_t^2 - x^2}{x^2}} - 1 = -\sqrt{x_t^2 - x_0^2} + x_t \log \left( \frac{x_t + \sqrt{x_t^2 - x_0^2}}{x_0} \right) \approx x_t \log \left( \frac{2x_t}{\epsilon x_0} \right), \]  

(19)

where we assumed \( x_0 \ll x_t \), and \( x_t^2 = \ell (\ell + 1) \approx (\ell + 1/2)^2 \). Furthermore \( (x_t/x_0)^2 = E_B/E \) where \( E \) is the decay energy and \( E_B = \hbar^2 (\ell + 1/2)^2/(2\mu R_0^2) \) is the potential energy at the inner radial turning point \( R_0 \). The reduced mass, angular momentum and turning point for the process in question then has to be inserted. The WKB transmission coefficient from Eq.(3) becomes

\[ T = \frac{1}{1 + \left( \frac{4E_B}{\epsilon^2 E} \right)^{\ell+1/2}} \approx \left( \frac{e^2 E}{4E_B} \right)^{\ell+1/2}, \]  

(20)

where the last approximation is valid when \( E \ll E_B \). For \( s \)-waves we get \( T \approx e k R_0 \), where \( k \) is the wave number associated with the energy \( E \). This agrees with the exact low-energy result, \( 4k/K \), for a square well [1] when the wave number \( K \) inside the well and the inner classical turning point are related by \( KR_0 = 4/e \) which is around 1.5. Furthermore, for finite \( \ell \) we also get the correct low-energy threshold behavior \( T \propto E^{\ell+1/2} \) [1].

Using Eqs.(16), (17), (18) and (20) we then arrive at the WKB transmission coefficients for direct and two-step sequential decays, i.e.

\[ T = \left( \frac{2m \rho_0^2 E}{\hbar^2 (\ell_{12,3} + \ell_{12} + 2)^2} \right)^{\ell_{12,3}+\ell_{12}+2} \approx \left( \frac{2 \sum_{i<k} m_i m_k R_0^2 E}{M \hbar^2 (\ell_{12,3} + \ell_{12} + 2)^2} \right)^{\ell_{12,3}+\ell_{12}+2} \]  

(21)
\[ T_{12,3} = \frac{2\mu_{12,3}R_{12,3}^2(E - E_{12})}{\hbar^2(\ell_{12,3} + 1/2)^2} \cdot \ell_{12,3} + 1/2 \approx \left( \frac{4\mu_{12,3}R_{0}^2(E - E_{12})}{3\hbar^2(\ell_{12,3} + 1/2)^2} \right)^{\ell_{12,3} + 1/2}, \tag{22} \]

\[ T_{12} = \frac{2\mu_{12}R_{12}^2E_{12}}{\hbar^2(\ell_{12} + 1/2)^2} \cdot \ell_{12} + 1/2 \approx \left( \frac{2\mu_{12}R_{0}^2E_{12}}{\hbar^2(\ell_{12} + 1/2)^2} \right)^{\ell_{12} + 1/2}, \tag{23} \]

where we again used the appropriate reduced masses, angular momenta and radii for the different processes. We assumed the initial state roughly is a triangle with equal side length \( R_0 \). Furthermore we expressed \( \rho_0 \) in Eq.(21) in terms of \( R_0 \) from an equation analogous to Eq.(1), see [14].

Thus, again when \( T_{12,3} > T \) the sequential decay is dominating and vice versa. The exponent \( (\ell_{12,3} + \ell_{12} + 2) \) in the \( T \)-expression is larger than the sum of the other two exponents reflecting that a centrifugal barrier exists for the three-body system even when all relative angular momenta are zero, see Eq.(16) with \( K = 0 \). Each decay rate increases with increasing reduced mass and available energy, but first of all with decreasing angular momentum.

Combining Eqs.(21), (22) and (15) we obtain

\[
\frac{T}{T_{12,3}} \approx \left( \frac{3}{2} \right)^{\ell_{12,3} + 1/2} \left( \frac{2E\mu_{12,3}R_{0}^2}{\hbar^2} \right)^{\frac{\ell_{12} + 3/2}{\ell_{12,3} + 1}} \left( \sum \frac{m_i m_k}{M \mu_{12,3}} \right)^{\ell_{12,3} + \ell_{12} + 2} \left( \frac{\ell_{12,3} + 1/2}{\ell_{12,3} + \ell_{12} + 2} \right)^{2\ell_{12,3} + 1} \left( \ell_{12,3} + 1/2 \right)^{2\ell_{12,3} + 2\ell_{12} + 4}. \tag{24} \]

The \( \epsilon_a \) term is always larger than one favoring direct decay. We see again that for an energy \( E_{12} \) approaching \( E \), the direct decay is preferred independent of the masses. The inverse energy factor multiplying \( E \) is for nuclei of the order of a few MeV\(^{-1} \), which means that \( E \) smaller than about 2 MeV tends to prefer sequential decay. The mass average \( \sum_i \frac{m_i m_k}{M \mu_{12,3}} \) is always larger than one and reduces for equal masses to 3/2. For vanishing \( m_3 \) direct decay is preferred whereas large \( m_3 \) favors sequential decay. Emission of the large mass first is preferred. The angular momentum factor tends to be much smaller than unity (1/32 already for \( s \)-waves) and therefore favoring sequential decay. Then a finite angular momentum requires a large energy to overcome the centrifugal barrier.

### 3.5 Important examples

The general expressions and discussions can be much better appreciated with examples from existing systems. We therefore select a few particularly important cases. We first look at three identical particles of charge \( Z_0 e \) and mass \( A_0 m_n \) where \( m_n \) for example is the neutron mass. For \( \alpha \)-particles combined
in the first excited $0^+$ state of $^{12}$C where $E \approx 3E_{12}$, we get $b_{12}/b = 0.24$, $b_{12.3}/b = (2/3)^{3/2}$, $\epsilon_a \approx \sqrt{3/2}$, $\epsilon_b \approx \sqrt{3}$, and therefore $R_a \approx 2/3$. Thus sequential decay is preferred in agreement with [15].

The higher-lying $0^+$-excitation has $E \approx 40E_{12}$, $\epsilon_a \approx 1$, $\epsilon_b \approx 6.5$, and therefore $R_a \approx 0.5$. Sequential decay is even more favored. A higher-lying intermediate two-body state may also produce sequential decay. The $2^+$-state in $^8$Be corresponds to $E \approx 2E_{12}$, $\epsilon_a \approx \epsilon_b \approx \sqrt{2}$, and therefore now $R_a \approx 0.7$, still favoring sequential decay through this branch over the direct decay.

The ratio of these two sequential transmission coefficients from Eq.(13) then tremendously favors decay through the two-body $0^+$-state. This expectation is in agreement with the analysis of available experimental data showing that at most 4% of this decay occurs without going through the $0^+$ ground state of $^8$Be [16]. However, one complication is that the two $0^+$ states are coherently populated in experiments. In principle then both states produce unseparable contributions in the relevant energy region. The attractive two-body interaction may also play a role.

Another example is two protons and a core corresponding to two-proton decay from proton dripline nuclei. Two different sequential decays can occur, i.e. first emission of either the proton or the core. The relevant mass and charge ratios in the two cases are found to be $(b_{12}/b, b_{12.3}/b) \approx (0.5/\sqrt{2}, 0.5/\sqrt{2}),(0.25/Z_c, 1)$, respectively. Thus sequential proton emission is preferred over sequential core emission. Direct decay is only favored when $E_{12}$ is close to $E$, i.e. with the present estimates when $7E/8 < E_{12} < E$.

If one proton is replaced by a neutron the specifically favored two-body structures become decisive. In general we consider one neutral and two charged particles. For nuclei the neutron is then one of the three particles. Let the other two particles have charges $Z_c e$, $Z_p e$ and masses $A_c m_n$, $A_p m_n$. There are three possible sequential decays, i.e. first emission of one of the three particles. In all three cases one decay is Coulomb dominated and the other essentially controlled by the centrifugal barriers. The Coulomb WKB exponents in Eqs.(10) and (11) in general are much larger than the logarithmic terms in Eq. (19). Therefore the path with the largest probability is found for the division of smallest Coulomb $S$-values. From Eqs.(12), (10) and (11) we obtain

\[ S_{pc} = \frac{\pi}{2} \sqrt{\frac{2m_n Z_p Z_c e^2}{\hbar^2 E_{pc}}} \sqrt{\frac{A_p A_c}{A_p + A_c}}, \]

\[ S_{nc,p} = \frac{\pi}{2} \sqrt{\frac{2m_n Z_p Z_c e^2}{\hbar^2 \sqrt{E - E_{nc}}}} \sqrt{\frac{(A_c + 1)A_p}{A_p + A_c + 1}}, \]

\[ (A_c + 1)A_p \]
\[ S_{np,c} = \frac{\pi}{2} \sqrt{\frac{2m_n}{h^2}} \frac{Z_p Z_c e^2}{\sqrt{E - E_{np}}} \sqrt{\frac{(A_p + 1)A_c}{A_p + A_c + 1}}, \quad (27) \]

\[ S = \frac{\pi}{2} \sqrt{\frac{2m_n}{h^2}} \frac{Z_p Z_c e^2}{\sqrt{E}} \sqrt{\frac{A_c A_p}{A_p + A_c + 1}}. \quad (28) \]

We immediately conclude that all these quantities are rather similar except for the energy factors. Thus, both \( S_{np,c} \) and \( S_{nc,p} \) exceed \( S \), i.e. direct decay is always preferred. The heavier core mass favors sequential emission of the proton over the core. In both cases high energy emission is favored.

If two neutrons (mass \( m_n \)) surround a core the centrifugal barriers are decisive. Two sequential decays are possible, i.e. neutron or core emission first. For a relatively large core mass we get for neutron emission from Eq. (24)

\[
\frac{T}{T_{12,3}} \approx \left( \frac{4Em_nR_0^2}{h^2} \right)^{\ell_{12}+3/2} \left( \frac{3E}{E - E_{12}} \right)^{\ell_{12,3}+1/2} \times \frac{(\ell_{12,3} + 1/2)^{2\ell_{12,3}+1}}{(\ell_{12,3} + \ell_{12} + 2)^{2\ell_{12,3}+2\ell_{12}+4}}. \quad (29)
\]

For core emission this expression should be divided by \( 2^{\ell_{12,3}+1/2} \). For \( s \)-waves Eq. (29) is approximately

\[
\frac{T}{T_{12,3}} \approx \left( \frac{E}{7\text{MeV}} \right)^{3/2} \sqrt{\frac{E}{E - E_{12}}}. \quad (30)
\]

which as always for energies \( E_{12} \) very close to \( E \) favor direct decay, but sequential decay is favored for \( E_{12} \) relatively close to \( E \) for moderate energies up to a few MeV.

4 Summary and conclusions

We investigate the decay of low-lying continuum states into three particle final states. The initial state may be populated in reactions or by other decays like beta-decay. We assume that the decay mechanism is independent of how the initial state was formed. We consider the partial decay of a many-body resonance state into three specific fragments. We assume that these three fragments are formed with a certain probability before they enter regions of larger spatial extension on their way to total separation as observed in the final state. The analogy is \( \alpha \)-decay where a preformation factor describes the probability that the \( \alpha \)-particle exists before its attempts to penetrate the barrier. This relates to definitions of three-body spectroscopic factors.
We choose the hyperradius as the adiabatic coordinate which is an average radial coordinate obtained as a mass weighted mean square average of distances between pairs of particles. We can then study the structure along the path defined by any of the adiabatic potentials. Then one particle may increase its distance to the two other particles which first choose to stay together and at some later point also separate. This is naturally called sequential decay. The system may also prefer to increase all pairwise distances simultaneously. This is called direct decay. Thus we can study the conditions for choosing the different decay mechanisms.

We use a schematic model with only Coulomb and centrifugal barrier terms effective at distances larger than a minimum hyperradius. We discuss how to classify and characterize the sequential and direct decay mechanisms. With different geometric configurations we compare analytically the probabilities for sequential and direct decays. When all three particles are charged the Coulomb potential dominates over the centrifugal barrier. Sequential decay by emission of a high energy particle of small mass and small charge is most favorable simply because the corresponding barrier then is smaller. The short-range interaction are expected to provide the intermediate stepping stone of a favorable configuration like a two-body resonance or perhaps only by exploiting an attraction efficient at short distances. When only a large centrifugal barrier term is present sequential decay by emission of the large mass seems to be most likely, but if this barrier is relatively small the intermediate configuration cannot be reached before the full decay has taken place. These conclusions can be modified or changed by inclusion of the short-range interaction as discussed in the following companion paper.

References


