NOTES ON COARSE GRAININGS AND FUNCTIONS OF OBSERVABLES

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Abstract. Using the Naimark dilation theory we investigate the question under what conditions an observable which is a coarse graining of another observable is a function of it. To this end, conditions for the separability and for the Boolean structure of an observable are given.

Keywords: semispectral measure, Naimark dilation, coarse graining, separable observable, Boolean observable.

1. Introduction

Let \((\Omega, \mathcal{A})\) be a measurable space, \(\mathcal{H}\) a complex Hilbert space, \(\mathcal{L}(\mathcal{H})\) the set of bounded operators on \(\mathcal{H}\), and \(E : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})\) a normalized positive operator measure, that is, a semispectral measure. We call such measures observables of a physical system described by \(\mathcal{H}\).

Let \((\mathcal{K}, \tilde{E}, V)\) be a Naimark dilation of \(E\) into a spectral measure \(\tilde{E}\), that is, \(\tilde{E} : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{K})\) is a projection measure acting on a Hilbert space \(\mathcal{K}\) and \(V : \mathcal{H} \rightarrow \mathcal{K}\) an isometric linear map such that \(E(X) = V^* \tilde{E}(X)V\) for all \(X \in \mathcal{A}\). We say that an observable \(E : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})\) is separable if it has a Naimark dilation \((\mathcal{K}, \tilde{E}, V)\) which is separable, that is, the range \(\tilde{E}(\mathcal{A})\) of \(\tilde{E}\) is a separable Boolean sub-\(\sigma\)-algebra in the projection lattice \(\mathcal{P}(\mathcal{K})\) of the Hilbert space \(\mathcal{K}\). (We use the lattice theoretical terminology as introduced in [13].)

We recall that a Boolean sub-\(\sigma\)-algebra \(\mathcal{B}\) of \(\mathcal{P}(\mathcal{K})\) is separable, if there exists a countable subset \(B\) such that the smallest Boolean sub-\(\sigma\)-algebra of \(\mathcal{B}\) containing \(B\) is \(\mathcal{B}\). The importance of such sub-\(\sigma\)-algebras of \(\mathcal{P}(\mathcal{K})\) lies in the following fact: a Boolean sub-\(\sigma\)-algebra \(\mathcal{R}\) of \(\mathcal{P}(\mathcal{K})\) is the range of a real projection measure \(F : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{K})\), that is, \(\mathcal{R} = F(\mathcal{B}(\mathbb{R}))\) if and only if \(\mathcal{R}\) is separable [13 Lemma 3.16]. Furthermore, in that case, if \(\mathcal{R}_1\) is a Boolean sub-\(\sigma\)-algebra contained in \(\mathcal{R}\), then there is a real Borel function \(f\) such that \(\mathcal{R}_1 = F^f(\mathcal{B}(\mathbb{R}))\), where \(F^f(X) = F(f^{-1}(X))\) [13 Theorem 3.9], see also [2 Lemma 4.11].

Consider now two observables \(E_1\) and \(E\) defined on the \(\sigma\)-algebras \(\mathcal{A}_1\) and \(\mathcal{A}\) of the measurable spaces \((\Omega_1, \mathcal{A}_1)\) and \((\Omega, \mathcal{A})\), respectively,
and taking values in $\mathcal{L}(\mathcal{H})$. We say that $E_1$ is a function of $E$ if there is a measurable function $f : \Omega \rightarrow \Omega_1$ such that $E_1 = Ef$, that is, $E_1(X) = E(f^{-1}(X))$ for all $X \in \mathcal{A}_1$. Clearly, if $E_1$ is a function of $E$, then the range of $E_1$ is contained in the range of $E$. In general, for any two observables $E_1$ and $E$, if $E_1(\mathcal{A}_1) \subset E(\mathcal{A})$ we say that $E_1$ is a coarse graining of $E$.

Assume that $E_1$ is a coarse graining of $E$. If $(\mathcal{K}, \tilde{E}, V)$ is a Naimark dilation of $E$, we let $\mathcal{R}_1$ be the set of all projections $P \in \tilde{E}(\mathcal{A})$ such that $V^*PV \in E_1(\mathcal{A}_1)$. Then

$$E_1(\mathcal{A}_1) = V^*\mathcal{R}_1V \subset E(\mathcal{A}) = V^*\tilde{E}(\mathcal{A})V.$$ 

Calling two observables equivalent if their ranges are the same we observe that if $\tilde{E}(\mathcal{A})$ is a separable Boolean sub-$\sigma$-algebra of $\mathcal{P}(\mathcal{K})$, then $\tilde{E}$ is equivalent to a real projection measure $F : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{K})$. If, in addition, $\mathcal{R}_1$ is a Boolean sub-$\sigma$-algebra of $\tilde{E}(\mathcal{A})$ then it can be expressed as $\mathcal{R}_1 = F^f(\mathcal{B}(\mathbb{R}))$ for some Borel function $f$. In this case observables $E_1$ and $E$ are equivalent to the two real functionally related semispectral measures $E^*_r$ and $E^*$, where $E^*_r(X) = V^*F^f(X)V$ and $E^*(X) = V^*F(X)V$ for all $X \in \mathcal{B}(\mathbb{R})$.

The questions of interest for this study are the following. First, under what conditions is an observable separable? Secondly, if an observable is a coarse graining of another observable, when is it a function of the latter? Sections 2 and 3 are devoted to the separability questions whereas in Section 4 we study the question of functional relations between observables.

**Remark 1.** For positive operator measures $E_1$ and $E$, the condition $E_1(\mathcal{A}_1) \subset E(\mathcal{A})$ need not imply that $E_1$ is a function of $E$. However, $E_1$ and $E$ may still be functionally related (functionally coexistent) so that there is a positive operator measure $F$ with measurable functions $f$ and $g$ such that $E_1 = F \circ f^{-1}$ and $E = F \circ g^{-1}$. Indeed, as an illustration of this phenomenon, consider the real scalar measures $E$ and $E_1$ concentrated, respectively, on the sets $\{x_1, x_2, x_3, x_4\}$ and $\{y_1, y_2, y_3, y_4\}$ such that $E(\{x_1\}) = E(\{x_2\}) = 1/8, E(\{x_3\}) = E(\{x_4\}) = 3/8$, and $E_1(\{y_1\}) = E_1(\{y_2\}) = E_1(\{y_3\}) = 1/8, E_1(\{y_4\}) = 5/8$. Clearly, the range of $E_1$ is contained in that of $E$, but there is no function $f : \{x_1, x_2, x_3, x_4\} \rightarrow \{y_1, y_2, y_3, y_4\}$ such that $E_1(Y) = E(f^{-1}(Y))$. Indeed, if such a function exists, we must have $E_1(\{y_1\}) = E(f^{-1}(\{y_1\})) = 1/8$, which gives $f^{-1}(\{y_1\}) = \{x_1\}$, or $f^{-1}(\{y_1\}) = \{x_2\}$, and $E_1(\{y_4\}) = E(f^{-1}(\{y_4\}))$, which yields $f^{-1}(\{y_1\}) = \{x_1, x_2, x_3\}$ or $f^{-1}(\{y_1\}) = \{x_1, x_2, x_4\}$. Both $E$ and $E_1$ are, however, functions of the observable $\{z_i\} \mapsto F(\{z_i\}) = 1/8$, $i = 1, \ldots, 8$. 
2. Separable Boolean σ-algebras

In this section we collect, for the reader's convenience, some basic observations in the context of separable Boolean sub-σ-algebras of the projection lattice of a Hilbert space. The proofs follow readily from known facts and the results themselves may be part of the folklore of the subject though hard to find in the literature.

Let $\mathcal{B}$ be a Boolean algebra. An atom of $\mathcal{B}$ is any non-zero element $a$ of $\mathcal{B}$ such that $b \leq a$ for $b \in \mathcal{B}$ implies $b = 0$ or $b = a$. Let $\text{At}(\mathcal{B})$ be the set of all atoms of $\mathcal{B}$. If $\text{At}(\mathcal{B}) = \emptyset$, $\mathcal{B}$ is said to be atomless. If $a$ and $b$ are two different atoms of $\mathcal{B}$, then they are disjoint, $a \land b = 0$.

If $\mathcal{B}_i = (\mathcal{B}_i; 0_i, 1_i, ^i)$, $i = 1, 2$, are Boolean σ-algebras, then their Cartesian product $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$ is again a Boolean σ-algebra with operations defined coordinatewise, the least and the greatest elements being $0 = (0_1, 0_2)$ and $1 = (1_1, 1_2)$, respectively.

**Proposition 2.** Let $\mathcal{B}$ be a Boolean σ-algebra such that every system of mutually orthogonal non-zero elements of $\mathcal{B}$ is at most countable. Then $\mathcal{B}$ can be decomposed in the form $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$, where $\mathcal{B}_1$ is a Boolean σ-algebra isomorphic with the power set $2^N$, where $N$ is a finite or countable cardinal, and $\mathcal{B}_2$ is an atomless Boolean σ-algebra.

**Proof.** Let $\text{At}(\mathcal{B})$ be the set of all atoms of $\mathcal{B}$. Since any two different atoms $a$ and $b$ of $\mathcal{B}$ are mutually orthogonal, $a \leq b'$, $0 \leq |\text{At}(\mathcal{B})| \leq \aleph_0$.

Define $a_0 := \bigvee \{a : a \in \text{At}(\mathcal{B})\}$; if $\text{At}(\mathcal{B}) = \emptyset$, we put $a_0 := 0$. For any element $a \in \mathcal{B}$, we have the decomposition

$$a = (a \land a_0) \lor (a \land a'_0). \quad (1.1)$$

Define $\mathcal{B}_1 := \{a \in \mathcal{B} : a \leq a_0\}$ and $\mathcal{B}_2 := \{a \in \mathcal{B} : a \leq a'_0\}$. Then $\mathcal{B}_1 = (\mathcal{B}_1; 0, a_0, \lor a_0)$, where $x \lor a_0 := x' \land a_0$ for $x \in \mathcal{B}_1$, and $\mathcal{B}_2 = (\mathcal{B}_2; 0, a'_0, \lor a'_0)$, where $x \lor a'_0 := x' \land a'_0$ for $x \in \mathcal{B}_2$, are Boolean σ-algebras such that $\mathcal{B}_1$ is isomorphic with the σ-algebra $2^N$, where $N = |\text{At}(\mathcal{B})|$, and $\mathcal{B}_2$ is atomless. In view of (1.1) we have the decomposition $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$. \hfill $\square$

The set $\mathcal{P}(\mathcal{H})$ of all projections on $\mathcal{H}$ forms a complete orthomodular lattice with respect to the operator order and orthocomplementation $P \mapsto P^\perp := I_\mathcal{H} - P$, with $I_\mathcal{H} = I$ and $O_\mathcal{H} = O$ being the identity and zero operators on $\mathcal{H}$.

**Theorem 3.** Let $\mathcal{H}$ be a complex separable Hilbert space and let $\mathcal{B}$ be a Boolean sub-σ-algebra of $\mathcal{P}(\mathcal{H})$. Then $\mathcal{B}$ is separable. In particular, if $\mathcal{H}$ is finite dimensional, then $\mathcal{B} = 2^N$, where $N$ is an integer such that $1 \leq N \leq \dim \mathcal{H}$. 
Proof. Using Proposition 2 we decompose the \( \sigma \)-algebra \( \mathcal{B} \) in the form \( \mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \), where \( \mathcal{B}_1 \) is isomorphic with \( 2^N \), \( N = |\text{At}(\mathcal{B})| \), and \( \mathcal{B}_2 \) is atomless. Let \( P_0 = \bigvee \{ P : P \in \text{At}(\mathcal{B}) \} \) and denote \( \mathcal{H}_0 = P_0(\mathcal{H}) \).

Assume \( \dim \mathcal{H} = \aleph_0 \). If \( P_0 = I_{\mathcal{H}} \), then \( \mathcal{B} = \mathcal{B}_1 \), and \( \mathcal{B} \) is separable. If \( P_0 \neq I_{\mathcal{H}} \), then \( I_{\mathcal{H}} - P_0 \neq 0 \), and since \( \mathcal{B}_2 \) is atomless, we have \( \dim(\mathcal{H}_0) = \aleph_0 \). In addition, \( \mathcal{B}_2 \) is therefore a Boolean \( \sigma \)-algebra which is a subalgebra of \( \mathcal{P}(\mathcal{H}_0^\perp) \). Let \( \mathbb{B}_2 \) be the von Neumann algebra generated by \( \mathcal{B}_2 \). Then \( \mathbb{B}_2 \) is a commutative von Neumann algebra acting in the infinite-dimensional complex separable Hilbert space \( \mathcal{H}_0^\perp \) and the projection lattice of \( \mathbb{B}_2 \) coincides with \( \mathcal{B}_2 \) which is atomless. Therefore, by [12, Theorem III.1.22], \( \mathbb{B}_2 \) is isomorphic with the von Neumann algebra \( L^\infty(0, 1) \) (the space of all essentially bounded functions on the unit interval \( (0, 1) \) with respect to the Lebesgue measure). Since the projections from \( L^\infty(0, 1) \) are only characteristic functions, they have a countable generator, consequently, \( \mathcal{B}_2 \) has a countable generator. Because \( \mathcal{B}_1 \) is generated by the countable set of atoms, in view of \( \mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \), \( \mathcal{B} \) is separable.

Assume now \( \dim \mathcal{H} < \infty \). Then \( P_0 = I_{\mathcal{H}} \) and therefore, \( \mathcal{B} = \mathcal{B}_1 = 2^N \). □

3. Separable observables

A Naimark dilation \((\mathcal{K}, \tilde{E}, V)\) of a semispectral measure \( E : \mathcal{A} \to \mathcal{L}(\mathcal{H}) \) is minimal if \( \mathcal{K} \) is the closed linear span of \( \{ \tilde{E}(X) \mid X \in \mathcal{A} \} \). As is well known, a minimal dilation always exists and it is unique up to an isometric isomorphism [10].

Lemma 4. Let \((\Omega, \mathcal{A})\) be a measurable space with a separable \( \sigma \)-algebra \( \mathcal{A} \) and let \( E : \mathcal{A} \to \mathcal{L}(\mathcal{H}) \) be a normalized positive operator measure acting on a complex separable Hilbert space \( \mathcal{H} \). If \((\mathcal{K}, \tilde{E}, V)\) is a minimal Naimark dilation of \( E \), then \( \mathcal{K} \) is separable.

Proof. Let \( \mathcal{F} \) be a countable collection of subsets of \( \Omega \) which generates the \( \sigma \)-algebra \( \mathcal{A} \), and let \( \mathcal{R} \) be the ring generated by \( \mathcal{F} \). Since \( \mathcal{F} \) is countable, the ring \( \mathcal{R} \) is countable [3, Theorem I.5.C]. Let \( \mathcal{C} \) be the complex linear span of the characteristic functions \( \chi_X \) of the sets \( X \in \mathcal{R} \), and let \( \tilde{\mathcal{C}} \) be its closure in the set of bounded functions \( \Omega \to \mathcal{C} \) (with respect to the sup-norm). \( \tilde{\mathcal{C}} \) is a separable commutative \( C^* \)-algebra. Let \( \Phi : \tilde{\mathcal{C}} \to \mathcal{L}(\mathcal{H}) \) be the positive linear map corresponding to the normalized positive operator measure \( E : \mathcal{A} \to \mathcal{L}(\mathcal{H}) \), \( \Phi(f) = \int f \, dE \). Then \( \Phi \) is completely positive [10, Theorem 3.10]. Let \((\mathcal{K}, \pi, V)\) be its minimal Stinespring dilation. The Hilbert space \( \mathcal{K} \) is separable [10, p. 46]. Let \( P_o : \mathcal{R} \to \mathcal{L}(\mathcal{K}) \) be the projection-valued set function
defined by $P_o(X) = \pi(\chi_X)$ for all $X \in \mathcal{R}$. Then $V^*P_o(X)V = E(X)$ for all $X \in \mathcal{R}$. From its construction it easily follows that $P_o$ is weakly $\sigma$-additive on $\mathcal{R}$.

For any $\varphi \in \mathcal{K}$ and $X \in \mathcal{R}$ denote $\mu_{\varphi,\varphi}^o(X) = \langle \varphi \vert P_o(X)\varphi \rangle$. Since $\mu_{\varphi,\varphi}^o$ is $\sigma$-additive on $\mathcal{R}$, it has a unique extension to a (positive) measure $\mu_{\varphi,\varphi}$. For any $\phi \in \mathcal{K}$ and $X \in \mathcal{R}$ denote $\mu_{\varphi,\phi}^o(X) = \langle \phi \vert P_o(X)\phi \rangle$. Since $\mu_{\varphi,\phi}^o$ is $\sigma$-additive on $\mathcal{R}$, it has a unique extension to a (positive) measure $\mu_{\phi,\phi}$. For any $\phi,\psi \in \mathcal{K}$, we let $\mu_{\phi,\psi}^o = \frac{1}{4}\sum_{k=1}^4 i^k \mu_{\varphi+i\psi,\varphi+i\psi}$. Elementary estimates show that the map $(\phi,\psi) \mapsto \mu_{\phi,\psi}(X)$ is a bounded sesquilinear form for each $X \in \mathcal{A}$, and we get a positive operator measure $\tilde{P} : \mathcal{A} \to \mathcal{L}(\mathcal{K})$ which extends $P_o$.

It remains to be shown that the map $\tilde{P}$ is a projection measure. We denote by $M(\mathcal{R})$ the monotone class generated by $\mathcal{R}$. The class $\{X \in \mathcal{A} \mid \tilde{P}(X)^2 = \tilde{P}(X)\}$ contains $\mathcal{R}$ and is easily seen to be a monotone class and so it equals $\mathcal{A}$ [3, Theorem I.6.B]. Clearly, $V^*\tilde{P}(X)V = E(X)$ for all $X \in \mathcal{A}$ and $(\mathcal{K}, \tilde{P}, V)$ constitutes a minimal dilation of $E$ and $\mathcal{K}$ is separable.

An alternative approach would be to use in the above proof Naimark’s dilation theory [11, Appendix, Theorem 1] instead of Stinespring’s.

**Remark 5.** A physically relevant dilation $(\mathcal{K}, \tilde{E}, V)$ of a quantum observable $E$ is typically not minimal, see e.g. [8]. An interesting example of a dilation acting on a nonseparable Hilbert space appears in [9] for the canonical phase observable.

**Corollary 6.** Let $(\Omega, \mathcal{A})$ be a measurable space with a separable $\sigma$-algebra $\mathcal{A}$ and let $\mathcal{H}$ be a complex separable Hilbert space. Any normalized positive operator measure $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is separable.

**Proof.** Let $(\mathcal{K}, \tilde{E}, V)$ constitute a minimal Naimark dilation of $E$. The set $\{\tilde{E}(X)\varphi \mid X \in \mathcal{A}, \varphi \in \mathcal{H}\}$ is dense in $\mathcal{K}$. By Lemma [4] $\mathcal{K}$ is separable. Therefore, by Theorem [3] $\tilde{E}(\mathcal{A})$ is a separable Boolean sub-$\sigma$-algebra of $\mathcal{P}(\mathcal{K})$. □

### 4. Boolean observables

The Boolean structure of the range of an observable plays an important role in the functional calculus of observables. We therefore recall the following results. Here $\mathcal{E}(\mathcal{H})$ denotes the set of effect operators on $\mathcal{H}$, i.e., $\mathcal{E}(\mathcal{H}) = \{A \in \mathcal{L}(\mathcal{H}) : \ O \leq A \leq I\}$.

**Proposition 7.** The range $E(\mathcal{A})$ of an observable $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is a Boolean subalgebra of the set $\mathcal{E}(\mathcal{H})$ of effects if and only if $E$ is projection valued.
Proof. For any $X \in \mathcal{A}$ the product $E(X)E(X')$ is a positive lower bound of $E(X)$ and $E(X')$. If $E(\mathcal{A})$ is Boolean then $E(X)\land E(X') = O$, and thus $E(X)E(X') = O$, that is, $E(X)^2 = E(X)$. On the other hand, if $E$ is projection valued, then the claim follows from the multiplicativity of the spectral measure and from the fact that for any two projections $P$ and $R$ their greatest lower bound and smallest upper bound in $\mathcal{E}(\mathcal{H})$ are the same as in $\mathcal{P}(\mathcal{H})$, that is, $P \land R$ and $P \lor R$, respectively.

The order structure of the set of effects $\mathcal{E}(\mathcal{H})$ is highly complicated. For instance, if $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is an observable, then for any $X, Y \in \mathcal{A}$, the effect $E(X \cap Y)$ is a lower bound of the effects $E(X)$ and $E(Y)$, but these effects need not have the greatest lower bound $E(X) \lor E(Y)$ and even if $E(X) \lor E(Y)E(Y)$ exists it need not coincide with $E(X \cap Y)$. When the order and the complement of $\mathcal{E}(\mathcal{H})$ are restricted to the range $E(\mathcal{A})$ of $E$ it is possible that the system $(E(\mathcal{A}), \preceq, \lor)$ is a Boolean $\sigma$-algebra without $E$ being projection valued. To express that option it is useful to introduce two further concepts. We say that an observable $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ is regular if for any $O \neq E(X) \neq I$, neither $E(X) \leq E(X')$ nor $E(X') \leq E(X)$, and it is $\Delta$-closed if for any triple of pairwise orthogonal elements $A, B, C \in E(\mathcal{A})$, the sum $A + B + C$ is in $E(\mathcal{A})$.

From [5, 1, 7] the following results are then obtained.

**Proposition 8.** For any observable $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ the following three conditions are equivalent.

a) $(E(\mathcal{A}), \preceq, \lor)$ is a Boolean $\sigma$-algebra.

b) $E$ is regular.

c) $E$ is $\Delta$-closed.

Consider now two observables $E_1$ and $E$ defined on the $\sigma$-algebras $\mathcal{A}_1$ and $\mathcal{A}$ of the measurable spaces $(\Omega_1, \mathcal{A}_1)$ and $(\Omega, \mathcal{A})$, respectively, and taking values in $\mathcal{L}(\mathcal{H})$, with $\mathcal{H}$ being complex and separable. Assume that $E_1$ is a coarse graining of $E$, that is, $E_1(\mathcal{A}_1) \subset E(\mathcal{A})$. Let $(\mathcal{K}, \tilde{E}, V)$ be a Naimark dilation of $E$, with separable $\mathcal{K}$, and let $\mathcal{R}_1$ be again the set of projections $P \in \tilde{E}(\mathcal{A})$ such that $V^*PV \in E_1(\mathcal{A}_1)$.

**Proposition 9.** With the above notations, $\mathcal{R}_1$ is a Boolean sub-$\sigma$-algebra of $\mathcal{P}(\mathcal{K})$ if and only if there is a real Borel function $f$ and a real semispectral measure $E_r$ such that $E$ is equivalent with $E_r$ and $E_1$ is equivalent with $E_r^f$.

**Proof.** If $\mathcal{R}_1$ is a Boolean sub-$\sigma$-algebra of $\mathcal{P}(\mathcal{K})$ then, as a subset of $\tilde{E}(\mathcal{A})$, it is also separable. Thus by the results [13, Lemma 3.16, Theorem 3.9] there is a real projection measure $F_r$ and a real Borel
function $f$ such that $\tilde{E}(A) = F_r(B(\mathbb{R}))$ and $\mathcal{R}_1 = F_r^f(B(\mathbb{R}))$. The semispectral measures $E_r := V^*F_rV$ and $E_r^f := V^*F_r^fV$ are now as required. The other direction is immediate. □

We say that an observable $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ has the $V$-property with respect to a subset $Q$ of $E(\mathcal{A})$ if for each $X, Y \in \mathcal{A}$ and $C \in Q$ the inequality $E(X) \leq C \leq E(Y)$ implies that there is a $Z \in \mathcal{A}$ such that $X \subseteq Z \subseteq Y$ and $C = E(Z)$. The importance of this property is in the fact that for any two (real) observables $E_1$ and $E$, if $E_1(\mathcal{A}) \subseteq E(\mathcal{A})$ and if $E$ has the $V$-property on $E_1(\mathcal{A})$, then $E_1$ is a function of $E$. □

Lemma 10. With the above notations, $O_K, I_K \in \mathcal{R}_1$, and if $P \in \mathcal{R}_1$ then also $P^\perp \in \mathcal{R}_1$. Moreover, for any $P, R \in \mathcal{R}_1$, if $P \leq R$, then $V^*PV \leq V^*RV$. In addition, the observable $\tilde{E}$ has the $V$-property on $\mathcal{R}_1$.

Proof. If $P \in \mathcal{R}_1$, then $V^*PV = E_1(X)$ for some $X \in \mathcal{A}_1$ and thus $E_1(X') = I_\mathcal{H} - E_1(X) = V^*V - V^*PV = V^*(I_K - P)V$, so that $P^\perp \in \mathcal{R}_1$. If $P \leq R$, then for any $\psi \in \mathcal{K}$, $\langle \psi | PV \psi \rangle \leq \langle \psi | RV \psi \rangle$, and thus, in particular, for any $\varphi \in \mathcal{H}$, $\langle \varphi | E_1(X) \varphi \rangle = \langle \varphi | V^*PV \varphi \rangle = \langle V^*P \varphi | V^*V \varphi \rangle \leq \langle V^*P \varphi | RV \varphi \rangle = \langle \varphi | V^*RV \varphi \rangle = \langle \varphi | E_1(Y) \varphi \rangle$. To demonstrate the $V$-property, let $X, Y \in \mathcal{A}$, $X \subseteq Y$, so that $\tilde{E}(X) \leq \tilde{E}(Y)$. Assume that $P \in \mathcal{R}_1$ is such that $\tilde{E}(X) \leq P \leq \tilde{E}(Y)$. Let $Z \in \mathcal{A}$ be such that $\tilde{E}(Z) = P$. Then for $Z_1 = X \cup (Y \setminus Z)$ we have $X \subseteq Z_1 \subseteq Y$, and $\tilde{E}(Z_1) = \tilde{E}(X) \lor (\tilde{E}(Y) \land \tilde{E}(Z)) = (\tilde{E}(X) \lor \tilde{E}(Y)) \land (\tilde{E}(X) \lor P) = \tilde{E}(Y) \land P = P$. □

Remark 11. The assumption that $\tilde{E}$ has the $V$-property on $\mathcal{R}_1$ does not imply that $E$ has the $V$-property on $E_1(\mathcal{A})$. For an illustration, see Remark 1.

Proposition 12. With the above notations, if $E_1$ is projection valued, then $\mathcal{R}_1$ is a Boolean sub-$\sigma$-algebra of $\tilde{E}(\mathcal{A})$.

Proof. For any $P \in \mathcal{P}(\mathcal{K})$, $V^*PV \in \mathcal{P}(\mathcal{H})$ if and only if $VV^*P = PVV^*$. Let $P, R \in \mathcal{R}_1$ so that there are $X, Y \in \mathcal{A}_1$ such that $V^*PV = E_1(X)$ and $V^*RV = E_1(Y)$. Then

$$V^*P \land RV = V^*PRV = V^*VV^*PRV = V^*PVV^*RV = E_1(X)E_1(Y) = E_1(X \cap Y)$$

showing that $\mathcal{R}_1$ is closed under $\land$. By the de Morgan laws, the same is true for $\lor$. If $(P_n)_{n=1}^\infty$ is a sequence of mutually orthogonal projections
of $\mathcal{R}_1$, that is, $P_n \leq P_m^\perp$ for all $n \neq m$, then also $E_1(X_n) \leq E_1(X_m)^\perp = E_1(X_m')$. Therefore,

$$V^*(\sqrt{P_n}V = V^*(\sum P_n) = \sum V^*P_n V = \sum E_1(X_n) = E_1(\bigcup X_n)$$

(where the series converge weakly) which shows the $\sigma$-property of $\mathcal{R}_1$. □

**Corollary 13.** Let $\Omega_1$ and $\Omega$ be complete separable metric spaces and let $\mathcal{B}(\Omega_1)$ and $\mathcal{B}(\Omega)$ be their respective Borel $\sigma$-algebras. Assume that $\Omega_1$ and $\Omega$ have the cardinality of $\mathbb{R}$. Consider the observables $E_1 : \mathcal{B}(\Omega_1) \rightarrow \mathcal{L}(\mathcal{H})$ and $E : \mathcal{B}(\Omega) \rightarrow \mathcal{L}(\mathcal{H})$ such that $E_1$ is a coarse graining of $E$. If $E_1$ is projection valued, then $E_1 = E^f$ for some Borel function $f : \Omega \rightarrow \Omega_1$.

**Proof.** Since $\Omega_1$ and $\Omega$ are complete separable metric spaces with the cardinality of $\mathbb{R}$, according to [4, Remark (ii), p. 451], there are bijections $\alpha : \Omega \rightarrow \mathbb{R}$ and $\beta : \Omega_1 \rightarrow \mathbb{R}$ which are such that $\alpha, \alpha^{-1}, \beta, \beta^{-1}$ are Borel measurable. Now $E^\alpha$ and $E_1^\beta$ are real observables with the same ranges as $E_1$ and $E$, respectively. By [13, Theorem 3.9] there is a measurable function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that $E_1^\alpha(X) = E_1^\alpha(g^{-1}(X)), X \in \mathcal{B}$. Putting $X = \beta(Z), Z \in \mathcal{B}(\Omega)$, we obtain $E_1(Z) = E_1^\alpha(\beta(Z)) = E_1^\alpha(g^{-1}(\beta(Z))) = E_1^f(Z)$, where $f = \beta^{-1} \circ g \circ \alpha : \Omega \rightarrow \Omega_1$. □

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