Supersymmetric AdS Backgrounds
in String and M-theory

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Abstract

We first present a short review of general supersymmetric compactifications in string and M-theory using the language of $G$-structures and intrinsic torsion. We then summarize recent work on the generic conditions for supersymmetric AdS$_5$ backgrounds in M-theory and the construction of classes of new solutions. Turning to AdS$_5$ compactifications in type IIB, we summarize the construction of an infinite class of new Sasaki–Einstein manifolds in dimension $2k + 3$ given a positive curvature Kähler–Einstein base manifold in dimension $2k$. For $k = 1$ these describe new supergravity duals for $\mathcal{N} = 1$ superconformal field theories with both rational and irrational R-charges and central charge. We also present a generalization of this construction, that has not appeared elsewhere in the literature, to the case where the base is a product of Kähler–Einstein manifolds.

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1 Introduction

In this paper we aim to review first the general framework of supersymmetric solutions of string or M-theory, where spacetime is a product $E \times X$ of an external manifold $E$ and an internal manifold $X$, and then, secondly, two interesting classes of examples where $E$ is five-dimensional anti-de Sitter (AdS$_5$) space and $X$ is five- or six-dimensional. This latter work was first presented in three papers, refs. [1], [2] and [3]. Such backgrounds are central in string theory first, when $E$ is four-dimensional Minkowski space, as a way to construct semi-realistic supersymmetric models of particle physics, and second, when $E$ is an AdS space, as gravitation duals of quantum conformal field theories, via the AdS-CFT correspondence (for a review see ref. [4]).

We consider string or M-theory in the low-energy supergravity limit where the condition for a supersymmetric solution requires the existence of a constant spinor with respect to a particular Clifford algebra-valued connection $D^X$, perhaps supplemented with additional algebraic conditions on the spinor. When certain fields, so-called $p$-form fluxes, in the supergravity are zero, $D^X$ is equal to the Levi–Civita connection and hence supersymmetry translates into a condition of special holonomy. However, in many cases one wants to include non-trivial flux. In the first part of the paper we review how this translates into the existence of a $G$-structure $P$ and how the fluxes are encoded in the intrinsic torsion of $P$. We also comment on the relation to generalized holonomy and generalized calibrations. By way of an example we concentrate on the case of $d = 11$ supergravity on a seven-dimensional $X$ with $SU(3)$-structure, and type IIB supergravity on a six-dimensional $X$ and only five-form flux with an $SU(3)$-structure.

The second part of the paper, based on ref. [1], discusses first the general conditions on the geometry of $X$ in supersymmetric AdS$_5 \times X$ solutions of $d = 11$ supergravity, and second a large family of explicit regular solutions of this form characterized by $X$ being complex. Previously, a surprisingly small number of explicit solutions were known. Most notable was that of Maldacena and Nuñez [5] describing the near horizon limit of fivebranes wrapping constant curvature holomorphic curves in Calabi–Yau three-folds. The new solutions can be viewed as corresponding to a more general type of embedded holomorphic curve. They fall into two classes where $X$ is a fibration of a two-sphere over either a four-dimensional Kähler–Einstein manifold or a product of constant-curvature Riemann surfaces.

The third part of the paper relates to Sasaki–Einstein (SE) manifolds. These arise as the internal manifold $X$ in supersymmetric type IIB AdS$_5 \times X$ solutions. We review a new construction [2,3] of an infinite class of SE manifolds in any dimension $n = 2k+3$ based on an underlying $2k$-dimensional positive curvature Kähler–Einstein manifold. All SE spaces have a constant norm Killing vector $K$ (see, for instance, refs. [6] and [7]) and can be characterized by whether the orbits of $K$ are closed (so-called regular and quasi-regular cases) or not (the irregular case). The new class of solutions includes quasi-regular and irregular cases. Again, previously, surprisingly few explicit SE metrics were known: the homogeneous regular cases have been classified [3]; several quasi-regular examples had been constructed using algebraic geometry techniques but without an explicit metric; and no irregular examples were known. Finally we give a straightforward extension of the construction to the case where the underlying manifold is a product of Kähler–Einstein spaces. This is new material and leads to new AdS$_4 \times X_7$ solutions of M-theory.
The history of considering supersymmetric backgrounds of supergravity theories with non-trivial fluxes is a comparatively long one. The use of $G$-structures to classify such backgrounds was first proposed in ref. [9]. This was based partly on earlier work by Friedrich and Ivanov [10], though these authors did not consider the supergravity equations of motion. The relationship between background supersymmetry conditions and generalized calibrations [11] was first discussed slightly earlier in ref. [12] and shown to be generic in ref. [13, 14].

These techniques have subsequently been developed and extended in a number of directions. First, one can use $G$-structures to classify all supersymmetric solutions of a given supergravity theory. This has been carried out for the most generic case of a single preserved supersymmetry in $d = 11$ supergravity in ref. [13]. A similar classification has now also been carried out for simpler supergravity theories in four [15], five [16], six [17] and seven [18] dimensions. Note that this work extends older work of Tod [19] which classified supersymmetric solutions of four-dimensional supergravity using techniques specific to four dimensions. An important open problem in, for instance, the $d = 11$ case is to refine the classification presented in ref. [13] and determine the extra conditions required for solutions to preserve more than one supersymmetry. There has been some recent progress on this using $G$-structures, partly implementing some suggestions in ref. [13], in the context of seven [20] and eleven dimensions [21]. Note that the case of maximal supersymmetry can be analysed using different techniques and this has been carried out for type IIB and $d = 11$ supergravity in ref. [22]. A quite different attempt at classification, first advocated in refs. [23] and subsequently studied in ref. [24], is to use the notion of “generalized holonomy”. We comment on the relation to $G$-structures in the next section.

A second application is to use $G$-structures to analyse supersymmetric “flux compactifications” in string theory. These are supersymmetric backgrounds where the external space $E$ is flat Minkowski space and $X$ is often, but not always, compact. This is a large field with a rich literature, starting with that of Strominger [25] and Hull [26] in the context of the heterotic string (see also ref. [27]). More recently, starting with Polchinski and Strominger [28] as well as ref. [29], several authors have analyzed flux backgrounds for the special case where $X$ is a special holonomy manifold (for early work see refs. [30]–[34]) and the resulting low-energy effective theories on $E$ (a large field, see, for instance, the references in ref. [35] or ref. [37]). Let us concentrate on the use of $G$-structure techniques to analyse cases when $X$ does not have special holonomy. In ref. [35] it was argued that the mirror of a Calabi–Yau threefold with three-form $H$-flux is a manifold with a “half-flat” $SU(3)$-structure. Further work in this direction appears in refs. [36]. For the heterotic string, Strominger’s and Hull’s results imply $X$ has a non-Kähler $SU(3)$ structure and these have been analyzed in refs. [14, 37, 42]. Such flux compactifications have only $H$-flux, and these were completely classified, including type II backgrounds, in ref. [14]. (Note that ref. [14] corrects a sign in ref. [25], disqualifying the putative Iwasawa solutions in ref. [37].) Flux compactifications on more general $SU(3)$-structures have been considered in refs. [43, 46]. General type II compactifications with more general fluxes have been addressed, for instance, in refs. [47]–[53]. General $d = 11$ flux compactifications have been discussed in terms of $G$-structures in several papers [54], [55]–[62].

A third connected application is to spacetime solutions dual to supersymmetric field theories via the AdS-CFT correspondence. The basic case of interest is when $E$ is AdS
since the solution is then dual to a supersymmetric conformal field theory. There are more general kinds of solutions, however, that are dual to other types of field theories, as well as to renormalisation group flows (see the review [4]). Again this is very large field. Aside from the initial paper [9] (which focussed on solutions dual to “little string theories”) and the work [11, 2, 3] on which this paper is based, G-structures have been used to analyze AdS solutions in refs. [5, 8] and [60]. Very recently an interesting class of half-supersymmetric solutions has been found [63]. Note that there is also a related approach to finding special sub-classes of solutions initiated by Warner and collaborators (see for instance refs. [64]).

Finally, we comment on some work related to G-structure classifications and generalised calibrations. Calibrations are important in string theory backgrounds with vanishing fluxes since the calibrated cycles are the cycles static probe branes can wrap whilst preserving supersymmetry. Generalised calibrations [11] are the natural generalisation to backgrounds when the fluxes are non-vanishing. Important work relating calibrations and the superpotential of the effective theory on $E$ first appeared in ref. [65]. Starting with the work [12] and subsequent work including refs. [13, 14, 58] it has become clear that the conditions placed on supersymmetric backgrounds often have the useful physical interpretation as generalised calibrations. The reason for this is simply that the backgrounds can arise when branes wrap calibrated cycles after taking into account the back-reaction (see ref. [14] for further discussion). Such wrapped brane solutions were first found in ref. [5] and a review can be found in ref. [66]. The relationship between wrapped and intersecting brane solutions and generalised calibrations has been studied in refs. [67, 68, 69]. The classification of supersymmetric solutions using G-structures has also led to a further exploration of generalised calibrations for non-static brane configurations [70]. Further work, specifically on the relationship between supersymmetry and generalized calibrations in flux compactifications, has appeared in refs. [12, 14, 71].

2 Supersymmetry and G-structures

2.1 Some supergravity

Let us start by characterising the type of problem we are trying to solve. First we summarise a few relevant parts of the supergravity theories which arise in string theory and then describe the notion of a supersymmetric compactification or reduction. We will concentrate on two fairly generic examples in ten and eleven dimensions.

We start with a supergravity theory on a $d$-dimensional Lorentzian spin manifold $M$. This is an approximation to the full string theory valid in the limit where the curvature of the manifold is small compared with the intrinsic string scale. The supergravity is described in terms a number of fields, including the bosonic fields

$$
\begin{align*}
g & \quad \text{Lorentzian metric,} \\
\Phi & \in C^\infty(M) \quad \text{dilaton}, \\
F^{(p)} & \in C^\infty(\Lambda^p T^* M) \quad p\text{-form fluxes,}
\end{align*}
$$

for certain values of $p$, satisfying equations of motion which are generalisations of Einstein’s and Maxwell’s equations. Particular $p$-form fluxes are also sometimes labeled $G$ or $H$. For
the cases we will consider, the dilaton $\Phi$ is either not present in the theory or assumed to be zero. Since the theory is supersymmetric these fields are paired with a set of fermionic fields transforming in spinor representations. However, these will all be set to zero in the backgrounds we consider. A bosonic solution of the equations of motion is called a supergravity background.

We would like to characterise supersymmetric backgrounds. Let $S \rightarrow M$ be a spin bundle. (Precisely which spinor representation we have depends on the dimension $d$ and the type of supergravity theory.) The supergravity theory defines a particular connection

$$D : C^\infty(S) \rightarrow C^\infty(S \otimes T^* M)$$

in terms of the metric, dilaton and $p$-form fluxes. A background is supersymmetric if we have a non-trivial solution to

$$D \epsilon = 0$$

for $\epsilon \in C^\infty(S)$. If we have $n$ independent solutions then the background is said to preserve $n$ supersymmetries. (Often the supergravity also defines a map $P \in C^\infty(\text{End}(S))$ in terms of the dilaton and $F^{(p)}$ and a supersymmetric solution must simultaneously satisfy the “dilatino equation” $P \epsilon = 0$. For our particular examples either $P$ is not present in the supergravity theory or is assumed to be identically zero.)

The two cases we will consider are (1) $d = 11$ supergravity with four-form flux $G$ and (2) $d = 10$ Type IIB supergravity keeping only a self-dual five-form flux $F^{(5)} = *F^{(5)}$. The corresponding supergravity connections are given, in components, by

$$D = \nabla^g + \frac{1}{12} G \Gamma^{(5)} + \frac{1}{6} \Gamma^{(3)} \otimes G$$

$$D = \nabla^g \otimes \text{id} - \frac{1}{8} \Gamma^{(4)} \otimes F^{(5)} \otimes i\sigma_2$$

where $\nabla^g$ is Levi–Civita connection for $g$. In the first case, the spinor $\epsilon$ is a 32-dimensional real representation $\Delta^{R}_{10,1}$ of Spin$(10,1)$ while in the second case $\epsilon$ is a pair of spinors $(\epsilon_1, \epsilon_2)$ each in the 16-dimensional real, chiral representation $\Delta^{+R}_{9,1}$ of Spin$(9,1)$. The gamma matrices $\Gamma$ generate Cliff$(d-1,1)$ and $\Gamma^{(p)}$ is the antisymmetrised product of $p$ gamma matrices. The matrices $\text{id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ act on the doublet of spinors $\epsilon = (\epsilon_1, \epsilon_2)$. In index notation we have $(v \otimes w)_{M_{p+1}...M_q} = \frac{1}{p!} \epsilon^{M_1...M_p} \epsilon_{M_1...M_p M_{p+1}...M_q}$.

2.2 The problem

It is interesting to determine what the existence of solutions to the Killing spinor equation implies about the geometry of $M$, in general. For example, this has been studied in ref. [13] for the most general solutions of supergravity in eleven dimensions. However, here we are concerned with a more restricted problem.

First we assume we have a compactification, where the topology of $M$ is taken to be a product

$$M = E \times X$$

of a $(d-n)$-dimensional external manifold $E$ and an $n$-dimensional internal manifold $X$. Although compact $X$ is often of most interest, by an abuse of terminology, we will also allow
for non-compact \( X \). Next, the metric is taken to be a warped product. In particular we consider two cases
\[
\begin{align*}
g &= e^{2\Delta} \eta_{d-n} + g_X \quad \text{flat space}, \\
g &= e^{2\lambda} \phi_{d-n} + g_X \quad \text{AdS space},
\end{align*}
\]
where \( g_X \) is a Riemannian metric on \( X \) while \( \eta_r \) is the flat Minkowski metric on \( E = \mathbb{R}^{r-1,1} \) and \( \phi_r \) is the constant curvature metric on anti-de Sitter space \( E = \text{AdS}_r \). In the latter case
\[ \text{Ric}_{\phi_r} = -(r-1)m^2 \phi_r \]
where \( m \) is the inverse radius of the AdS space. In each case \( \lambda, \Delta \in C^\infty(X) \).

Finally the dilaton and fluxes are assumed to be given by objects on \( X \), so that in the two cases we have
\[
\begin{align*}
F^{(p)} &= \begin{cases} 
  e^{(d-n)\Delta} \text{vol}_{\eta_{d-n}} \wedge f + h, & f \in C^\infty(\Lambda^{n-d+p}T^*X), \\
  e^{(d-n)\lambda} \text{vol}_{\phi_{d-n}} \wedge f + h, & h \in C^\infty(\Lambda^pT^*X), \\
  \Phi \in C^\infty(X)
\end{cases},
\end{align*}
\]
where \( \text{vol}_{g_E} \) is the volume form corresponding to the metric \( g_E \in \{\eta_r, \phi_r\} \) on \( E \) and \( f \) and \( h \) are sometimes referred to the as the electric and magnetic fluxes.

Physically such solutions are interesting because, first, in the flat-space case with \( d - n = 4 \), the space \( E \) is a model for four-dimensional particle physics. Secondly, the AdS-CFT correspondence \[4\] implies that such AdS geometries should be gravitational duals of conformal field theories in \( d - n - 1 \) dimensions. The particular internal \( X \) geometry encodes the content of the particular conformal field theory.

Given this product ansatz the Killing spinor equation \[2.3\] reduces to equations on a spinor \( \psi \) of \( \text{Spin}(d-n-1,1) \) on \( E \) and a spinor, not necessarily irreducible, \( \xi \) of \( \text{Spin}(n) \) on \( X \). The exact decomposition of \( \epsilon \in C^\infty(S) \) depends on the dimensions \( d \) and \( n \). In all cases one takes \( \psi \) to satisfy the standard Killing spinor equation on \( E \), that is
\[
\begin{align*}
\nabla^\eta \psi &= 0 \quad \text{flat space}, \\
(\nabla^\phi - \frac{1}{2}m\rho) \psi &= 0 \quad \text{AdS space},
\end{align*}
\]
where \( \nabla^\eta \) and \( \nabla^\phi \) are the Levi-Civita connections for \( \eta_{d-n} \) and \( \phi_{d-n} \) respectively and \( \rho \) are gamma matrices for \( \text{Spin}(d-n-1,1) \). If \( S^X \to X \) is the spin bundle on \( X \) coming from the decomposition of \( S \), the Killing spinor equation \[2.3\] then has the form\[1\]
\[
D^X \xi = 0, \quad Q^X \xi = 0, \quad \text{reduced Killing spinor eqns.}
\]
where the connection \( D^X : C^\infty(S^X) \to C^\infty(S^X \otimes T^*X) \) and the map \( Q^X : C^\infty(S^X) \to C^\infty(S^X) \) each are defined in terms of flux, dilaton, and \( \Delta \) or \( \lambda \) and \( m \). The condition \( D^X \xi = 0 \) comes from the reduction of \( D\epsilon = 0 \) on \( X \) and \( Q^X \xi = 0 \) from the reduction on \( E \).

Our basic question is then
\[1\]If there was also originally a \( P\epsilon = 0 \) condition this also reduces to a further condition \( P^X \xi = 0 \) with \( P^X \in C^\infty(\text{End}(S^X)) \).
what does the existence of solutions to the reduced Killing spinor equations imply about the geometry of $X$ and the form of the fluxes and dilaton?

In general we want to translate the Killing spinor conditions into some convenient set of necessary and sufficient conditions, such as, for instance, $X$ has a particular almost complex or contact structure or a particular Killing vector.

Let us end with a couple of further comments. First note that there is a connection between the two types of compactification (2.7). Consider $E \times X = \text{AdS}_{d-n} \times X$. Locally we can write the AdS metric $\phi_{d-n}$ in Poincaré coordinates

$$\phi_{d-n} = e^{-2mr} \eta_{d-n-1} + \text{dr} \otimes \text{dr}. \quad (2.12)$$

Thus we have

$$g = e^{2\lambda} \phi_{d-n} + g_X = e^{2\lambda} e^{-2mr} \eta_{d-n-1} + \left( e^{2\lambda} \text{dr} \otimes \text{dr} + g_X \right)$$

$$\equiv e^{2\Delta} \eta_{d-n} + g_{X'}, \quad (2.13)$$

where

$$\Delta = \lambda - mr,$$

$$g_{X'} = e^{2\lambda} \text{dr} \otimes \text{dr} + g_X. \quad (2.14)$$

and hence an AdS compactification on $X$ to $\text{AdS}_{d-n}$ is really a special case of a flat space compactification on $X' = X \times \mathbb{R}^+$ to $\mathbb{R}^{d-n-2,1}$. This will be particularly useful for deriving the conditions on the geometry of AdS compactifications in what follows.

Next, recall that to be a true background the fields also have to satisfy the supergravity equations of motion. Part of these are a set of Bianchi identities involving the exterior derivatives of $F^{(p)}$. In general one can derive equations involving the Ricci tensor and derivatives of the fluxes by considering integrability conditions, such as $D^2 \epsilon = 0$, for the Killing spinor equations. One can show, following ref. [27, 13] that, for product backgrounds of the form (2.7), once one imposes the Bianchi identities and the equation of motion for the flux the other equations of motion follow from these integrability conditions. In fact, for the cases we consider, the flux equation of motion is also implied by the supersymmetry conditions and so if we have a solution of the Killing spinor equation (2.3) and in addition the Bianchi identity

$$dG = 0 \quad \text{or} \quad dF^{(5)} = 0 \quad \text{Bianchi identity,} \quad (2.15)$$

then we have a solution of the equations of motion. When $E = \text{AdS}$, at least for the cases considered here, the supersymmetry conditions are even stronger: any solution of the Killing spinor equations is necessarily a solution of the equations of motion [1].

To have truly a string or M-theory background as opposed to a supergravity solution there is also a “quantisation” condition on the fluxes. For $n < 8$ the equations of motion for $G$ gives $d \ast G = 0$ while $d \ast F^{(5)} = 0$ is implied by the Bianchi identity since $F^{(5)}$ is self-dual. Hence in both cases the fluxes are harmonic. To be a true string or M-theory background, we have the quantisation condition $G \in H^4(X, \mathbb{Z})$ or $F^{(5)} \in H^5(X, \mathbb{Z})$. More precisely the fluxes represent classes in K-theory [72]. In the AdS-CFT correspondence, these integer classes are related to integral parameters in the field theory.
2.3 \textit{G}-structures

Our approach for analysing what solutions to the reduced Killing spinor equations (2.11) imply about the geometry of $X$ will use the language of \textit{G}-structures. Let us start with a brief review. For more information see for instance refs. \cite{73} or \cite{74}.

Let $F$ be the frame bundle of $X$, then a \textit{G}-structure is a principle sub-bundle $P$ of $F$ with fibre $G \subset GL(n, \mathbb{R})$.

For example if $G = O(n)$, the sub-bundle is interpreted as the set of orthonormal frames and defines a metric. Let $\nabla$ be a connection on $F$ or equivalently the corresponding connection on $TM$. One finds

\begin{enumerate}
\item given a \textit{G}-structure, all tensors on $X$ can be decomposed into $G$ representations;
\item if $\nabla$ is \textit{compatible} with the \textit{G}-structure, that is, it reduces to a connection on $P$, then $\text{Hol}(TX, \nabla) \subseteq G$;
\item there is an \textit{obstruction} to finding torsion-free compatible $\nabla$, measured by the \textit{intrinsic torsion} $T_0(P)$, which can be used to classify $G$-structures.
\end{enumerate}

The intrinsic torsion is defined as follows. Given a pair $(\nabla', \nabla)$ of compatible connections, viewed as connections on $P$ we have $\nabla' - \nabla \in C^\infty(\text{ad} P \otimes T^*X)$. Let $T(\nabla) \in C^\infty(TX \otimes \Lambda^2 T^*X)$ be the torsion of $\nabla$. We can then define a map $\sigma_P : C^\infty(\text{ad} P \otimes T^*X) \to C^\infty(TX \otimes \Lambda^2 T^*X)$ given by

$$\alpha = \nabla' - \nabla \mapsto \sigma_P(\alpha) = T(\nabla') - T(\nabla),$$

and hence we have the quotient bundle $\text{Coker} \sigma_P = TX \otimes \Lambda^2 T^*X / \alpha(\text{ad} P \otimes T^*X)$. Let the intrinsic torsion $T_0(P)$ be the image of $T(\nabla)$ in $\text{Coker} \sigma_P$ for any compatible connection $\nabla$.

By definition it is the part of the torsion independent of the choice of compatible connection and only depends on the $G$-structure $P$.

We will be interested in the particular class of $G$-structures where

\begin{enumerate}
\item $P$ can be defined in terms of a finite set $\eta$ of $G$-invariant tensors on $X$,
\item $G \subset O(n)$.
\end{enumerate}

Prime examples of the former condition are an almost complex structure with $G = GL(k, \mathbb{C}) \subset GL(2k, \mathbb{R})$, or an $O(n)$-structure defined by a metric $g$. The sub-bundle of frames $P$ is defined by requiring the tensors to have a particular form. For instance, for the $O(n)$-structure we define $P$ as the set of frames such that the metric $g$ has the form

$$g = e_1 \otimes e_1 + \cdots + e_n \otimes e_n.$$  

These restrictions imply a number of useful results. From the first condition it follows that

$$\nabla \text{ is compatible with } P \iff \nabla \Xi = 0, \ \forall \Xi \in \eta.$$  

The second condition implies that $P$ defines a metric $g$ and hence an $O(n)$ structure $Q$. A key point, given $\text{ad} Q \cong \Lambda^2 T^*X$, is that $\sigma_Q$ is in fact an isomorphism and hence
an $O(n)$-structure with metric $g$ has a unique compatible torsion-free connection, namely the Levi–Civita connection $\nabla^g$.

Any $P$-compatible connection $\nabla$ can then be written as $\nabla = \nabla^g + \alpha + \alpha^\perp$ where $\alpha$ is a section of $\text{ad} P \otimes T^*X$ while $\alpha^\perp$ is a section of $(\text{ad} P)^\perp \otimes T^*X$ with $(\text{ad} P)^\perp = \text{ad} Q/\text{ad} P$. Furthermore $\text{Coker } \sigma_P \cong (\text{ad} P)^\perp \otimes T^*X$ and given the isomorphism $\sigma_Q$, we see that $T_0(P)$ can be identified with $\alpha^\perp$. Equivalently, since by definition $\nabla \Xi = (\nabla^g + \alpha^\perp)\Xi = 0$ for any $\Xi \in \eta$, we have

$$T_0(P) \text{ can be identified with the set } \{\nabla^g \Xi : \Xi \in \eta\}. \quad (2.19)$$

Finally, if $T_0(P) = 0$ then $\nabla^g$ is compatible with $P$ and $X$ has special holonomy, that is, for $G \subset O(n)$,

$$T_0(P) = 0 \iff \text{Hol}(X) \subseteq G, \quad (2.20)$$

where $\text{Hol}(X) \equiv \text{Hol}(TX, \nabla^g)$.

A number of examples of such $G$-structures, familiar from the discussion of special holonomy manifolds, are listed in table 1. Except for $g$ in $\text{Spin}(7)$ all the elements of $\eta$ are forms, where, in the table, the subscript denotes the degree. Consider for instance the case $G = SU(k)$ in dimension $n = 2k$. This includes Calabi-Yau $k$-folds in the special case that $T_0(P) = 0$. The elements of $\eta$ are the fundamental two-form $J$ and the complex $k$-form $\Omega$. The structure $P$ is defined as the set of frames where $J$ and $\Omega$ have the form

$$J = e^1 \wedge e^2 + \cdots + e^{n-1} \wedge e^n,$n

$$\Omega = (e^1 + ie^2) \wedge \cdots \wedge (e^{n-1} + ie^n). \quad (2.21)$$

The two-form $J$ is invariant under $Sp(k, \mathbb{R}) \subset GL(2k, \mathbb{R})$ and $\Omega$ is invariant under $SL(k, \mathbb{C}) \subset GL(2k, \mathbb{R})$. The common subgroup is $SU(k) \subset SO(2k)$. Thus the pair $J$ and $\Omega$ determine a metric. For $SU(k)$-holonomy we then require that the intrinsic torsion vanishes or equivalently $\nabla^g J = \nabla^g \Omega = 0$ and $J$ is then the Kähler form and $\Omega$ the holomorphic $k$-form. By considering the corresponding $SU(k)$-representations, it is easy to show [14] that

$$T_0(P) \text{ can be identified with the set } \{dJ, d\Omega\}, \quad (2.22)$$

so that $\text{Hol}(X) \subseteq SU(k)$ is equivalent to $\{dJ = 0, d\Omega = 0\}$. This result that $T_0(P)$ is encoded in the exterior derivatives $d\Xi$ for $\Xi \in \eta$ is characteristic of all the examples in table 1.

<table>
<thead>
<tr>
<th>dimension $n$</th>
<th>special hol. space $X$</th>
<th>$G \subset SO(n)$</th>
<th>$\eta$</th>
<th>no. of supersysms.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 2k$</td>
<td>Calabi-Yau</td>
<td>$SU(k)$</td>
<td>${J_2, \Omega_k}$</td>
<td>$d_e/2^{k-1}$</td>
</tr>
<tr>
<td>$n = 4k$</td>
<td>hyper-Kähler</td>
<td>$Sp(k)$</td>
<td>${J_2^{(1)}, J_2^{(2)}, J_2^{(3)}}$</td>
<td>$d_e/2^k$</td>
</tr>
<tr>
<td>$n = 7$</td>
<td>$G_2$</td>
<td>$G_2$</td>
<td>${\phi_3}$</td>
<td>$d_e/8$</td>
</tr>
<tr>
<td>$n = 8$</td>
<td>$\text{Spin}(7)$</td>
<td>$\text{Spin}(7)$</td>
<td>${g, \Psi_4}$</td>
<td>$d_e/16$</td>
</tr>
</tbody>
</table>

Table 1: $G$-structures and supersymmetry
2.4 Supersymmetry and $G$-structures

We can now use the language of $G$-structures to characterise the constraints on the geometry of $X$ due to the existence of solutions to the Killing spinor equations (2.11). We define the space of solutions

$$C = \{ \xi \in \mathcal{C}^\infty(S^X) : D^X \xi = 0, \ Q^X \xi = 0 \}, \quad (2.23)$$

which defines a sub-bundle of $S^X$. The basic idea is that the existence of $C$ implies that there is a sub-bundle $P$ of the frame bundle and hence a $G$-structure.

First note that since we have spinors we have an $SO(n)$-structure $Q$ defined by the metric and orientation and a spin structure $(\tilde{Q}, \pi)$, where $\tilde{Q}$ is a $Spin(n)$ principle bundle and $\pi : \tilde{Q} \to Q$ is the covering map modelled on the double cover $Spin(n) \to SO(n)$. For any $n$ the Clifford algebra $Cliff(n)$ is equivalent to a general linear group acting on the vector space of spinors $\xi$ and implying we can also define a $Cliff(n)$ principle bundle $\tilde{\Gamma}$ with $\tilde{\Lambda} \subset \tilde{\Gamma}$.

Recall that $D^X$ is a Clifford connection defined on $\tilde{\Gamma}$ and generically does not descend to a connection on $\tilde{Q}$. Let $K_x \subset Cliff(n)$ be the stabilizer group in the Clifford algebra of $C|_x$, the set of solutions $C$ evaluated at the point $x \in X$. Since $D^X$ is a $Cliff(n)$ connection, by parallel transport $K = K_x$ is independent of $x \in X$, and hence $C$ defines a $K$ principle sub-bundle $\tilde{\Lambda} \subset \tilde{\Gamma}$ built from those elements of $\tilde{\Gamma}$ leaving $C$ invariant. We can equally well consider the stabilizer $\tilde{G}_x \subset Spin(n)$ of $C|_x$ in the spin group. Since $D^X$ does not descend to $\tilde{Q}$ in general $\tilde{G}_x$ is not independent of $x \in X$ and hence the stabilizer does not define a sub-bundle of $\tilde{Q}$. However, since there is only a finite number of possible stabiliser groups, we can still define a unique $\tilde{G} = \tilde{G}_x$ with $x \in U$ for some open subset of $U \subset X$ (with possibly non-trivial topology). Or alternatively we can restrict our considerations to $C$ such that $\tilde{G}$ is globally defined. In this way $C$ defines a sub-bundle $\tilde{P} \subset \tilde{Q}$ of the spin bundle. The double cover $\pi$ then restricts to a projection $\pi : \tilde{P} \to P \subset Q$ and hence we have a $G$-structure $P$ where $G$ is the projection of $\tilde{G}$. (In fact in all cases we consider $\tilde{G} = G$ and $P \cong \tilde{P}$.)

In conclusion we see that

1. $C$ defines a $G$-structure $P$ over (at least) some open subset $U \subset X$ where $G \subset SO(n)$.

The different structures and groups can be summarized as follows

$$\begin{array}{ccc}
\tilde{\Gamma} & \leftarrow & \tilde{\Lambda} \\
\uparrow & & \uparrow \\
\tilde{Q} & \leftarrow & \tilde{P} \\
\pi \downarrow & & \pi \downarrow \\
Q & \leftarrow & P \\
& & \downarrow \pi \\
& & SO(n) \\
& & \leftarrow \downarrow \pi \\
& & G \\
\end{array}$$

Note we can equivalently think of defining $\tilde{P}$ as the intersection $\tilde{\Lambda} \cap \tilde{Q}$ as embeddings in $\tilde{\Gamma}$. Generically this is not a bundle defined over the whole of $X$ since the fibre group can change, reflecting the fact that $P$ is generically only defined over $U \subset X$. Note also that, by construction,

$$\text{Hol}(S^X, D^X) \subseteq K. \quad (2.25)$$
This corresponds to the notion of generalised holonomy introduced by Duff and Liu \([23]\). Note, however, that this misses the important information that there is also a spin structure \(\tilde{Q} \subset \tilde{\Gamma}\) in the Clifford bundle. In other words the full information is contained in the pair \((\tilde{\Lambda}, \tilde{Q})\), which at least in a patch \(U\) translates into the \(G\)-structure \(P\).

To see explicitly that \(C\) defines a \(G\)-structure recall that the Clifford algebra gives us a set of maps \(w_p : C^\infty(S^X \otimes S^X) \to C^\infty(\Lambda^pTX)\) given by

\[
(\xi, \chi) \mapsto w_p(\xi, \chi) = \bar{\xi} \gamma(\xi) \chi,
\]

where \(\gamma(\xi)\) is the antisymmetric product of \(p\) gamma matrices generating the Clifford algebra \(\text{Cliff}(n)\). By construction if \(\xi, \chi \in C\) then \(w_p(\xi, \chi)\) is invariant under \(G\). The invariant forms \(\Xi \in \eta\) defining \(P\) are then generically constructed from combinations of bilinears of the form \(w_p(\xi, \chi)\). Specific examples will be given in the next section.

Finally, since \(D^X\) is determined by the flux, dilaton, and \(\Delta\) or \(\lambda\) and \(m\), from the discussion of the last section, we have our second result

(2) the intrinsic torsion \(T_0(P)\) is determined in terms of the flux, dilaton, and \(\Delta\) or \(\lambda\) and \(m\).

Generically, however, there may be components of, for instance, the flux which are not related to \(T_0(P)\). Thus we see that the existence of solutions to the Killing spinor equations (2.11) translates into the existence of a \(G\)-structure \(P\) with specific intrinsic torsion \(T_0(P)\).

As mentioned above, in some cases the \(G\)-structure is globally defined. On the other hand, in some cases the \(G\)-structure is only defined locally in some open set, and possibly only in a topologically trivial neighbourhood. Of course in such a neighbourhood the structure group of the frame bundle can always be reduced to the identity structure. However, the key point is that supersymmetry defines a canonical \(G\)-structure that can be used to give a precise characterisation of the local geometry of the solution. In turn, as we shall see, this provides an often powerful method to construct explicit local supersymmetric solutions. Furthermore, the global properties of such solutions can then be found by determining the maximal analytic extension of the local solution (this is a standard technique used in the physics literature).

Let us now see how this description in terms of \(G\)-structures works in a couple of specific examples relevant to the new solutions we will discuss later.

**Example 1: \(n = 6\) in Type IIB**

Consider the case of type IIB supergravity with \(M = \mathbb{R}^{3,1} \times X\) and only the self-dual five-form non-vanishing (the most general case is considered in ref. \([51]\)). First we need the spinor decomposition. Recall that \(\epsilon = (\epsilon_i^\alpha)\) is a section of \(S = S_+ \oplus S_\perp\) where the spin bundle \(S_+\) corresponds to the real (positive) chirality spinor representation \(\Delta_{9,1}^+\) of \(\text{Spin}(9, 1)\). In general we have that the complexified representation decomposes as

\[
(\Delta_{9,1}^+)_{\mathbb{C}} = \Delta_{3,1}^+ \otimes \Delta_6^+ + \overline{\Delta_{3,1}^+} \otimes \overline{\Delta_6^+},
\]  

(2.27)
where $\Delta_{3,1}^+$ and $\Delta_{6}^+$ are the complex positive chirality representations of $Spin(3,1)$ and $Spin(6)$. The bar denotes the conjugate representation. Let $S_{3,1}^+$ and $S_{6}^+$ be the corresponding spin bundles. To ensure that the $\epsilon_i$ are real we decompose

$$\epsilon_i = \psi \otimes e^{\Delta/2} \xi_i + \psi^c \otimes e^{\Delta/2} \xi^c_i;$$

where $\psi \in C^\infty(S_{3,1}^+)$, $\xi_i \in C^\infty(S_{6}^+)$ while $\psi^c$ and $\xi^c_i$ are the complex conjugate spinors and we have included factors of $e^{\Delta/2}$ in the definition of $\xi_i$ for convenience. We then define the combinations $\xi^\pm = \xi_1 \pm i \xi_2$.

The self-dual five-form flux ansatz (2.9) can be written as

$$F^{(5)} = e^{4\Delta} \text{vol}_n \wedge f - *_X f,$$

where $*_X$ is the Hodge star defined using $g_X$ on $X$. Decomposing the Killing spinor equations it is easy to show that either $\xi^+ = 0$ or $\xi^- = 0$. Let us assume that $\xi^- = 0$ then, defining $\xi = \xi^+$ with $S_X = S_{6}^+$, we have

$$D^X = \nabla g_X + \frac{1}{8} f \lrcorner \gamma^{(2)},$$

$$Q^X = \gamma^{(1)} \lrcorner d\Delta + \frac{1}{4} \gamma^{(1)} \lrcorner f.$$  

(2.30)

Note that $D^X$ involves only $\gamma^{(2)}$ and so in this case it does descend to a metric compatible connection $\nabla$ on $TX$. Thus, in this case, the $G$-structure to be discussed next, is in fact globally defined.

We will consider the case of the minimum number of preserved supersymmetries where the set of solutions $C$ is one-dimensional, corresponding to non-zero multiples of some fixed solution $\xi \in C$. The stabiliser of a single spinor is $SU(3)$ and thus we have

$$X \text{ has } SU(3)-\text{structure.}$$

(2.31)

It is easy to show that $\nabla g_X (\bar{\xi} \xi) = 0$. If we choose to normalise such that $\bar{\xi} \xi = 1$, it then follows that the elements of $\eta$ fixing the $SU(3)$ structure are given by the bilinears

$$J = -i \bar{\xi} \gamma^{(2)} \xi, \quad \Omega = \xi^c \gamma^{(3)} \xi.$$  

(2.32)

We next calculate the intrinsic torsion. Recall that this is contained in $dJ$ and $d\Omega$. From (2.30) one finds

$$d(e^{4\Delta} f) = -e^{4\Delta} f,$$

$$d(e^{2\Delta} J) = 0,$$

$$d(e^{3\Delta} \Omega) = 0,$$

(2.33)

which completely determines the intrinsic torsion, as well as the flux, in terms of $d\Delta$. This implies that $g_X$ is conformally Calabi–Yau, that is we can write

$$g_X = e^{-2\Delta} g_6,$$  

(2.34)

where $g_6$ has integrable $SU(3)$-structure. In addition

$$f = -4d\Delta,$$  

(2.35)
Note that the Bianchi identity for $F^{(5)}$ is satisfied provided we have $d \ast X f = 0$. This translates into the harmonic condition
\[ \nabla_{g_6}^2 e^{-4\Delta} = 0, \]  
(2.36)
where $\nabla_{g_6}^2$ is the Laplacian for $g_6$ on $X$. Thus we have completely translated the conditions for a supersymmetric background into a geometrical constraint (2.34) together with a solution of the Laplacian (2.36).

**Example 2: $n = 7$ in $d = 11$**

Now consider the case of $d = 11$ supergravity on $M = \mathbb{R}^{3,1} \times X$. (Here we are following the discussion of ref. [57].) Again we start with the spinor decomposition. Recall that the $d = 11$ spinor $\epsilon$ is a section of a spin bundle corresponding to the real 32-dimensional representation $\Delta_R^{10,1}$ of $Spin(10,1)$. Under $Spin(3,1) \times Spin(7)$ the complexified representation decomposes as
\[ (\Delta_R^{10,1})_C = \Delta_{3,1}^+ \otimes (\Delta_7)^C + \overline{\Delta}_{3,1}^+ \otimes (\Delta_7^R)^C, \]  
(2.37)
where $\Delta_{3,1}^+$ and $\Delta_7^R$ are the complex positive chirality representation of $Spin(3,1)$ and real representation of $Spin(7)$ respectively. Let $S_{3,1}^+$ and $S_7^R$ be the corresponding spin bundles.

To ensure that the $\epsilon_i$ are real we decompose
\[ \epsilon = \psi \otimes (\xi_1 + i\xi_2) + \psi^c \otimes (\xi_1 - i\xi_2), \]  
(2.38)
where $\psi \in C^\infty(S_{3,1}^+)$, $\xi_i \in C^\infty(S_7^R)$.

In the flux ansatz (2.9) we assume $G$ is pure magnetic so
\[ G \in C^\infty(\Lambda^4 T^* X). \]  
(2.39)

Defining $\xi$ as the doublet $\xi = (\xi_1, \xi_2)$ with $S^X = S_{3,1}^+ \oplus S_7^R$, we find
\[ D^X = \nabla^{gX} \otimes id + \frac{1}{12} \gamma^{(2)} \ast_X G \otimes i\sigma_2 + \frac{1}{6} i\gamma^{(3)} \ast G \otimes i\sigma_2, \]
\[ Q^X = \gamma^{(1)} \ast d\Delta \otimes id + \frac{1}{6} i\gamma^{(4)} \ast G \otimes i\sigma_2, \]  
(4.0)
where $i\sigma_2 = (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})$. Note that $D^X$ does not descend to a metric compatible connection $\nabla$ on $TX$.

Again we are interested in the minimum number of preserved supersymmetries so the set of solutions $C$ is one-dimensional, corresponding to non-zero multiples of some fixed solution $\xi \in C$. In addition we will assume the $\bar{\xi}_i$ in $\xi$ are each non-zero and more importantly
\[ \bar{\xi}_1 \xi_2 = 0. \]  
(2.41)
(Note that the generic conditions, without this assumption, were derived in ref. [60].) It is then easy to show that the $e^{-\Delta/2}\xi_i$ have constant norm. Together with (2.41) this then implies that the stabiliser of $\xi = (\xi_1, \xi_2)$ is $G = SU(3)$ independent of $x \in X$ and hence
\[ X \text{ has } SU(3) \text{-structure.} \]  
(2.42)
Note that this is an $SU(3)$-structure in seven dimensions.

If we normalise $\xi \in C$ such that $e^{-\Delta}\bar{\xi}_1\xi_1 = e^{-\Delta}\bar{\xi}_2\xi_2 = 1$ then the elements of $\eta = \{J, \Omega, K\}$ fixing the $SU(3)$ structure are given by

$$
J = -e^{-\Delta}\bar{\xi}_1\gamma^{(2)}\xi_2,
$$

$$
\Omega = -\frac{1}{2}e^{-\Delta}(\bar{\xi}_1\gamma^{(3)}\xi_1 - \bar{\xi}_2\gamma^{(3)}\xi_2) + ie^{-\Delta}\bar{\xi}_1\gamma^{(3)}\xi_2,
$$

$$
K = -ie^{-\Delta}\bar{\xi}_1\gamma^{(1)}\xi_2,
$$

(2.43)

where we are using the convention $i\gamma^{(7)} = \text{vol}_X \text{id}$. The one-form $K$ defines a product structure $R \subset Q$ with fibre $SO(6) \subset SO(7)$ and then $J$ and $\Omega$ define the $G$-structure $P \subset R$ with fibre $SU(3)$.

As in six dimensions the intrinsic torsion of an $SU(3)$-structure in seven dimensions is completely determined by the exterior derivatives of $K$, $J$ and $\Omega$. One finds

$$
d(e^{2\Delta}K) = 0,
$$

$$
d(e^{4\Delta}J) = e^{-4\Delta} \ast_X G,
$$

$$
d(e^{3\Delta}\Omega) = 0,
$$

$$
d(e^{2\Delta}J \wedge J) = -2e^{2\Delta}G \wedge K.
$$

(2.44)

These equations were derived in ref. [57] (up to a factor in the last equation differs, as discussed in ref. [1]). It was argued in ref. [54] that these are the necessary and sufficient conditions for a geometry to admit a single Killing spinor. Furthermore, the second equation implies the $G$ equation of motion and thus, given an integrability argument as in ref. [13], only the Bianchi identity $dG$ need be imposed to give a solution to the full equations of motion.

**2.5 Relation to generalised calibrations**

In turns out that there is a very interesting relation between the torsion conditions, such as (2.33) and (2.44), one derives for supersymmetric backgrounds and the notion of a “generalised calibrations” introduced in ref. [11]. This gives a very physical interpretation of the conditions in terms of string theory “branes”. Here we will only touch on this relation briefly.

Let us start by recalling the notion of calibrations and a calibrated cycle [73, 74] (for a review see refs. [73, 66]). Suppose we have a Riemannian manifold $X$ with metric $g_X$ and let $\xi \subset T_xX$ be an oriented $p$-dimensional tangent plane at any point $x \in X$. We can then define $\text{vol}_\xi$ as the volume form on $\xi$ built from the restriction $g_X|_\xi$ of the metric to $\xi$. A $p$-form $\Xi$ is then a calibration if

$$
(i) \quad \Xi|_\xi \leq \text{vol}_\xi \quad \forall \, \xi,
$$

$$
(ii) \quad d\Xi = 0.
$$

(2.45)

Furthermore, given a $p$-dimensional oriented submanifold $C_p$, we say $C_p$ is calibrated if

$$
\text{calibrated submanifold: } \Xi|_{T_xC_p} = \text{vol}_{T_xC_p} \quad \forall \, x \in C_p.
$$

(2.46)
If $C'_p$ is another submanifold in the same homology class we have

$$\int_{C'_p} \text{vol}_{C'_p} = \int_{C'_p} \Xi|_{TC'_p} = \int_{C'_p} \Xi|_{TC'_p} \leq \int_{C'_p} \text{vol}_{C'_p},$$

(2.47)

and we get the main result that a calibrated submanifold has minimum volume in its homology class.

Now suppose we have a set of invariant tensors $\eta$ defining a $G$-structure $P$ with $G \subset SO(n)$. One finds that, for $p$-forms $\Xi$

$$\Xi \in \eta \Rightarrow \text{calibration condition (i)},$$

$$T_0(P) = 0 \Rightarrow \text{calibration condition (ii)}.$$ 

(2.48)

Thus the vanishing intrinsic torsion of the $G$-structure (i.e. special holonomy $G$) corresponds to the closure of the calibration forms. It is natural, then, to try to interpret our intrinsic torsion conditions (2.33) and (2.44) as defining “generalised calibrations” [11]. Obviously calibrated sub-manifolds will no longer be volume minimising, but one might ask if there is some more general notion of the “energy” of the submanifold which is minimised by calibrated sub-manifolds.

String theory provides precisely such an interpretation. It contains a number of extended $p+1$-dimensional objects which embed into the spacetime and are known as “$p$-branes”:

- A simple example is the two-dimensional string itself. Each brane has a particular energy functional depending both on the volume of the embedded submanifold and crucially the flux and dilaton. Differential conditions such as (2.33) and (2.44) then imply that the corresponding brane energy is minimised when the submanifold is calibrated by a generalised calibration.

Consider for instance our $d=11$ example with $M = \mathbb{R}^{3,1} \times X$. The relevant branes in eleven dimensions are the “M2-brane” and the “M5-brane” and are described by embeddings of the worldvolumes $\Sigma \hookrightarrow M$. In particular, we can take $\Sigma = \mathbb{R}^{r,1} \times C_s$ with $r + s \in \{2, 5\}$, where $C_s$ is a $p$-dimensional submanifold of $X$. One then says that the brane is “wrapped” on $C_s$. Each of the conditions (2.44) can then be interpreted as generalised calibrations for different types of wrapped brane. We have

$$d(e^{2\Delta} K) = 0 \quad \text{M2-brane on } C_1,$$

$$d(e^{4\Delta} J) = e^{-4\Delta} *_X G \quad \text{M5-brane on } C_2,$$

$$d(e^{3\Delta} \Omega) = 0 \quad \text{M5-brane on } C_3,$$

$$d(e^{2\Delta} J \wedge J) = -2 e^{2\Delta} G \wedge K \quad \text{M5-brane on } C_4.$$ 

(2.49)

Note that the power of $e^\Delta$ appearing in each expression counts the $q+1$ unwrapped dimensions of the brane. Roughly, the fluxes appearing on the right hand side can be understood by noting that M2-branes couple to electric $G$-flux, that M5-branes couple to magnetic $G$-flux and that we have only kept certain components of $G$ in our ansatz (for example the electric $G$-flux vanishes). The flux appearing in the last expression in (2.49) arises from the fact that there can be induced M2-brane charge on the M5-brane. For more on this correspondence see refs. [12, 13, 14, 58].
3 New $AdS_5$ solutions in M-theory

We now turn to the specific problem of finding, first, the generic structure of the minimal supersymmetric configurations of $D = 11$ supergravity with $M = AdS_5 \times X$ and, second, a class of particular solutions of this form. Such backgrounds are of particular interest because, via the AdS-CFT correspondence, they are dual to $\mathcal{N} = 1$ superconformal field theories in four-dimensions. This work was first presented in ref. [1].

3.1 General differential conditions

As we have seen $AdS_5 \times X$ geometries are special cases of $\mathbb{R}^{3,1} \times X'$ where $X' = X \times \mathbb{R}$ with metrics and warp factors related as in eqns. (2.14). Thus we can actually obtain the general conditions for supersymmetric $AdS_5$ compactifications from the corresponding $n = 7$ $SU(3)$ conditions given in eq. (2.44). (One might be concerned that these latter conditions are not completely generic, nonetheless one can show [1] that they give the generic conditions for $AdS_5$.)

To derive the conditions explicitly, let us denote the $SU(3)$ structure on $X'$ by the primed forms $(K', J', \Omega')$. The radial unit one-form $e^\lambda dr$ is generically not parallel to $K'$; instead we can write

$$e^\lambda dr = -\sin \zeta K' - \cos \zeta W',$$

where $W'$ is a unit one-form orthogonal to $K'$. We can then define two other unit mutually orthogonal one-forms

$$K^1 = \cos \zeta K' - \sin \zeta W'$$

$$K^2 = V' = J' \cdot W'$$

where $K^1$ is the orthogonal linear combination of $K'$ and $W'$ and $K^2$ is defined using $J^a_b$ the almost complex structure on $X'$. We can then define real and imaginary two-forms from the parts of $J$ and $\Omega$ orthogonal to $W'$ and $V'$, that is

$$J = J' - W' \wedge V'$$

$$\Omega = i_{W'+iV'} \Omega'.$$

Note that $\Omega$ is not strictly a two-form on $X$ but is a section of $\Lambda^2 T^* X$ twisted by the complex line bundle defined by $W' + iV'$. This implies that the set $(K^1, K^2, J, \Omega)$ actually defines a local $U(2)$ structure on the six-dimensional manifold $X$, rather than an $SU(2)$ structure as would be the case if $\Omega$ were truly a two-form. Note that the structure is only local since, in particular, it breaks down when $K'$ is parallel to $dr$, that is $\cos \zeta = 0$, in which case we cannot define $K^1$ and $K^2$. Using these definitions the constraints (2.44) become

$$d(e^{3\lambda} \sin \zeta) = 2m e^{2\lambda} \cos \zeta K^1,$$

$$d(e^{4\lambda} \cos \zeta \Omega) = 3m e^{3\lambda} \Omega \wedge (-\sin \zeta K^1 + iK^2),$$

$$d(e^{5\lambda} \cos \zeta K^2) = e^{5\lambda} \ast G + 4m e^{4\lambda} (J - \sin \zeta K^1 \wedge K^2),$$

$$d(e^{3\lambda} \cos \zeta J \wedge K^2) = e^{3\lambda} \sin \zeta G + m e^{2\lambda} (J \wedge J - 2 \sin \zeta J \wedge K^1 \wedge K^2).$$

(Note that the $SU(2)$ structure here differs from that used in ref. [1] by a conformal rescaling.)
To ensure we have a solution of the equations of motion, in general one also needs to impose the equation of motion and Bianchi identity for $G$. The connection with the $n = 7$ results gives us a quick way of seeing that, in fact, provided $\sin \zeta$ is not identically zero, both conditions are a consequence of the supersymmetry constraints (3.4)–(3.7). As already noted, the equation of motion for $G$ follows directly from the exterior derivative of the second equation in (2.44). For the Bianchi identity one notes that, given the ansatz for the $n = 7$ metric and $G$, the first and last equations in (2.44) imply in general that

$$\sin \zeta dG \wedge dr = 0$$

(3.8)

since $dG$ lies solely in $X$. This implies that $dG = 0$ provided $\sin \zeta$ is not identically zero – which can only occur only when $m = 0$ (from (3.4)). Thus we see that the constraints (3.4)–(3.7) are necessary and sufficient both for supersymmetry and for a solution of the equations of motion.

### 3.2 Local form of the metric

By analysing the differential conditions (3.4)–(3.7) on the forms, after some considerable work, one can derive the necessary and sufficient conditions on the local form of the metric and flux. Here we will simply summarize the results referring to ref. [1] for more details.

First one notes that as a vector $e^{-\lambda} \cos \zeta K_2$ is Killing and that coordinates can be chosen so that

$$\frac{\partial}{\partial \psi} = \frac{1}{3m} e^{-\lambda} \cos \zeta K_2$$

(3.9)

In addition the Lie derivatives $L_{\partial/\partial \psi} G = L_{\partial/\partial \psi} \lambda = 0$ vanish so in fact acting with $L_{\partial/\partial \psi}$ preserves the full solution. This reflects the fact that the dual field theory has a $U(1)_R$ symmetry.

Second, one can introduce a coordinate $y$ for $K_1$ given by

$$2my = e^{3\lambda} \sin \zeta$$

(3.10)

so that

$$K^1 = e^{-2\lambda} \sec \zeta dy.$$  

(3.11)

While we could eliminate either $\lambda$ or $\zeta$ from the following formulae, for the moment it will be more convenient to keep both.

The metric then takes the form

$$g_X = e^{-4\lambda} \left( \hat{g} + \sec^2 \zeta dy \otimes dy \right) + \frac{1}{9m^2} e^{2\lambda} \cos^2 \zeta (d\psi + \rho) \otimes (d\psi + \rho)$$

(3.12)

where $i_{\partial_y} \rho = i_{\partial_\psi} \rho = 0$. We also have, with $\hat{J} = e^{4\lambda} J$,

- (a) $\partial/\partial \psi$ is a Killing vector
- (b) $\hat{g}$ is a family of Kähler metrics on $M_4$ parameterized by $y$
- (c) the corresponding complex structure $\hat{J}^i_j$ is independent of $y$ and $\psi$. 

(3.13) (3.14) (3.15)
and

\[ \begin{align*}
(d) & \quad 2my = e^{3\lambda} \sin \zeta \\
(e) & \quad \rho = \hat{P} + \hat{J} \cdot d_4 \log \cos \zeta
\end{align*} \] (3.16) \hspace{1cm} (3.17)

where, in complex co-ordinates, \( \hat{P} = \frac{1}{2} \hat{J} \cdot d \log \sqrt{\hat{g}} \) is the canonical connection defined by the Kähler metric, satisfying \( \hat{R} = d\hat{P} \) where \( \hat{R} \) is the Ricci form. Finally we have the conditions

\[ \begin{align*}
(f) & \quad \partial_y \hat{J} = -\frac{2}{3} y d_4 \rho \\
(g) & \quad \partial_y \log \sqrt{\text{det} \hat{g}} = -3y^{-1} \tan^2 \zeta - 2\partial_y \log \cos \zeta .
\end{align*} \] (3.18) \hspace{1cm} (3.19)

The four-form flux \( G \) is given by

\[ G = -(\partial_y e^{-6\lambda}) \hat{\text{vol}}_4 - e^{-10\lambda} \sec \zeta (\hat{\ast}_4 d_4 e^{6\lambda}) \wedge K^1 - \frac{1}{3m} e^{-\lambda} \cos^3 \zeta (\hat{\ast}_4 \partial_y \rho) \wedge K^2 + e^\lambda \left[ \frac{1}{3m} \cos^2 \zeta \hat{\ast}_4 d_4 \rho - 4me^{-6\lambda} \hat{J} \right] \wedge K^1 \wedge K^2 \] (3.20)

and is independent of \( \psi \) — that is, \( \mathcal{L}_{\partial/\partial\psi} G = 0 \). As discussed previously the equations of motion for \( G \) and the Bianchi identity are implied by expressions (3.13)–(3.19).

To summarize, we have given the local form of the generic \( \mathcal{N} = 1 \) AdS\(_5\) compactification in \( d = 11 \) supergravity. Any \( d = 11 \) AdS-CFT supergravity dual of a \( d = 4 \) superconformal field theory will have this form.

### 3.3 Complex Ansatz

In this section we consider how the conditions on the metric specialise for solutions where the six-dimensional space \( X \) is a complex manifold. Crucially, the supersymmetry conditions simplify considerably and we are able to find many solutions in closed form. Globally, the new regular compact solutions that we construct are all holomorphic \( \mathbb{C}P^1 \) bundles over a smooth four-dimensional Kähler base \( M_4 \). Using a recent mathematical result on Kähler manifolds \([77]\), we are able to classify completely this class of solutions (assuming that the Goldberg conjecture is true). In particular, at fixed \( y \) the base is either (i) a Kähler–Einstein (KE) space or (ii) a non-Einstein space which is the product of two constant curvature Riemann surfaces.

More precisely we specialize to the case where

\[ g_X \text{ is a Hermitian metric on a complex manifold } X, \]

where we define the complex structure, compatible with \( g_X \) and the local \( U(2) \)-structure, given by the holomorphic three-form \( \Omega_3 = \Omega \wedge (K^1 + iK^2) \). Requiring this complex structure to be integrable, that is \( d\Omega_3 = A \wedge \Omega_3 \) for some \( A \), implies that

\[ \begin{align*}
d_4 \zeta &= 0 \quad d_4 \lambda = 0 \quad \partial_y \rho = 0.
\end{align*} \] (3.21)
In addition one finds that the connection $\rho$ is simply the canonical connection defined by the Kähler metric $\hat{g}$, that is

$$\rho = \hat{P}$$

(3.22)

together with the useful condition that

at fixed $y$, the Ricci tensor on $\text{Ric}_\hat{g}$ has two pairs of constant eigenvalues.

We would like to find global regular solutions for the complex manifold $X$. Our construction is as follows. We require that $\psi$ and $y$ describe a holomorphic $\mathbb{C}P^1$ bundle over a smooth Kähler base $M_4$

$$\mathbb{C}P^1_{y,\psi} \longrightarrow X \quad \downarrow \quad M_4$$

(3.23)

For the $(y, \psi)$ coordinates to describe a smooth $\mathbb{C}P^1$ we take the Killing vector $\partial/\partial \psi$ to have compact orbits so that $\psi$ defines an azimuthal angle and $y$ is taken to lie in the range $[y_1, y_2]$ with $\cos \zeta(y_i) = 0$. Thus $y_\pm$ are the two poles where the $U(1)$ fibre shrinks to zero size. It turns out that the metric $g_X$ gives a smooth $S^2$ only if we choose the period of $\psi$ to be $2\pi$.

Given the connection (3.22), we see that, as a complex manifold,

$$X = \mathbb{P}(\mathcal{O} \oplus \mathcal{L})$$

(3.24)

where $\mathcal{L}$ is the canonical bundle and $\mathcal{O}$ the trivial bundle on the base $M_4$.

Let us now consider the Kähler base. A recent result on Kähler manifolds (Theorem 2 of ref. [77]) states that, if the Goldberg conjecture\footnote{The Goldberg conjecture says that any compact Einstein almost Kähler manifold is Kähler-Einstein i.e. the complex structure is integrable. This has been proven for non-negative curvature [78].} is true, then a compact Kähler four-manifold whose Ricci tensor has two distinct pairs of constant eigenvalues is locally the product of two Riemann surfaces of (distinct) constant curvature. If the eigenvalues are the same the manifold is by definition Kähler–Einstein. The compactness in the theorem is essential, since there exist non-compact counterexamples. However, for AdS/CFT purposes, we are most interested in the compact case (for example, the central charge of the dual CFT is inversely proportional to the volume).

From now on we will consider only these two cases. One then finds that the conditions (3.18) and (3.19) can be partially integrated. In summary we have two cases:

**case 1:**

$$\hat{g} = \frac{1}{3} (b - ky^2) \tilde{g}_k$$

**case 2:**

$$\hat{g} = \frac{1}{3} (a_1 - k_1 y^2) \tilde{g}_{k_1} + \frac{1}{3} (a_2 - k_2 y^2) \tilde{g}_{k_2}$$

(3.25)

where $k, k_i \in \{0, \pm 1\}$, and the (two- or four-dimensional) Kähler–Einstein metrics $\tilde{g}_k$ satisfy

$$\text{Ric}_{\tilde{g}_k} = k \tilde{g}_k$$

(3.26)
and are independent of $y$. The remaining equation (3.19), implies

$$\begin{align*}
\text{case 1:} & \quad m^2(1 + 6y\partial_y \lambda) = \frac{k}{b - ky^2} (e^{6\lambda} - 4m^2y^2), \\
\text{case 2:} & \quad m^2(1 + 6y\partial_y \lambda) = \frac{k_2a_1 + k_1a_2 - 2k_1k_2y^2}{2(a_1 - k_1y^2)(a_2 - k_2y^2)} (e^{6\lambda} - 4m^2y^2).
\end{align*}$$

(3.27)

### 3.4 New compact solutions

#### 3.4.1 Case 1: KE base

We start by considering the case where the base is Kähler–Einstein (KE). The remaining supersymmetry condition (3.27) can be integrated explicitly. One finds,

$$\begin{align*}
e^{6\lambda} &= \frac{2m^2(b - ky^2)^2}{2kb + cy + 2k^2y^2} \\
\cos^2 \zeta &= \frac{a_1a_2 - 3(k_2a_1 + k_1a_2)y^2 - 2cy^3 - 3k_1k_2y^4}{(a_1 - k_1y^2)(a_2 - k_2y^2)}
\end{align*}$$

(3.28)

where $c$ is an integration constant. Without loss of generality by an appropriate rescaling of $y$ we can set $b = 1$ and $c \geq 0$.

Assuming $X$ has the topology given by (3.24), we find this leads to a smooth metric at the $y = y_i$ poles of the $\mathbb{C}P^1$ fibres provided we take $\psi$ to have period $2\pi$. One then finds our first result

for $0 \leq c < 4$ we have a one-parameter family of completely regular, compact, complex metrics $g_X$ with the topology of a $\mathbb{C}P^1$ fibration over a positive curvature KE base.

For negative ($k = -1$) and zero ($k = 0$) curvature KE metrics $\hat{g}$ there are no regular solutions. Since four-dimensional compact Kähler-Einstein spaces with positive curvature have been classified [79, 80], we have a classification for the above solutions. In particular, the base space is either $S^2 \times S^2$ or $\mathbb{C}P^2$, or $\mathbb{C}P^2\#_n\mathbb{C}P^2$ with $n = 3, \ldots, 8$. For the first two examples, the KE metrics are of course explicitly known and this gives explicit solutions of M-theory. The remaining metrics, although proven to exist, are not explicitly known, and so the same applies to the corresponding M-theory solutions.

#### 3.4.2 Case 2: product base

Next consider the case where the base is a product of constant curvature Riemann surfaces. Again the remaining supersymmetry condition (3.27) can be integrated explicitly giving

$$\begin{align*}
e^{6\lambda} &= \frac{2m^2(a_1 - k_1y^2)(a_2 - k_2y^2)}{(k_2a_1 + k_1a_2) + cy + 2k_1k_2y^2} \\
\cos^2 \zeta &= \frac{a_1a_2 - 3(k_2a_1 + k_1a_2)y^2 - 2cy^3 - 3k_1k_2y^4}{(a_1 - k_1y^2)(a_2 - k_2y^2)}
\end{align*}$$

(3.29)
where $c$ is an integration constant giving a three-parameter family of solutions. Note that on setting $a_1 = a_2 = b$, $k_1 = k_2 = k$ these reduce to the KE solutions considered above. Again we have a smooth metric at the $y = y_i$ poles of the $\mathbb{CP}^2$ fibres provided we take $\psi$ to have period $2\pi$. The full metric $g_X$ is regular if the base is $S^2 \times T^2$, $S^2 \times S^2$ or $S^2 \times H^2$. However the final case is not compact.

Summarizing the compact cases, for $S^2 \times T^2$ without loss of generality we can take $k_2 = 0$, $a_2 = 3$ and, by scaling $y$, we can set $c = 1$ or $c = 0$. We find

for $0 < a < 1$ and $c \neq 0$ we have a one-parameter family of completely regular, compact, complex metrics $g_X$ where $X$ is a topologically trivial $\mathbb{CP}^1$ bundle over $S^2 \times T^2$. A single additional solution of this type is obtained when $c = 0$ and $a \neq 0$.

For the $S^2 \times S^2$ topology again, generically, one parameter can be scaled away and we find

for various ranges of $(a_1, a_2, c)$ there are completely regular, compact, complex metrics $g_X$ where $X$ is topologically a $\mathbb{CP}^1$ bundle over $S^2 \times S^2$.

In particular there are solutions when $a_1$ is not equal to $a_2$ and hence this gives a broader class of solutions than in the Kähler-Einstein case considered above. The existence of regular solutions is rather easy to see if one sets $c = 0$. Note that we can also recover the well-known Maldecena–Nuñez solution [5] when the base has topology $S^2 \times H^2$, though the topology is slightly different from the ansatz here. More details are given in ref. [1].

The $S^2 \times T^2$ solutions are of particular interest since they lead to new type IIA and type IIB supergravity solutions. Type IIA supergravity arises from $d = 11$ supergravity reduced on a circle. Since these solutions have two Killing directions on the $T^2$ base we can trivially reduce on one circle in $T^2$ to give a IIA solution. Given the second Killing vector we can then use T-duality to generate a IIB solution. (T-duality is a specific map between IIA and IIB supergravity backgrounds which exists when each background has a Killing vector which also preserves the flux and dilaton, and also the Killing spinors if the map is to preserve supersymmetry at the level of the supergravity solution.) The resulting IIB background has the form $\text{AdS}_5 \times Z$ with non-trivial $F^{(5)}$ flux. As we will see, this implies that $Z$ is a Sasaki–Einstein manifold. The geometry of these manifolds will be the subject of the following section.

4 A new infinite class of Sasaki–Einstein solutions

By analogy with the previous section let us now turn to the case of $\text{AdS}_5 \times X$ solutions in IIB supergravity with non-trivial $F^{(5)}$. It is a well-known result that $X$ must then be Sasaki–Einstein [81]. As noted above, the $d = 11$ solutions on $S^2 \times T^2$ potentially give new $n = 5$ Sasaki–Einstein solutions. In this section we discuss the structure of these solutions. In fact we will show the general result that

for every positive curvature $2n$-dimensional Kähler–Einstein manifold $B_{2n}$, there is a countably infinite class of associated compact, simply-connected, spin, Sasaki–Einstein manifolds $X_{2n+3}$ in dimension $2n + 3$.  

20
4.1 Sasaki–Einstein spaces

Let us start by showing directly that for IIB backgrounds of the form $\text{AdS}_5 \times X$ with $F^{(5)}$ flux $X$ must be Sasaki–Einstein. As before we will consider the reduction from $\mathbb{R}^{3,1} \times X'$ to $\text{AdS}_5 \times X$ of the backgrounds given in eqns. (2.33). Let $(J', \Omega')$ denote the $SU(3)$ structure on $X'$. Picking out the radial one-form $R \equiv e^\lambda dr$ globally defines a second one-form $K = J' \cdot R$. One then has the real and complex two-forms given by

$$J = J' - K \wedge R$$

$$\Omega = i_{K+iR} \Omega'$$

(4.1)

Note that globally $\Omega$ is not strictly a two-form on $X$ but is a section of $\Lambda^2 T^*X$ twisted by the complex line bundle defined by $K + iR$. For this reason $(K, J, \Omega)$ define only a $U(2)$ (or almost metric contact) structure on $X$ rather than $SU(2)$.

Reducing the condition on $(J', \Omega')$ one finds that $\lambda$ is constant and we set it to zero without loss of generality. We then have that $K$ is unit norm and that

$$dK = 2mJ$$

$$d\Omega = i3mK \wedge \Omega$$

(4.2)

with the five-form flux given by $F^{(5)} = 4m(\text{vol}_{\text{AdS}_5} + \text{vol}_{X_5})$. Clearly $\mathcal{L}_K J = 0$ and $\mathcal{L}_K \Omega = i3m\Omega$ so that $K$ is a Killing vector. The second condition in (4.2) implies that we have an integrable contact structure. The first condition implies that the metric is actually Sasaki–Einstein. (For more details see for example refs. [82] and [83].)

The Killing condition means that locally we have

$$K = d\psi' + \sigma$$

(4.3)

where $d\sigma = 2mJ$ and that we can write the metric in the form

$$g_X = \hat{g} + K \otimes K$$

(4.4)

where $\hat{g}$ is a positive curvature Kähler–Einstein metric. Note that, by definition, the metric cone over $g_X$ is Calabi–Yau. All these results generalize without modification to $(2k + 1)$-dimensional Sasaki–Einstein manifolds $X$.

Finally note that one can group Sasaki–Einstein manifolds by the nature of the orbits of the Killing vector $K$. If the orbits close, then we have a $U(1)$ action. Since $K$ is nowhere vanishing, it follows that the isotropy groups of this action are all finite. Thus the space of leaves of the foliation will be a positive curvature Kähler–Einstein orbifold of complex dimension $k$. Such Sasaki–Einstein manifolds are called quasi-regular. If the $U(1)$ action is free, the space of leaves is actually a Kähler–Einstein manifold and the Sasaki–Einstein manifold is then said to be regular. Moreover, the converse is true: there is a Sasaki–Einstein structure on the total space of a certain $U(1)$ bundle over any given Kähler–Einstein manifold of positive curvature [54]. A similar result is true in the quasi-regular case [6]. If the orbits of $K$ do not close, the Sasaki–Einstein manifold is said to be irregular.

Although there are many results in the literature on Sasaki–Einstein manifolds explicit metrics are rather rare. Homogeneous regular Sasaki–Einstein manifolds are classified: they
are all $U(1)$ bundles over generalized flag manifolds. This result follows from the classification of homogeneous Kähler–Einstein manifolds. Inhomogeneous Kähler–Einstein manifolds are known to exist and so one may then construct the associated regular Sasaki–Einstein manifolds. However, until recently, there have been no known explicit inhomogeneous simply-connected manifolds in the quasi-regular class. Moreover, no irregular examples were known at all.

The family of solutions we construct in the following thus gives not only the first explicit examples of inhomogeneous quasi-regular Sasaki–Einstein manifolds, but also the first examples of irregular geometries.

4.2 The local metric

Let $B$ be a (complete) $2n$-dimensional positive curvature Kähler–Einstein manifold, with metric $g_B$ and Kähler form $J_B$ such that $\text{Ric}_{g_B} = \lambda g_B$ with $\lambda > 0$. It is thus necessarily compact and simply-connected. We construct the local Sasaki–Einstein metric in two steps. First, following refs. and , consider the local $2n + 2$-dimensional metric

$$\hat{g} = \rho^2 g_B + U^{-1} d\rho \otimes d\rho + \rho^2 U(d\tau - A) \otimes (d\tau - A)$$

where

$$U(\rho) = \frac{\lambda}{2n + 2} - \frac{\Lambda}{2n + 4} \rho^2 + \frac{\Lambda}{2(n + 1)(n + 2)} \left(\frac{\lambda}{\Lambda}\right)^{n+2} \frac{\kappa}{\rho^{2n+2}}$$

$\kappa$ is a constant, $\Lambda > 0$ and $\text{d}A = 2J_B$.

or, in other words, we can take $A = 2P_B/\lambda$ where $P_B$ is the canonical connection defined by $J_B$. By construction $\hat{g}$ is a positive curvature Kähler–Einstein metric with

$$\hat{J} = \rho^2 J_B + \rho(d\tau - A) \wedge d\rho$$

and $\text{Ric}_{\hat{g}} = \Lambda \hat{g}$. (Clearly $\hat{J}$ is closed. If we let $\hat{\Omega}$ be the corresponding $(n + 1, 0)$ form, then we calculate $d\hat{\Omega} = i\hat{P} \wedge \hat{\Omega}$, leading to a Ricci-form given by $\hat{R} \equiv d\hat{P} = \Lambda \hat{J}$.)

In ref. it was shown that the local expression describes a complete metric on a manifold if and only if $\kappa = 0$, $B$ is $\mathbb{C}P^n$ and the total space is $\mathbb{C}P^{n+1}$ the latter each with the canonical metric. Here we consider adding another dimension to the metric above – specifically, the local Sasaki–Einstein direction. We define the $(2n + 3)$-dimensional local metric, as in

$$g_X = \hat{g} + (d\psi' + \sigma) \otimes (d\psi' + \sigma)$$

where $d\sigma = 2\hat{J}$. As is well-known (see for example ref. for a recent review), such a metric is locally Sasaki–Einstein. The curvature is $2n + 2$, provided $\Lambda = 2(n + 2)$. An appropriate choice for the connection one-form $\sigma$ is

$$\sigma = \frac{\lambda}{\Lambda} A + \left(\frac{\lambda}{\Lambda} - \rho^2\right) (d\tau - A).$$

By a rescaling we can, and often will, set $\lambda = 2$.

\[^3\text{One can obtain quasi-regular geometries rather trivially by taking a quotient of a regular Sasaki–Einstein manifold by an appropriate finite freely-acting group. Our definition of Sasaki-Einstein will always mean simply-connected.}\]
4.3 Global analysis

We next show that the metrics (4.9) give an infinite family of complete, compact Sasaki–Einstein metrics on a $2n+3$-dimensional space $X$. Topologically $X$ will be given by $S^1$ bundles over $\mathbb{P}(\mathcal{O} \oplus \mathcal{L}_B)$ where $\mathcal{O}$ is the trivial bundle and $\mathcal{L}_B$ the canonical bundle on $B$. However, it should be noted that the complex structure of the Calabi-Yau cone is not compatible with that on $\mathbb{P}(\mathcal{O} \oplus \mathcal{L}_B)$ – we use the latter notation only as a convenient way to represent the topology.

The first step is to make a very useful change of coordinates which casts the local metric (4.9) into a different $(2n+2)+1$ decomposition. Define the new coordinates

$$\alpha = -\tau - \frac{\Lambda}{\lambda} \psi'$$

and $(\Lambda/\lambda)\psi' = \psi$. We then have

$$g_X = \rho^2 g_B + U^{-1} d\rho \otimes d\rho + q(d\psi + A) \otimes (d\psi + A)$$

$$+ w(d\alpha + C) \otimes (d\alpha + C)$$

where

$$q(\rho) = \frac{\lambda^2 \rho^2 U(\rho)}{\Lambda^2 w(\rho)}$$

$$w(\rho) = \rho^2 U(\rho) + (\rho^2 - \lambda/\Lambda)^2$$

$$C = f(r)(d\psi + A).$$

and

$$f(r) \equiv \frac{\rho^2(U(\rho) + \rho^2 - \lambda/\Lambda)}{w(\rho)}.$$

The metric is Riemannian only if $U \geq 0$ and hence $w \geq 0$ and $q \geq 0$. This implies that we choose the range of $\rho$ to be

$$\rho_1 \leq \rho \leq \rho_2$$

where $\rho_i$ are two appropriate roots of the equation $U(\rho) = 0$. As we want to exclude $\rho = 0$, since the metric is generically singular there, we thus take $\rho_i$ to be both positive (without loss of generality). Considering the roots of $U(\rho)$ we see that we need only consider the range

$$-1 < \kappa \leq 0$$

so that

$$0 \leq \rho_1 < \sqrt{\frac{\lambda}{\Lambda}} < \rho_2 \leq \sqrt{\frac{\lambda(n+2)}{\Lambda(n+1)}}.$$

The limiting value $\kappa = 0$ in (4.5) gives a smooth compact Kähler–Einstein manifold if and only if $B = \mathbb{C}P^n$, in which case $X$ is $S^n$ (or a discrete quotient thereof). As a consequence, we can focus on the case where the range of $\kappa$ is $-1 < \kappa < 0$.

Topologically we want $X$ to be a $S^1$ fibration over $Y = \mathbb{P}(\mathcal{O} \oplus \mathcal{L}_B)$

$$S^1_\alpha \longrightarrow X \quad \mathbb{C}P^1_{\rho, \psi} \longrightarrow Y$$

$$\downarrow \quad \downarrow$$

$$Y \quad B$$

(4.18)
As indicated, we construct the $S^1$ fibre from the $\alpha$ coordinate and the $\mathbb{C}P^1$ fibre of the base bundle $Y$ from the $(\rho, \psi)$ coordinates, where $\psi$ is the azimuthal angle and $\rho = \rho_i$ are the north and south poles. To make this identification we first need to check that the metric is smooth at the poles $\rho = \rho_i$. It is easy to show that this is true provided $\psi$ has period $2\pi$ (with $\lambda = 2$ and $\Lambda = 2(n + 2)$).

Next consider the $X \to Y$ circle fibration. Suppose $\alpha$ has period $2\pi \ell$. The question is then, can we choose $\ell$ and the parameter $\kappa$ such that $g_X$ is a regular globally defined metric on $X$. The only point to check is that the term $C$ in the expression for the metric \ref{metric} is in fact a connection on a $U(1)$ bundle. The necessary and sufficient condition is that the periods of the corresponding curvature are integral, that is

\[
\frac{1}{2\pi} F = \frac{1}{2\pi \ell} dC \in H^2_{\text{de Rham}}(Y, \mathbb{Z}). \tag{4.19}
\]

(Note that since $B$ is simply connected, so is $Y$ and hence also $H^2(Y, \mathbb{Z})$ is torsion free. Thus, the periods of $F$ in fact completely determine the $U(1)$ bundle.)

To check the condition \ref{integral_periods}, we need a basis for the torsion-free part of $H^2(Y, \mathbb{Z})$. First note that since $Y$ is a projectivised bundle over $B$, we can use the results of sec. 20 of ref. \cite{8} to write down the cohomology ring of $Y$ in terms of $B$. In particular, we have $H^2(Y, \mathbb{Z}) \cong \mathbb{Z} \oplus H^2(B, \mathbb{Z})$, where the first factor is generated by the cohomology class of the $S^2$ fibre. Let $\{\Sigma_i\}$ be a set of two-cycles in $B$ such that the homology classes $[\Sigma_i]$ generate the torsion-free part of $H^2(B, \mathbb{Z})$. Next define a submanifold $\Sigma \cong S^2$ of $Y$ corresponding to the fibre of $Y$ at some fixed point on the base $B$. Finally we also have the global section $\sigma^N : B \to Y$ corresponding to the “north pole” ($\rho = \rho_1$) of the $S^2$ fibres. Together we can then construct the set $\{\Sigma, \sigma^N \Sigma_i\}$ which forms a representative basis generating the free part of $H^2(Y, \mathbb{Z})$.

Calculating the periods of $F$ we find

\[
\int_\Sigma \frac{F}{2\pi} = \frac{f(\rho_1) - f(\rho_2)}{\ell},
\]

\[
\int_{\sigma^N \Sigma_i} \frac{F}{2\pi} = \frac{f(\rho_2)c_{(i)}}{\ell}, \tag{4.20}
\]

where

\[
c_{(i)} = \int_{\Sigma_i} \frac{dA}{2\pi} = \langle c_1(L_B), [\Sigma_i] \rangle \in \mathbb{Z} \tag{4.21}
\]

are the periods of the canonical bundle $L_B$. Thus we have integral periods if and only if $f(\rho_1)/f(\rho_2) = p/q \in \mathbb{Q}$ is rational with $p, q \in \mathbb{Z}$ and $\ell = f(\rho_2)/q = f(\rho_1)/p$. The periods of $\frac{1}{2\pi} F$ are then $\{p - q, qc_{(i)}\}$. Rescaling $\ell$ by \(h = \text{hcf}\{p - q, qc_{(i)}\}\) gives a special class of solutions where the integral periods $\{h^{-1}(p - q), h^{-1}qc_{(i)}\}$ have no common factor. This is the class we will concentrate on from now on since in that case $X$ is simply-connected (see refs. \cite{2} \cite{3}).

Notice that, using the expression \ref{integral_periods}, we have

\[
R(\kappa) \equiv \frac{f(\rho_1)}{f(\rho_2)} = \frac{\rho_1^2(\rho_2^2 - \lambda/\Lambda)}{\rho_2^2(\rho_1^2 - \lambda/\Lambda)}. \tag{4.22}
\]
This is a continuous function of $\kappa$ in the interval $(-1, 0]$. Moreover, it is easy to see that $R(0) = 0$ and $R(-1) = -1$. Hence there are clearly a countably infinite number of values of $\kappa$ for which $R(\kappa)$ is rational and equal to $p/q$, with $|p/q| < 1$, and these all give complete Riemannian metrics $g_X$.

Locally, by construction, $g_X$ was a Sasaki–Einstein metric. In the $(\alpha, \psi)$ coordinates the unit-norm Killing vector $K$ is given by

$$K = \frac{\Lambda}{\lambda} \left( \frac{\partial}{\partial \psi} - \frac{\partial}{\partial \alpha} \right).$$

Given the topology of $X$ it is easy to see that this is globally defined. Hence, so is the corresponding one-form and also $J = \frac{1}{2} dK$. Thus the Sasaki–Einstein structure is globally defined.

Recall that our original problem was to find $X$ which admitted Killing spinors. For this we need $X$ to be spin. However, by construction this is the case irrespective of whether $B$ is spin or not (see ref. [3]). Since $X$ is also simply-connected we can then invoke theorem 3 of [89] to see that this implies we have global Killing spinors.

Finally we notice that the orbits of the Killing vector $K$ close if and only if $f(\rho_2) \in \mathbb{Q}$, in which case the Sasaki–Einstein manifold is quasi-regular. For generic $p$ and $q$ the space will be irregular. Determining when $f(\rho_2) \in \mathbb{Q}$ seems to be a non-trivial number-theoretic problem. (Though for the case $n = 1$ studied in ref. [2] one has to solve a quadratic diophantine, which can be done using standard methods.) Thus for the countably infinite number of values of $\kappa$ found here, the Kähler–Einstein “base” is at best an orbifold, and in the irregular case there is in fact no well-defined base at all.

### 4.4 Five-dimensional solutions

Of particular interest in string theory are five-dimensional Sasaki–Einstein solutions. For our class these are the backgrounds which are dual to the $S^2 \times T^2$ solutions in $d = 11$ supergravity discussed in the previous section.

To make the correspondence explicit we take $n = 1$, $\lambda = 2$ and $\Lambda = 6$ and introduce the new coordinate $\rho^2 = (\lambda/\Lambda)(1 - cy)$. The metric (4.12) then takes the form

$$g_X = \frac{1}{6}(1 - cy)\tilde{g} + w^{-1}q^{-1}dy \otimes dy + \frac{q}{9}(d\psi + \tilde{P}) \otimes (d\psi + \tilde{P})$$

$$+ w(d\alpha + C) \otimes (d\alpha + C)$$

where

$$q(y) = \frac{a - 3y^2 + 2cy^3}{a - y^2},$$

$$w(y) = \frac{2(a - y^2)}{1 - cy},$$

$$C = \frac{ac - 2y + cy^2}{6(a - y^2)}(d\psi + \tilde{P}).$$

with $\tilde{g}$ the canonical metric on $S^2$ with $\text{Ric}_{\tilde{g}} = \tilde{g}$ and $\tilde{P}$ the corresponding canonical connection. This is the form of the metric one obtains by making an explicit duality from the $d = 11$ solutions given in the previous section.
Topologically these spaces are all $S^2 \times S^3$. One finds both quasi-regular and irregular solutions depending on whether or not $f(\rho_2) \in \mathbb{Q}$. In the dual field theory this corresponds to rational or irrational R-charges and also central charge. Interestingly the irrational charges are at most quadratic algebraic, that is can be written in terms of square-roots of rational numbers. This matches a field theory argument due to Intriligator and Wecht.

4.5 A simple generalisation

We end by noting that there is a simple generalisation of the above construction to the case when the base manifold $B$ is a product of Kähler–Einstein manifolds.

More specifically, let $g_i$ with $i = 1, \ldots, p$ be a set of $2n_i$-dimensional positive curvature Kähler–Einstein metrics with Kähler forms $J_i$ and $\text{Ric}_{g_i} = \lambda_i g_i$. There is then a straightforward generalisation of the construction of refs. [86] and [87] allowing one to build a $(2n + 2)$-dimensional Kähler–Einstein metric $\hat{g}$ where $n = \sum_i n_i$. In analogy with (4.5) we write

$$\hat{g} = \sum_i \left( r^2 + \frac{\lambda_i}{\Lambda} \right) g_i + V(r)^{-1} dr \otimes dr + r^2 V(r)(d\tau - A) \otimes (d\tau - A) \quad (4.26)$$

and define the corresponding fundamental form

$$\hat{J} = \sum_i \left( r^2 + \frac{\lambda_i}{\Lambda} \right) J_i + r(d\tau - A) \wedge dr. \quad (4.27)$$

The metric is Kähler–Einstein with $\text{Ric}_{\hat{g}} = \Lambda \hat{g}$ provided

$$dA = 2 \sum_i J_i, \quad (4.28)$$

$$r^2 V(r) = -\frac{1}{2} \Lambda f (r^2; \lambda_i/\Lambda, \mu),$$

where

$$f(x; d_i, \mu) = \mu + \int_0^x dx' x' \prod_i (x' + d_i)^{n_i} \prod_i (x + d_i)^{n_i}. \quad (4.29)$$

and $\mu$ is an integration constant. Note that we have chosen a slightly different convention for the coordinate $r$ as compared to the case of $p = 1$ given in (1.6). They differ by $r^2 = \rho^2 - \lambda/\Lambda$.

Using the construction of sec. (4.2) above we can then obtain $(2n+3)$-dimensional metrics which are locally Sasaki–Einstein. For certain values of the constants $\lambda_i$ and $\mu$ one expects that these will give metrics on complete compact Sasaki–Einstein manifolds: the analysis will be a direct generalisation of that in ref. [3] but we leave the details for future work.

The case of most interest for M-theory is when the dimension of the resulting Sasaki–Einstein manifold is seven, as they can be used to obtain new $\text{AdS}_4 \times X$ solutions of $d = 11$ supergravity. It was shown in ref. [34] that the construction using (4.5) in case where the base is a direct product of two equal radius two-spheres generalises the well-known homogeneous Sasaki-Einstein metric $Q^{1,1,1}$ (see for example ref. [91] for a review). A further generalisation to the case where the spheres have different radii can be obtained from the generalised
contruction. Setting \( p = 2 \), \( n_1 = n_2 = 1 \) (so that \( g_i \) are round metrics on two-spheres) in (4.26) we obtain
\[
\hat{g} = \frac{1}{\Lambda} \left[ (1 + c_1 x) \lambda_1 g_1 + (1 + c_2 x) \lambda_2 g_2 \right] + \frac{\mathrm{d}x \otimes \mathrm{d}x}{F(x)} + \frac{F(x)}{\Lambda^2} (\mathrm{d}\beta - c_i A_i) \otimes (\mathrm{d}\beta - c_i A_i),
\]
(4.30)
where \( c_i = \Lambda / \lambda_i \), \( dA_i = \lambda_i J_i \) and
\[
F(x) = -\frac{\Lambda}{8} \frac{16 c_1 c_2 \mu + 8 x^2 + \frac{16}{3} (c_1 + c_2) x^3 + 4 c_1 c_2 x^4}{(1 + c_1 x)(1 + c_2 x)}.
\]
(4.31)

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References


