Aspects of generalized Calogero model

S.Meljanac\textsuperscript{a1}, M.Mileković \textsuperscript{b2}, A. Samsarov\textsuperscript{a3}

\textsuperscript{a} Rudjer Bošković Institute, Bijenička c.54, HR-10002 Zagreb, Croatia

\textsuperscript{b} Theoretical Physics Department, Faculty of Science, P.O.B. 331, Bijenička c.32, HR-10002 Zagreb, Croatia

Abstract

A multispecies model of Calogero type in $D \geq 1$ dimensions is constructed. The model includes harmonic, two-body and three-body interactions. Using the underlying conformal $SU(1,1)$ algebra, we find the exact eigenenergies corresponding to a class of the exact global collective states. Analysing corresponding Fock space, we detect the universal critical point at which the model exhibits singular behaviour.

PACS number(s): 03.65.Fd, 05.30.Pr

Keywords: multispecies Calogero model, $SU(1,1)$ algebra.

1 Introduction

The (rational) Calogero model \cite{1} is one of the most famous and exhaustively studied examples of exactly solvable systems. It describes $N$ identical particles on the line which interact through an inverse-square two-body interaction and are subjected to a common confining harmonic force. Starting from the inception more than thirty years ago, the model and its various descendents continue to be of interest for both physics and mathematics community, primarily because they are connected with a number of mathematical and physical problems, ranging from random matrices and
symmetric polynomials \[2\] to condensed matter systems \[3\], black hole physics \[4\] and 2D strings \[5\].

The Calogero model is defined by the following Hamiltonian \((i, j = 1, 2, \ldots N)\):

\[
H = -\frac{\hbar^2}{2m} \sum_i \frac{\partial^2}{\partial x_i^2} + \frac{m\omega^2}{2} \sum_i x_i^2 + \frac{\hbar^2 \nu(\nu - 1)}{2m} \sum_{i \neq j} \frac{1}{(x_i - x_j)^2},
\]

(1)

where \(\nu(\nu - 1) \geq -1/4\) is the dimensionless coupling constant, \(m\) is the mass of particles and \(\omega\) is the strength of a harmonic confinement potential. The ground state of the Hamiltonian (1) is of the well-known, highly correlated, Jastrow form

\[
\Psi_0(x_1, x_2, \ldots x_N) = \prod_{i<j} |x_i - x_j|^{\nu} e^{-\frac{m\omega}{\hbar} \sum_i x_i^2},
\]

(2)

with corresponding ground state energy, which depends on \(\nu\) explicitly

\[
E_0 = \omega \left( \frac{N}{2} + \frac{\nu N(N - 1)}{2} \right).
\]

(3)

One can exactly solve this model and find out the complete set of energy eigenvalues either by following the traditional approach \[1\] or by employing its underlying \(S_N\) (permutational) algebraic structure \[6\]. The later (operator) approach is considerably simpler than the original one, yields an explicit expression for the wavefunctions and emphasizes the interpretation in terms of generalized statistics. Namely, the inverse-square potential can be regarded as a pure statistical interaction and the model maps to an ideal gas of particles obeying fractional Haldane statistics \[7\], with coupling constant \(\nu\) playing the role of Haldane statistical parameter. In Haldane’s formulation, however, there is the possibility of having particles of different species with a mutual statistical parameter depending on the species coupled. This suggests the generalization of the single-species Calogero model to the multispecies Calogero model. Distinguishability of the species can be introduced by allowing particles to have different masses and different couplings to each other. The novel feature of 1D multispecies Calogero model is appearance of the long-range three-body interaction. Under certain conditions this three-body interaction can be eliminated from the Hamiltonian (but only in 1D!).

Further generalization of Calogero model (1) can be achieved by formulating the model in dimensions higher than one. In a case of a single-species model in \(D\) dimensions, some exact eigenstates (including the ground state) are known but the complete solution of the problem is still lacking. Some progress has been achieved only recently for a class of 2D models \[8\]. Usually, the inevitable appearance of the three-body interaction in \(D > 1\) is the main obstacle which makes any analysis of such a model(s) highly nontrivial.
The aim of the present article is to present and discuss some aspects of the model which incorporates both generalizations mentioned above. In Sec.2 we define a multispecies Calogero model in D dimensions. We succeeded in finding a class of, but not all, exact eigenstates of the model Hamiltonian which corresponded to global collective states. The analysis relied heavily on the SU(1, 1) algebraic structure of the Hamiltonian and utilized Fock space representation. In Sec.3 we briefly discuss some other interesting features of the model. We rewrite the Hamiltonian in a simple form and comment on universal critical point. Section 4 is concluding section.

2 A model Hamiltonian: some exact eigenstates and eigenenergies

Following the lines of reasoning already sketched in Introduction, we define the most general Calogero-type Hamiltonian in \( D \) dimensions [9], describing \( N \) distinguishable Calogero-like particles which interact with two-body and three-body interactions, as (we put \( \hbar = 1 \) and \( i, j = 1, 2, ..., N \))

\[
H = -\frac{1}{2} \sum_i \frac{1}{m_i} \vec{\nabla}_i^2 + \frac{\omega^2}{2} \sum_i m_i \dot{r}_i^2 + \frac{1}{2} \sum_{i<j} \nu_{ij} \left( \frac{\nu_{ij} + D - 2}{m_i + m_j} \right) \frac{1}{|\vec{r}_i - \vec{r}_j|^2} + \frac{1}{2} \sum_{i\neq j, i\neq k} \nu_{ij} \nu_{ik} \frac{(\vec{r}_i - \vec{r}_j)(\vec{r}_i - \vec{r}_k)}{m_i |\vec{r}_i - \vec{r}_j|^2 |\vec{r}_i - \vec{r}_k|^2}.
\]

(4)

Here, \( m_i \) are masses of the particles, \( \omega \) is the frequency of the harmonic potential and \( \nu_{ij} = \nu_{ji} \) are the statistical parameters between particles \( i \) and \( j \).

It can be shown that the ground state \( \Psi_0 \), obeying \( H \Psi_0 = E_0 \Psi_0 \), is of generalized Jastrow form (2)

\[
\Psi_0(\vec{r}_1, ..., \vec{r}_N) = \prod_{i<j} |\vec{r}_i - \vec{r}_j|^{\nu_{ij}} e^{-\frac{\omega}{2} \sum_i m_i \dot{r}_i^2} \equiv \Delta e^{-\frac{\omega}{2} \sum_i m_i \dot{r}_i^2} \quad (5)
\]

and the ground state energy \( E_0 \) generalizes Calogero result (3)

\[
E_0 = \omega \left( \frac{ND}{2} + \sum_{i<j} \nu_{ij} \right) \equiv \omega \epsilon_0 \quad (6)
\]

Notice that for \( \nu_{ij} = \nu \), \( m_i = m \) and \( D \neq 1 \) Eq.(5) smoothly goes to exact ground state of the Calogero-Marchioro model. For \( D = 1 \), the three-body term in (4) identically vanish if \( \nu_{ij} = \nu \) and \( m_i = m \) (single-species Calogero model [1]) or if there exists a certain relation between masses and coupling constants, of the form \( \nu_{ij} = \alpha m_i m_j \) (multispecies Calogero model [10]). This happens because identity
\[
\sum_{(x_i-x_j)(x_i-x_k)} \frac{1}{m_i} = 0, \text{ which holds in 1D only. Unlike in one dimension, however, it does not vanish in higher dimensions (there is no analogous identity for } D \neq 1) \text{ and plays a crucial role in the analysis that is to follow.}
\]

In order to simplify the analysis, we perform the non-unitary transformation on \( \Psi \), namely \( \tilde{\Psi} = \Delta^{-1} \Psi \). It generates a similarity transformation which leads to another Hamiltonian \( \tilde{H} = \Delta^{-1} H \Delta \). We find \( \tilde{H} \) as

\[
\tilde{H} = \omega^2 \left( \frac{1}{2} \sum_i m_i \vec{r}_i^2 \right) - \left( \frac{1}{2} \sum_i \frac{1}{m_i} \vec{\nabla}^2_i + \sum_{i<j} \nu_{ij} \frac{(\vec{r}_i - \vec{r}_j)}{|\vec{r}_i - \vec{r}_j|^2} \left( \frac{1}{m_i} \vec{\nabla}_i - \frac{1}{m_j} \vec{\nabla}_j \right) \right)
\equiv \omega^2 T_+ - T_-. \tag{7}
\]

In addition to \( T_\pm \), we also introduce dilatation operator \( T_0 = \frac{1}{2} \left( \sum_i \vec{r}_i \vec{\nabla}_i + \varepsilon_0 \right) \). The operators \( T_\pm, T_0 \) satisfy the \( SU(1,1) \) algebra.

It is convenient to introduce the center-of-mass coordinate \( \vec{R} \) and the relative coordinates \( \vec{\rho}_i \):

\[
\vec{R} = \frac{1}{M} \sum_i m_i \vec{r}_i, \quad \vec{\nabla}_R = \sum_i \vec{\nabla}_i, \\
\vec{\rho}_i = \vec{r}_i - \vec{R}, \quad \vec{\nabla}_{\rho_i} = \vec{\nabla}_i - \frac{m_i}{M} \vec{\nabla}_R. \tag{8}
\]

Using Eqs.(7) and (8) we define creation (+) and annihilation (-) operators

\[
\tilde{A}_1^\pm = \frac{1}{\sqrt{2}} \left( \sqrt{M} \omega \vec{r} \mp \frac{1}{\sqrt{M} \omega} \vec{\nabla}_R \right), \\
A_2^\pm = \frac{1}{2} \left( \frac{T_-}{\omega} + \omega T_+ \right) \mp T_0, \tag{9}
\]

which satisfy the following commutation relations \( (\alpha, \beta = 1, 2, \cdots D) \):

\[
[A_{1,\alpha}^-, A_{1,\beta}^+] = \delta_{\alpha\beta}, \quad [A_{1,\alpha}^-, A_{1,\beta}^-] = [A_{1,\alpha}^+, A_{1,\beta}^-] = 0, \\
[A_{2,\alpha}^-, A_{2,\beta}^+] = \tilde{H}, \quad [H, A_{1,\pm}^\pm] = \pm \omega A_{1,\pm}^\pm.
\]

They act on the Fock vacuum \( |\tilde{0}\rangle \propto \tilde{\Psi}_0(\vec{r}_1, ..., \vec{r}_N) \) as

\[
\tilde{A}_1^- |\tilde{0}\rangle = A_2^- |\tilde{0}\rangle = 0, \quad \langle 0|\tilde{0}\rangle = 1. \tag{11}
\]
The excited states in the Fock space, corresponding to global collective states, are of the form
\[
(A^+_{1,1})^{n_{1,1}} \cdots (A^+_{1,D})^{n_{1,D}} (A^+_{2})^{n_2} |\tilde{0}\rangle = \prod_{\alpha=1}^{D} (A^+_{1,\alpha})^{n_{1,\alpha}} (A^+_{2})^{n_2} |\tilde{0}\rangle,
\]
where \(n_{1,\alpha} = 0, 1, 2 \ldots (\forall \alpha)\) and \(n_2 = 0, 1, 2\ldots\)

The repeated action of the operators \(A^+_{1,\alpha}\) on the vacuum \(|\tilde{0}\rangle\) reproduces, in the coordinate representation, Hermite polynomials \(H_{n_{1,\alpha}}(R_{\alpha}\sqrt{M}\omega)\). Similarly, the repeated action of the operator \(A^+_{2}\) on the vacuum \(|\tilde{0}\rangle\) reproduces hypergeometric function, which reduces to associated Laguerre polynomials \(L_{n_2+\varepsilon_0-1}^{\varepsilon_0-1}(2\omega T_+^+)\) for certain values of parameters.

The states (12) are eigenstates of the \(\tilde{H}\) with the energy eigenvalues (cf. last two equations in Eqs.(10))
\[
E_{n_{1,\alpha}:n_2} = \omega \left( \sum_{\alpha=1}^{D} n_{1,\alpha} + 2n_2 + \varepsilon_0 \right).
\]

This is the part of the complete spectrum which corresponds to center-of-mass states and global dilatation states, respectively. We note in passing that this states, Eq.(12), are perfectly normalizable (i.e. quadratically integrable) and physically acceptable for both Hamiltonians \(\tilde{H}\) and \(H\), provided that \(\varepsilon_0 > D/2\).

### 3 Miscellaneous remarks

It is evident from Eqs.(10) that the modes described by \(\tilde{A}^\pm_{1}\) and \(A^\pm_{2}\) are still coupled. By introducing another set of of the creation and annihilation operators \(\{B^+_{2}, B^-_{2}\}\)
\[
B^\pm_{2} = A^\pm_{2} - \frac{1}{2} (\tilde{A}^\pm_{1})^2,
\]

one can show that the center-of-mass motion decouples completely
\[
[A_{1,\alpha}^\pm, B^\mp_{2}] = 0.
\]

The Hamiltonian \(\tilde{H}\) separates as \(\tilde{H} = \tilde{H}_{CM} + \tilde{H}_{R}\), with
\[
\tilde{H}_{R} = \omega[B^-_{2}, B^+_{2}], \quad [\tilde{H}_{R}, B^\pm_{2}] = \pm 2\omega B^\pm_{2},
\]
\[
\tilde{H}_{CM} = \frac{1}{2} \omega \sum_{\alpha=1}^{D} \{A^-_{1,\alpha}, A_{1,\alpha}^+\}_+, \quad [\tilde{H}_{CM}, \tilde{A}^\pm_{1}] = \pm \omega \tilde{A}^\pm_{1}.
\]
The corresponding Fock space (12) now splits into the CM-Fock space, spanned by $\prod_\alpha (A_{+\alpha}^{n_\alpha} |\tilde{0}\rangle)_{CM}$ and the R-Fock space, spanned by $B_2^{-n_2} |\tilde{0}\rangle_R$. The respective vacua are $|\tilde{0}\rangle_{CM} \propto e^{-\frac{1}{2} M \vec{R}^2}$ and $|\tilde{0}\rangle_R \propto e^{-\frac{1}{2} \sum_i m_i \vec{r}_i^2}$.

Closer inspection of the R-Fock space of the Hamiltonian $\tilde{H}_R$ reveals the existence of the universal critical point defined either by the null-vector

$$\frac{R\langle \tilde{0} | B_2^- B_2^+ | \tilde{0} \rangle_R}{R\langle \tilde{0} | \tilde{0} \rangle_R} = 0 \Rightarrow \frac{(N - 1)D}{2} + \frac{1}{2} \sum_{i \neq j} \nu_{ij} = 0,$$

or, equivalently, by the zero-energy condition

$$E_{0R} = \frac{(N - 1)D}{2} + \frac{1}{2} \sum_{i \neq j} \nu_{ij} = 0.$$

At the critical point the system described by $\tilde{H}_R$ collapses completely. This means that the relative coordinates, the relative momenta and the relative energy are all zero at this critical point. There survives only one oscillator, describing the motion of the centre-of-mass. This critical point was first noticed by analysing Gram matrices of the scalar products of Fock-space states in ordinary Calogero model [11] and further confirmed by large-N collective field theory approach to the same model in Ref.[12].

For the initial Hamiltonian (4), which is not unitary equivalent to $\tilde{H}$, the conditions (17,18) demand that some $\nu_{ij} < 0$ and, consequently, the norm of the wave function (5) blows up at the critical point. For $\nu_{ij}$ negative but greater than the critical values (17,18), the wave function is singular at coincidence points but still quadratically integrable.

4 Conclusion

In summary, we have studied a non-trivial many-body Hamiltonian of Calogero type in D dimensions (4), with two- and three-body interactions among non-identical particles. By applying the similarity transformation, we have obtained the simpler Hamiltonian $\tilde{H}$ (7), on which we have performed the Fock-space analysis and found some of its excited (collective) states (12) and their energies (13). The spectrum of collective modes is linear, equidistant and degenerate. By splitting the Fock space into the CM-Fock space and the R-Fock space (16) we have detected the universal critical point (17,18) at which the system exhibits singular behaviour. Unfortunately, within our algebraic treatment, we are unable to construct the rest.
of the eigenstates of the Hamiltonian (7) which parallel those in Calogero model (1). Therefore, the question of the full solvability of the model Hamiltonian (4) still remains open. Nevertheless, we hope that our analysis sheeds some light on the such kind of higher-dimensional models, that is the models with similar underlying conformal SU(1,1) symmetry.

Acknowledgments

One of the authors (M.M.) would like to thank Prof. Č. Burdik for kind invitation and hospitality during the XIII International Colloquium ”Integrable Systems and Quantum Groups”, Prague, June 17-19, 2004. This work was supported by the Ministry of Science and Technology of the Republic of Croatia under contracts No. 0098003 and No. 0119261.

References