Abstract
All superconducting accelerators to date have used magnets wound from Rutherford cable. In this first of two lectures, we describe the filamentary superconducting wires used within a Rutherford cable. To discuss the reasons for this fine filamentation of the superconductor, we first describe the critical state model and then use it to show how persistent currents in the superconductor produce magnetization effects, which cause field errors. Coupling between the filaments is then discussed, together with the criterion for twisting to de-couple the filaments. Finally the relationship between magnetization and ac losses in changing fields is described.

1. INTRODUCTION
This is the first of two linked lectures which describe Rutherford cable, the magnet conductor which has been used in all superconducting synchrotron accelerators to date. Fig 1 shows how Rutherford cable consists of a flat twisted cable comprising ~ 30 wires, each of which contains some 5000 – 20,000 fine filaments of superconductor embedded in a matrix of copper. In this first lecture we discuss why such fine subdivision of the superconductor is necessary and how it affects the magnetization and ac loss of the magnet winding. To provide a basis for that discussion, it will first be necessary to get acquainted with the critical state model, which describes how all high field Type II superconductors react to magnetic and electric fields.

Fig. 1: Rutherford cable, inset shows a single wire comprising ~ 10,000 filaments of NbTi superconductor in a copper matrix.
2. THE CRITICAL STATE MODEL

When a superconductor is exposed to changing magnetic fields, currents are induced to flow in such a way as to shield the interior of the superconductor from the changing field. They are like eddy currents but, because the superconductor has no resistance, they do not decay with time. The path and magnitude of these currents may be predicted using the critical state model, originally proposed by C.P. Bean [1]. The critical state model may be summarized in two statements:

a) all disturbances start at the surface of the superconductor and propagate inwards.

b) all regions within the superconductor have either zero current or current flowing at critical current density $J_c$.

It is common to discuss problems of this nature in terms of the simplified one dimensional slab model shown in Fig. 2. All the basic physical principles may be discussed in terms of one dimensional arguments; extension to the more realistic situation in two and three dimensions entails much more complication, but provides few further insights.

![Fig. 2: A field $B_y$ applied to a slab of superconductor induces currents $J_z$ to flow; right hand side shows a plot of $B_y$ through a cross section of the slab, which is infinite in the y and z directions.](image)

As shown in Fig. 2, the increasing field $B_y$ induces screening currents $\pm J_z$ to flow in the slab. According to the critical state model $J_z = J_c$. In one dimension we have:

$$\frac{\partial B_y}{\partial x} = -\mu_0 J_z = \mu_0 J_c$$  \hspace{1cm} (1)

Thus, when we plot magnetic field within the slab, we see a linear fall-off from the surface until the field reaches zero at the interior, after which the current density becomes zero and the field does not change. As the external field increases, the regions of $\pm J_c$ extend further into the slab until they reach the centre, at which point the slab is said to be fully penetrated. Further increases in field produce no change in the current pattern, merely a general increase in field throughout the slab.

Physically we may think of the process as follows. The changing magnetic field produces an electric field at the surface of the slab which drives current in the superconductor slightly beyond critical density until the resistive voltage matches the electric field. When the field change stops, current in the superconductor decays until it falls just below critical density, after which there is no reason for it to decay further, so it stays flowing at $J_c$.

If we now reduce the external field, patterns like those shown in Fig. 3 are induced. Once again we see the disturbance starting from the surface, where it induces negative currents, which
progressively penetrate into the slab ‘wiping out’ the original currents. Thus we see that the pattern of currents depends on the history and the direction in which the field is changing; the behaviour is *hysteretic*.

![Fig 3: pattern of screening current and internal field when the external field is progressively reduced.](image)

If the slab is thick, so that the screening currents produce a large difference in field between the surface and interior of the slab, the currents can become unstable and collapse. When this happens, magnetic flux rushes into the slab in a process known as a *flux jump*. Details of this process and the criterion of slab width below which the currents are stable, will be presented in the lectures on Stability by H. Ten Kate.

### 3. MAGNETIZATION

These persistent screening currents produce magnetic effects which can be detected outside the slab. In fact we can define a *magnetization* in the usual way as magnetic moment per unit volume, ie the summation of (current) × (area enclosed) per unit volume. For the infinite slab, when it is fully penetrated, we may integrate the current density across the slab, using the nomenclature sketched in Fig. 4.

![Fig 4: Nomenclature for the magnetization calculation](image)

Thus

$$M_{\text{slab}} = \sum_{v} \frac{IA}{V} = \frac{1}{a} \int_{0}^{a} J_c x dx = \frac{J_c a}{2}$$

(2)

Note that in deriving (2), we are assuming symmetry about the chain dotted line in Fig. 3, often known as the *electric centre line*. For the more practical case of a cylindrical wire aligned perpendicular to the field, the calculation is similar but more complicated and gives the result:

$$M_{\text{cyl}} = \frac{4}{3\pi} J_c a$$

(3)

Putting the sample into a magnetometer, we can measure its magnetization and plot a loop like that shown in Fig 5, where the field has be repeatedly swept between positive negative values, measuring the magnetization continuously. Of particular interest for synchrotrons is the branch labelled ‘inj’, for which the field was swept down from high to low positive values and then swept up again from the low value. Note how rapidly the magnetization changes at the beginning of the upward sweep – just at the time when the synchrotron is being injected.
Fig 5: Magnetization curve of a Nb Ti superconducting wire; captions show internal field pattern for each zone of the curve.

In order to reduce magnetization effects, all magnet conductors are made with the superconductor divided into fine filaments. To avoid flux jumping, it is necessary to subdivide below a characteristic diameter, which for NbTi is \( \sim 50 \mu m \) – and this is always done, not just for accelerator magnets. For the more stringent conditions of accelerator magnets however, finer subdivision is required, mainly to reduce the field errors at injection. In addition, for accelerators with a fast ramping cycle, finer subdivision is also needed to reduce the ac losses. There is really no lower limit to the degree of subdivision which would be beneficial, but increasing cost and falling critical current density usually impose a practical lower limit of \( \sim 6 - 10 \mu m \), and this has been used in all the superconducting accelerators built to date.

Finally, we emphasize that the magnetization discussed here is the \textit{irreversible} magnetization coming from the \textit{bulk} current density. It should not be confused with the \textit{reversible} magnetization, inherent to any type II superconductor and shown in Fig.6. For all practical purposes, the inherent
reversible magnetization may be neglected however because it is very much smaller than the irreversible component (much more so than indicated by Fig. 6, where the scale of the reversible component has been increased for visibility).

4. FILAMENTARY COMPOSITE WIRES

A single 6µm filament of NbTi in 6T field will carry about 75mA – clearly we need a lot of them in parallel to make a synchrotron conductor! For this reason, as well as mechanical robustness, conductors are made in the form of multi-filamentary composite wires containing 5000 – 20000 filaments in a matrix of copper. As will be seen in the lectures on stability and protection, the copper performs several valuable roles in promoting reliable operation of the magnet. Unfortunately it also brings a problem by coupling the filaments together in changing fields, as illustrated in Fig. 7.

Fig. 7: A composite wire containing 7 filaments of superconductor in a copper matrix; left hand shows screening currents and field profile with the filaments uncoupled, right hand shows the same with filaments coupled.

Ideally we would like the filaments to behave quite independently as shown on the left hand side of Fig. 7, but in changing fields the screening currents flow across the matrix at each end of the wire, thereby screening more field from the interior. When the field change stops, these coupling currents will eventually decay to the pattern shown on the right of Fig. 7, but the time constant for this decay will be days or even years for wires of more than a few metres long. The advantages of fine subdivision are thus completely negated during practical operating timescales.

Fortunately there is a simple solution to this problem – twist the wire so that the coupling currents are forced to reverse every twist pitch. In this way the time constant for decay of the coupling currents may be reduced to a few msec so that, for reasonable rates of change of field, they never have a chance of building up. We now develop the theory of time constants for a twisted multi-filamentary composite wire. Essential to the theory is the assumption that the screening effect of the induced currents is small, so that the rate of change of field within the wire is the same as that outside. Fig. 8 illustrates a twisted multi-filamentary wire exposed to a changing field $B' = dB/dt$. Coupling currents flow along the zig-zag path $PQR$ over many twist pitches before finally crossing over to $S$ at the ends of the wire and returning via a similar zig-zag path down the other side.
Provided the filaments are below their critical current, there can be no electric field along them, so the whole electric field induced by the changing magnetic field must lie along the vertical paths QR.

\[
\oint E \, dl = \int_{Q} B' \, a \cos \theta \, dz = \frac{2B' \, p \, Y}{2\pi}
\]

where \( a \) is the radius and \( p \) is the twist pitch. The expression holds for all \( y \) implying a uniform vertical electric field, which will produce a vertical current density:

\[
J_y = \frac{B' \, p}{2\pi \rho_t}
\]

where \( \rho_t \) is the effective transverse resistivity across the wire. We now consider how this current density through the matrix is collected from and donated to the ring of filaments.

We approximate the totality of filament currents by a cylindrical current sheet of linear (Am\(^{-1}\)) current density \( g_f \) and consider the current transferred to the matrix from a sector of angular extent \( \delta \theta \) over a length in which its azimuthal coordinate changes by \( \Delta \theta \).

\[
\Delta g_f \, a \, \delta \theta \cos \psi = \frac{B' \, p \, a \, \Delta \theta}{2\pi \rho_t \tan \psi} \cos \theta
\]

so that:
\[ \Delta g_f \propto \frac{B_i' p}{2 \pi \rho_i \cos \psi \tan \psi} \]  
\[ g_f(\theta) = \frac{B_i' p}{2 \pi \rho_i \cos \psi \tan \psi} a \cos \theta \]  
\[ g_z(\theta) = \frac{B_i' \left[ p \left( \frac{p}{2\pi} \right) \right]^2}{\rho_i} \cos \theta \]

As discussed in the lectures on Magnetic Field Design, a \( \cos \theta \) distribution of current around a cylinder produces a uniform field inside that cylinder. In this case, the induced screening currents produce a uniform field inside our cylinder of filaments, opposing the changing external field \( B' \). Thus we may write the internal field:

\[ B_i = B_e - \frac{\mu_0}{2} g_z(0) = B_e - \frac{\mu_0}{2} B_i' \left( \frac{p}{2\pi} \right)^2 \]

so that:

\[ B_i = B_e - B_i' \tau \]  
where
\[ \tau = \frac{\mu_0}{2 \rho_i} \left( \frac{p}{2\pi} \right)^2 \]

Finally we may integrate the (screening current) \( \times \) (area enclosed) to get the induced magnetization

\[ M_e = \frac{4}{\pi a^2} \int_0^{\pi/2} g_z(\theta) a \cos \theta a d\theta \]  
and
\[ M_e = \frac{2}{\mu_0} B_i' \tau \]

Note the resemblance between these equations and the standard eddy current formulae. Indeed these currents are really eddy currents in the copper matrix, whose amplitude has been greatly increased by the extra flux linkages through the superconducting filaments.

Thus we see that a filamentary composite wire has two components of magnetization:

a) steady state magnetization coming from the persistent currents flowing within filaments.

b) rate dependent magnetization coming from eddy currents flowing between filaments.

These two components are illustrated in Fig. 10, which plots the magnetization of a filamentary composite wire at different rates of change of field, the outermost loop being at the fastest rate of change. Note how the steady state component depends on \( B \) (via \( J_c \)) whereas the rate dependent component is independent of \( B \) and depends only on \( B' \).
We can see exactly the same behaviour in the field error of a magnet. Fig. 11 plots the skew quadrupole error of a niobium tin dipole [2] at various rates of change of field.

Fig. 11: Skew quadrupole error measured in a niobium tin dipole [2] at different sweep rates.

5. AC LOSSES

We have already seen in the critical state model that a changing magnetic field, by producing an electric field, drives the superconductor into the resistive state. It follows that the changing magnetic field must produce ac loss. In addition, the coupling currents through the copper matrix of a filamentary composite dissipate Ohmic heat. Because the work done to produce these losses is
supplied by the magnetic field, we may calculate them from the external field and magnetization of the sample. The work per unit volume when the sample is taken around a cycle is given by:

$$Q = \oint \mu_0 M dH = \oint \mu_0 H dM$$

If the magnetization is small compared with the external field (usually the case) we may also write:

$$Q = \oint M dB$$

For a general physics proof of the above, see any textbook on Thermodynamics, for an engineering proof see [3]. Thus the ac loss around any ramping cycle is given by the area enclosed within the magnetization loop. Note that the magnetization includes all components, i.e., persistent currents in the filaments plus coupling between the filaments. For the coupling component of magnetization, which is independent of $B$, the ac loss when field changes between $B_1$ and $B_2$ is simply:

$$Q_c = M_c (B_2 - B_1)$$

For the loss component coming from persistent currents within the filaments, we need to know how the critical current $J_c$ depends on field. This dependency is a function of pinning strength and varies between materials, but for NbTi at fields below ~ 6T, a reasonable fit is given by the Kim Anderson formula.

$$J_c(B) = \frac{J_o B_o}{B + B_o}$$

Thus we may integrate (3) to get the loss between $B_1$ and $B_2$:

$$Q_h = \frac{4a}{3\pi} \int_{B_1}^{B_2} J_c B_o \, dB = \frac{4a}{3\pi} J_o B_o \ln \left( \frac{B_2 + B_o}{B_1 + B_o} \right)$$

AC losses are not much of a problem in slow ramping storage rings like LHC, but for a fast cycling fixed target machine like the SIS recently proposed by GSI Darmstadt [4], they are a major design constraint.

6. SUMMARY

Persistent currents in superconductors cause problems of flux jumping and magnetization. We can describe them in terms of the critical state model, which also explains the hysteretic behaviour of magnetization. Magnetization in the superconductor is usually the greatest source of field error at injection in superconducting accelerators. Magnetization is proportional to filament size, so conductors for accelerators are made with very fine filaments - typically 5 - 10 $\mu$m diameter. Practical conductors are made in the form of filamentary composite wires with the superconductor embedded in a matrix of copper. In changing fields the filaments are coupled together through the copper, thereby losing the benefit of subdivision and introducing a second component of magnetization. Twisting the composite wire serves to reduce the coupling by forcing the coupling currents to reverse every twist pitch. Coupling currents produce a second component of magnetization, which adds to the field error in magnets. They behave like eddy currents with a time constant determined by the twist pitch, so shorter twist pitches mean smaller coupling currents.

In changing fields, e.g., a synchrotron, the superconductor is driven into a partially resistive state, which means that it produces ac loss and consequent heating. These losses can produce a significant
refrigeration load. Loss is proportional to the area enclosed by the magnetization loop, as plotted in the $BM$ plane. So to reduce losses, we must reduce both components of magnetization.

**REFERENCES**