ELECTROMAGNETIC DESIGN OF SUPERCONDUCTING ACCELERATOR MAGNETS

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Abstract
The design and optimization of the superconducting magnets for LHC is dominated by the requirement of an extremely uniform field, which is mainly defined by the layout of the coils. Even very small geometrical effects such as the trapezoidal shape of the cable and the grading of the current density in the cable due to varying cable compaction, as well as coil deformations due to collaring and cool down have to be considered in the field calculation. In particular for the 3D case, commercial software has proven inadequate to this task. The CERN field computation program ROXIE was therefore developed for the design optimization of the LHC magnets and includes the method of coupled boundary elements and finite-elements. As with this method the coils do not need to be represented in the finite element mesh, they can be modeled with the required accuracy. This report describes the mathematical foundations of superconducting accelerator magnet design and the fundamentals of numerical field computation methods implemented in the ROXIE code.

1. Introduction
The Large Hadron Collider (LHC) requires high-field superconducting magnets to guide the counter-rotating beams with the desired proton energy of 7 TeV in the existing LEP tunnel with a circumference of 26.6 km. The LHC magnet system consists of 1232 superconducting dipoles and 386 main quadrupoles together with about 20 different types of magnets for insertion and correction. The only way of obtaining a nominal dipole field of 8.3 T with niobium-titanium (NbTi) superconductors is to cool the magnets to 1.9 K where helium takes on the so-called super-fluid state, with zero viscosity and very large heat conductivity. This aids in the cooling of the superconducting wires while reducing the helium flow through the magnets. However, the heat capacity of the superconducting cables is reduced by nearly an order of magnitude (compared to 4.5 K), resulting in a higher temperature rise for a given deposit of energy. Therefore, any movement of the coil must be avoided by the use of an appropriate force-retaining structure, in particular as the forces and the stored energy in the magnets increase with the square of the magnetic field. Fig. 1 shows the critical current surface of the NbTi alloy as a function of current density and field with the load-line of the LHC dipoles. The maximum current is limited by the critical current density in the coils which are exposed to an about 3% higher field than the nominal field in the aperture of the magnet. Important for the magnet performance is the engineering current density $J_E$ which also takes into consideration the copper matrix, the filling factor in the cable, and the insulation.

Although a niobium-tin (Nb$_3$Sn) alloy allows, at 8 T, approximately twice the current density of NbTi, it was not considered for the LHC, as a series production of magnets would have to confront the brittle nature of the material, which requires a wind-and-react technique, whereby the conductor containing unreacted niobium and tin is first wound and then heat-treated at a reaction temperature of about 700°C to form the superconducting A15 phase of Nb$_3$Sn.

The coils of the LHC dipole and quadrupole magnets are wound of Rutherford-type cable of trapezoidal (keystoned) shape. The dipole coils consist of two layers with cables of the same height but of different width. Electrically the two layers are connected in series. The current density in the superconductor of the outer layer, being exposed to a lower magnetic field, is about 40% higher than in
the inner layer. The conductor for the inner layer consists of 28 strands of 1.065 mm in diameter, the one for the outer layer comprising 36 strands of 0.825 mm in diameter. The outer layer conductor of the dipole is used in both layers of the main quadrupoles.

The strands are made of thousands of filaments of NbTi material (6 and 7 μm in diameter) embedded in a copper matrix, which serves to stabilize the conductor and to take over the current in case of a quench, since superconductors have a high resistivity in the normal state. The filaments are made as small as possible in order to reduce the remanent magnetization effects and increase the stability against flux jumps during excitation, i.e., the release of fluxoids from their pinning centers. The keystoning of the cable is not sufficient to allow the cables to build up arc segments. Copper wedges are therefore inserted between the blocks of conductors. The size and shape of these wedges yield the necessary degree of freedom for optimizing the field quality produced by the coil. Since the field quality is extremely sensitive to coil positioning errors, each layer of the coil is polymerized in a mould at a temperature of 180 °C for 30 minutes in order to glue the turns firmly together and give the coil its final shape. The size and elastic modulus of each layer is measured to determine pole and coil-head shimming for the collaring. The required shim thickness is calculated such that the compression under the collaring press is about 120 MPa. After the collaring rods are inserted and external pressure is released, the residual coil pre-stress is about 50-60 MPa on both layers. The collars (made of stainless steel) are surrounded by an iron yoke which enhances the magnetic field by about 20%, reduces the stored energy and shields the fringe field. The dipole magnet, its connections, and the bus-bars are enclosed in the stainless steel shrinking cylinder closed at its ends and form the dipole cold-mass, a containment filled with static, pressurized (1 bar) superfluid helium at 1.9 K. The cold-mass, weighing about 24 tons, is assembled inside its cryostat, which comprises a support system, cryogenic piping, radiation insulation, and thermal shield, all contained within a vacuum vessel. The cross-section of the dipole magnet cold-mass (version of spring 2000) in its cryostat is shown in fig. 2.

1.1 Field quality in accelerator magnets

The magnetic field errors in the aperture of accelerator magnets can be expressed as the coefficients of the Fourier-series expansion of the radial field component at a given reference radius (in the 2-dimensional case). In the 3-dimensional case, the transverse field components are integrated over the entire length of the magnet. For beam tracking it is sufficient to consider the transverse field components, since the effect of the longitudinal component of the field (present only in the magnet ends) on the particle motion can be

Neglected. Assuming that the radial component of the magnetic flux density $B_r$ at a given reference radius $r = r_0$ inside the aperture of a magnet is measured or calculated as a function of the angular position $\varphi$, we get for the Fourier-series expansion of the field $B_r(r_0, \varphi) = \sum_{n=1}^{\infty} \left( B_n(r_0) \sin n\varphi + A_n(r_0) \cos n\varphi \right)$, with $A_n(r_0) = \frac{1}{\pi} \int_{-\pi}^{\pi} B_r(r_0, \varphi) \cos n\varphi d\varphi$ and $B_n(r_0) = \frac{1}{\pi} \int_{-\pi}^{\pi} B_r(r_0, \varphi) \sin n\varphi d\varphi$ for $n = 1, 2, 3, ...$

If the field components are related to the main field component $B_N$ we get for $N=1$ (dipole field), $N=2$ (quadrupole), etc.:

$$B_r(r_0, \varphi) = B_N(r_0) \sum_{n=1}^{\infty} \left( b_n(r_0) \sin n\varphi + a_n(r_0) \cos n\varphi \right).$$  \hspace{1cm} (1)

The $B_n$ are called the normal and the $A_n$ the skew components of the field given in Tesla, $b_n$ the normal relative, and $a_n$ the skew relative field components. The latter two are dimensionless and are usually given in units of $10^{-4}$ at a 17 mm reference radius. In practice, the $B_r$ components are calculated at discrete points $\varphi_k = \frac{k\pi}{P} - \pi, k = 0, 1, 2, \ldots, 2P-1$ in the interval $[-\pi, \pi]$ and a discrete Fourier transform is carried out:

$$A_n(r_0) \approx \frac{1}{P} \sum_{k=0}^{2P-1} B_r(r_0, \varphi_k) \cos n\varphi_k, \hspace{1cm} B_n(r_0) \approx \frac{1}{P} \sum_{k=0}^{2P-1} B_r(r_0, \varphi_k) \sin n\varphi_k.$$  \hspace{1cm} (2)
The description of the field quality by means of multipoles, Eq. (1), is perfectly in line with magnetic measurements using so-called harmonic coils, where the periodic variation of the flux linkage in radial or tangential rotating coils is analyzed with a Fast-Fourier Transformation (FFT).

The field of a real superconducting magnet deviates from the ideal shape. The three main error sources (geometrical effect, superconductor magnetization and ramp induced effects) can be associated with three types of errors: Systematic errors (average error over the whole LHC ring and in one single aperture) uncertainty (deviations of the systematic error per dipole magnet production line) and random effects (tolerances).

Systematic errors can be classified as follows:

- Errors caused by the shape of the coil winding that can only approximate the ideal \(\cos \Theta\) current distribution (section 2.73).
- Remanent fields caused by the so-called persistent currents, induced in the superconducting filaments during the ramp of the magnets to their nominal field value.
- Eddy currents in the multi-strand conductors (interstrand coupling currents).
- Errors from cross-talk in the asymmetric two-in-one magnet design with its common iron yoke and asymmetric iron saturation effects.
- Cool-down of the structure and resulting deformations of the nominal coil geometry.
- Effects from beam-screen, vacuum channel, cryostat, and fringe fields in the coil-end regions, including the effect of bus-bars and interconnections.
- Coil deformations under electromagnetic forces.

Uncertainty errors include:

- Systematic perturbations arising from manufacturing tooling.
- Variations of the properties of the superconducting cable due to different manufacturing procedures.
- Varying properties of steel in yoke and collar laminations depending on the batch.
- Different assembly procedures at the cold-mass manufacturers.
- Torsion and sagitta of the magnet cold-mass.

The random effects mainly arise from:

- Conductor placement errors due to tolerances on coil parts, e.g. insulation thickness, cable keystoning and size of copper wedges.
- Tolerances on yoke parts, e.g. collar outer shape and yoke laminations.
- Manufacturing tolerances and displacements of coil-blocks due to varying elastic modulus of the coil, coil winding procedure, curing, collaring, yoking, etc.
- Alignment tolerances of the magnet system.

1.2 The ROXIE program

The design and optimization of the LHC magnets is dominated by the requirement of an extremely uniform field (no skew field components \(a_n\), no higher-order normal field components \(b_n, n = 2, 3, \ldots\)), which is mainly defined by the layout of the superconducting coils. For the field calculation it is necessary to consider even very small geometrical effects, such as those produced by insufficient keystoning of the cable, insulation, coil deformations (due to collaring, cool down, and electro-magnetic forces) and variation of the current density in the cable caused by different cable compaction. If the coils had to be modeled in the finite-element mesh, as is the case in most commercial field computation software, it would be difficult to define the current density as this would require a further subdivision of the conductors into a number of radial layers.

For the 3D case in particular, commercial software has proven hardly appropriate for the field optimization of the LHC magnets. The ROXIE (Routine for the Optimization of magnet X-subsections,
Inverse field calculation and coil End design) program package was therefore developed at CERN for the design and optimization of the LHC superconducting magnets. The development of the program was driven by the following main objectives:

- To write an easy-to-use program for the design of superconducting coils in two and three dimensions considering field quality, quench margin, and persistent current multipoles.
- To provide for accurate field calculation routines that are specially suited for the investigation of superconducting magnets.
- To integrate the program into a mathematical optimization environment for field optimization and inverse problem solving.
- To integrate the program into the engineering design procedure through interfaces to Virtual Reality, to CAD/CAM systems (for the making of drawings and manufacturing of coil-end spacers), and through interfaces to commercial structural analysis programs.

The modeling capabilities of the ROXIE program, together with its interfaces to CAD/CAM and its mathematical optimization routines, have inverted the classical design process wherein numerical field calculation is performed for only a limited number of numerical models that only approximate the actual engineering design. ROXIE is now used as an approach towards an integrated design of superconducting magnets. The steps of the integrated design process are as follows:

- Feature-based geometry modeling of the coil and yoke, both in two and three dimensions using only a number of meaningful input data to be supplied by the design engineer. This is a prerequisite for addressing these data as design variables of the optimization problem.
- Conceptual coil design using a genetic algorithm, which allows the treatment of combined discrete and continuous problems (e.g. the change of the number of conductors per block). The applied niching method provides the designer with a number of local optima which can then be studied in detail.
- Subject to a varying magnetic field, currents that screen the interior of the superconducting filaments are generated in their outer region. The relative field errors caused by these currents are highest at injection field level and have to be calculated to allow a subsequent part-compensation by geometrical field errors. Deterministic search algorithms are used for the final optimization of the coil cross-section.
- Minimization of iron-induced multipoles using a two-dimensional version of the coupling method between boundary and finite elements (BEM-FEM) developed by ITE Stuttgart and R. Bosch GmbH, Germany.
- Calculation of the peak voltage and peak temperature during a quench.
- Sensitivity analysis of the optimal design through Lagrange-multiplier estimation and the set-up of payoff tables. This provides an evaluation of the hidden resources of the design.
- Tolerance analysis by calculating Jacobian-Matrices and estimation of the standard deviation of the multipole field errors.
- 3D coil-end geometry and field optimization including the modeling and optimization of the asymmetric connection side, ramp and splice region and external connections.
- 3D field calculation of the saturated iron yoke with 3D BEM-FEM computations.
- Production of drawings by means of a DXF interface for both the cross-sections and the 3D coil-end regions.
- End-spacer design and manufacture by means of interfaces to CAD/CAM (DXF, VDA), rapid prototyping methods (laser sinter techniques), and computer controlled 5-axis milling machines.
- Tracing of manufacturing errors from measured field imperfections, i.e., the minimization of a least-squares error function using the Levenberg-Marquard optimization algorithm.

The design process is described in the example of the LHC main quadrupole in [37]. In this report we
will focus on the foundations of the analytical and numerical field computation required for the design process.

2. Analytical Field Computation

The field in superconducting magnets is dominated by the current distribution within the coils. In particular for the higher order field components, the iron yoke plays the role of a mere shielding device. The design process can therefore be split in two parts: The calculation of the coil field using (semi) analytical methods and a subsequent optimization of the iron yoke.

The field in the magnet aperture is governed by the Laplace equation. As a consequence, knowing the normal and skew field components on a given reference radius as in Eq. (1) allows us to re-constitute the entire field in the aperture. The scaling laws so obtained are all we need for the post-processing of data from harmonic coil measurements.

The field harmonics on some reference radius in the aperture of the magnet can be calculated from the current distribution within each strand of the coil by means of Biot-Savart’s law and Taylor series expansion. The result combined with the imaging method for line currents in a cylindrical iron yoke reveals the influence of the yoke on the multipoles of different order, the sensitivity to manufacturing errors and symmetry conditions in the magnet. For practical magnet design, however, the analytical results for all strand currents (up to a few thousands) have to be summed-up, hence a semi-analytical method.

2.1 The field equations

For magnetostatic problems \( \frac{\partial}{\partial t} = 0 \), Maxwell’s equations reduce to

\[
\text{curl} \, \vec{H} = \vec{J}, \quad \text{div} \, \vec{B} = 0.
\]

which can be solved for a given current distribution if we consider the material relations (for non-moving, isotropic materials) which are also called the constitutive equations: \( \vec{B} = \mu_0(\vec{H} + \vec{M}) \) in case there is a remanent magnetization (permanent magnet material) to be considered. The permeability of free space has the value \( \mu_0 = 4\pi \cdot 10^{-7} \text{ H/m} \).

2.2 The boundary and interface conditions

Subsequently, the closed domain (either 2D or 3D) in which the electro-magnetic field is to be calculated will be denoted as \( \Omega \). The field quantities \( \vec{B} \) and \( \vec{H} \) satisfy boundary conditions on the boundary \( \Gamma \) of the domain \( \Omega \). Two types of boundary conditions (prescribed on the two disjunct parts denoted \( \Gamma_H \) and \( \Gamma_B \) with \( \Gamma = \Gamma_H \cup \Gamma_B \)) cover all practical cases:

- On the part \( \Gamma_B \) of the boundary the **normal** component of the magnetic flux density is given. On symmetry planes parallel to the field, on far boundaries or on outer boundaries of iron yokes surrounded by air (where it can be assumed that no flux leaves the outer boundary) the normal component of the flux density (denoted \( \vec{B}_n \)) is zero:

  \[
  \vec{B}_n = \vec{B} \cdot \vec{n} = 0 \quad \text{on} \, \Gamma_B.
  \]

- On the part \( \Gamma_H \) of the boundary the **tangential** component of the magnetic field is given. In many cases (as on symmetry planes perpendicular to the field) and on (infinitely) permeable iron poles (where the field enters at right angle) the tangential component of the field (denoted \( \vec{H}_t \)) is zero:

  \[
  \vec{H}_t = 0 \quad \rightarrow \quad \vec{H} \times \vec{n} = 0 \quad \text{on} \, \Gamma_H.
  \]
Consider now the interface between two permeable domains as displayed in fig. 3. If we apply Ampère’s law $\oint \mathbf{H} \cdot d\mathbf{s} = \int_A \mathbf{J} \cdot d\mathbf{A}$, to the loop displayed in fig. 3 (left), and let $h \to 0$, then the enclosed current is zero, as in an infinitesimal small rectangle there cannot be a current flow. Therefore the tangential field components are equal, $H_{t1} = H_{t2}$ which is equivalent to writing $\mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = 0$. Because of $\oint \mathbf{B} \cdot d\mathbf{A} = 0$ we get at the interface the continuity of the normal flux density $B_{n1} = B_{n2}$ which can also be written as $\mathbf{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$. Now from basic geometry (see fig. 3 right):

$$\tan \alpha_1 / \tan \alpha_2 = \frac{B_{n1}}{B_{n2}} = \mu_1 H_{t1} / \mu_2 H_{t2} = \frac{\mu_1}{\mu_2}.$$  \hspace*{1cm} (6)

For $\mu_2 \gg \mu_1$ it follows that $\tan \alpha_2 \gg \tan \alpha_1$. Therefore for all angles $\pi/2 > \alpha_2 > 0$ we get $\tan \alpha_1 \approx 0$, i.e., the magnetic field exits vertically from a highly permeable medium into a medium with low permeability.

### 2.3 The Lemmata of Poincaré

From vector-analysis it is known that the curl of an arbitrary vector field is source free, i.e., $\text{div} \, \text{curl} \, \mathbf{g} = 0$ and that an arbitrary gradient field is curl free, i.e., $\text{curl} \, \text{grad} \, \phi = 0$. Reversal of these statements yields the Lemmata of Poincaré:

A source free field $\mathbf{b}$ can be expressed through a vector potential $\mathbf{a}$.

$$\text{div} \, \mathbf{b} = 0 \quad \Rightarrow \quad \mathbf{b} = \text{curl} \, \mathbf{a}.$$  \hspace*{1cm} (7)

A curl free field $\mathbf{h}$ can be expressed through a scalar potential $\phi$.

$$\text{curl} \, \mathbf{h} = 0 \quad \Rightarrow \quad \mathbf{h} = \text{grad} \, \phi.$$  \hspace*{1cm} (8)

It is necessary that the domain (usually denoted $\Omega$) is topologically not too difficult (precisely, the domain must be star shaped).

### 2.4 Magnetic potentials

In the aperture of an accelerator magnet (current free region) both the magnetic scalar-potential as well as the vector-potential can be used to solve the field problem as the field is divergence and curl free. For two-dimensional field problems it follows:

$$\mathbf{H} = - \text{grad} \, \Phi = - \frac{\partial \Phi}{\partial x} \mathbf{e}_x - \frac{\partial \Phi}{\partial y} \mathbf{e}_y,$$  \hspace*{1cm} (9)

$$\mathbf{B} = \text{curl} \, (A_z \mathbf{e}_z) = \frac{\partial A_z}{\partial y} \mathbf{e}_x - \frac{\partial A_z}{\partial x} \mathbf{e}_y.$$  \hspace*{1cm} (10)

Fig. 3: Interface conditions for permeable media.
As will be shown below, both formulations lead to a scalar Laplace equation. The field governed by the Laplace equation is called *harmonic* and can be expressed by the fundamental solutions of the Laplace equation. Lines of constant vector-potential give the direction of the magnetic field and lines of constant scalar potential define the ideal pole shapes of conventional magnets.

### 2.41 Reduced magnetic scalar potential

Every vector field can be split into a source free and a curl free part. In case of the magnetic field with \( \vec{H} = \vec{H}_s + \vec{H}_m \) the curl free part \( \vec{H}_m \) arises from the induced magnetism in ferromagnetic materials and the source free part \( \vec{H}_s \) is the field generated by the prescribed sources and can be calculated directly by means of Biot Savart’s law. With \( \text{curl} \vec{H}_m = 0 \), it follows that \( \vec{H} = -\text{grad} \Phi_m + \vec{H}_s \), were \( \Phi_m \) is called the *reduced* magnetic scalar potential. We get:

\[
\begin{align*}
\text{div} \vec{B} &= 0, \\
\text{div} \mu (-\text{grad} \Phi_m + \vec{H}_s) &= 0, \\
\text{div} \mu \text{grad} \Phi_m &= \text{div} \mu \vec{H}_s.
\end{align*}
\]

The boundary conditions read:

\[
\begin{align*}
\text{grad} \Phi_m \times \vec{n} &= \vec{H}_s \times \vec{n} & \text{on } \Gamma_H, \\
\text{grad} \Phi_m \cdot \vec{n} = \frac{\partial \Phi_m}{\partial n} &= \vec{H}_s \cdot \vec{n} & \text{on } \Gamma_B.
\end{align*}
\]

While a solution of the above boundary value problem is possible, the two parts of the magnetic field \( \vec{H}_m \) and \( \vec{H}_s \) tend to be of similar magnitude (but opposite direction) in non-saturated magnetic materials, so that cancellation errors occur in the computation.

### 2.42 Total magnetic scalar potential

For regions where the current density is zero, \( \text{curl} \vec{H} = 0 \) and the field can be represented by a *total* scalar potential \( \vec{H} = -\text{grad} \Phi_m \). It follows for regions free of magnetic material and \( \mu = \mu_0 \):

\[
\begin{align*}
\mu_0 \text{div} \text{grad} \Phi_m &= 0, \\
\nabla^2 \Phi_m &= 0,
\end{align*}
\]

which is the Laplace equation for the scalar potential. The boundary conditions reduce to:

\[
\begin{align*}
\vec{H} \times \vec{n} &= \text{grad} \Phi_m \times \vec{n} = 0 & \text{on } \Gamma_H, \\
\frac{1}{\mu_0} \vec{B}_n &= \text{grad} \Phi_m \cdot \vec{n} = \frac{\partial \Phi_m}{\partial n} = 0 & \text{on } \Gamma_B.
\end{align*}
\]

Eq. (18), from which follows that \( \Phi_m \) is constant on the boundary, is called the homogeneous *Dirichlet* boundary condition on \( \Gamma_H \) where \( \vec{H} \) is *normal* to the boundary. Eq. (19) that specifies the normal derivative of the system variable is called the homogeneous *Neumann* boundary condition where \( \vec{B} \) is *parallel* to the boundary \( \Gamma_B \).

### 2.43 Vector-potential formulation with the magnetization seen as an effective current density

With the vector-potential formulation \( \vec{B} = \text{curl} \vec{A} \) we get

\[
\begin{align*}
\text{curl} \vec{A} &= \mu_0 (\vec{H} + \vec{M}), \\
\vec{H} &= \frac{1}{\mu_0} \text{curl} \vec{A} - \vec{M}, \\
\frac{1}{\mu_0} \text{curl} \text{curl} \vec{A} &= \vec{J} + \text{curl} \vec{M}, \\
\frac{1}{\mu_0} (-\nabla^2 \vec{A} + \text{grad} \text{div} \vec{A}) &= \vec{J} + \text{curl} \vec{M}.
\end{align*}
\]
The boundary conditions read:
\[ \vec{H} \times \vec{n} = \frac{1}{\mu} (\text{curl} \, \vec{A}) \times \vec{n} = 0 \quad \text{on} \quad \Gamma_H, \]  
\[ \vec{B}_n = \vec{B} \cdot \vec{n} = \text{curl} \, \vec{A} \cdot \vec{n} = 0 \quad \text{on} \quad \Gamma_B. \]  

The condition (25) is equivalent to \( \vec{A}_i = 0 \), i.e., \( \vec{A} \times \vec{n} = 0 \) on \( \Gamma_B \). This is the homogeneous Neumann boundary condition on \( \Gamma_B \). Surface current densities do not appear as long as we have finite conductivity and continuous time dependency. Since the curl (rotation) of a gradient field is zero, the vector-potential is not unique. The gradient of any (differentiable) scalar field \( \psi \) can be added without changing the curl of \( \vec{A} \):
\[ \vec{A}_0 = \vec{A} + \text{grad} \, \psi. \]  

Eq. (26) is called a gauge-transformation between \( \vec{A}_0 \) and \( \vec{A} \). \( \vec{B} \) is gauge-invariant as the transformation from \( \vec{A} \) to \( \vec{A}_0 \) does not change \( \vec{B} \). The freedom given by the gauge-transformation can be used to set the divergence of \( \vec{A} \) to zero \( \text{div} \, \vec{A} = 0 \) which (together with additional boundary conditions) makes the vector-potential unique and is is called the Coulomb gauge, as it leads to a Poisson type equation for the magnetic vector-potential. From Eq. (23) we get after incorporating the Coulomb gauge:
\[ \nabla^2 \vec{A} = -\mu_0 (\vec{J} + \text{curl} \, \vec{M}) \]  


### 2.44 Vector-potential formulation for field dependent permeability

An equivalent (but different) formulation for the total vector-potential can be obtained from
\[ \text{curl} \, \frac{1}{\mu} \text{curl} \, \vec{A} = \vec{J}. \]  

where the iron magnetization is taken into account in a multiplicative way through the permeability \( \mu \) which now depends nonlinearly on the magnetic field and hence \( \vec{B} = \mu(\vec{H})\vec{H} \). Introducing a penalty term [34] subtracted from Eq. (28), yields
\[ \text{curl} \, \frac{1}{\mu} \text{curl} \, \vec{A} - \text{grad} \, \frac{1}{\mu} \text{div} \, \vec{A} = \vec{J}. \]  


### 2.5 The Laplace equation

From vector-analysis we know that \( \nabla^2 \vec{A} \) reads in Cartesian coordinates:
\[ \nabla^2 \vec{A} = (\nabla^2 A_x) \vec{e}_x + (\nabla^2 A_y) \vec{e}_y + (\nabla^2 A_z) \vec{e}_z. \]  

The Laplace operator acting on a vector in Cartesian coordinates yields a vector that can also be obtained through the application of the operator on the components, i.e., \( \nabla^2 A_x = -\mu_0 (J_x + (\text{curl} \, \vec{M})_x) \) and so on, whereas this is not the case in cylindrical coordinates:
\[ \nabla^2 \vec{A} = (\nabla^2 A_r - \frac{1}{r^2} A_r - \frac{2}{r^2} \frac{\partial A_r}{\partial r}) \vec{e}_r + (\nabla^2 A_\varphi - \frac{1}{r^2} A_\varphi + \frac{2}{r^2} \frac{\partial A_\varphi}{\partial \varphi}) \vec{e}_\varphi + \nabla^2 A_z \vec{e}_z. \]  

In the two-dimensional case with no dependence on \( z \), \( \frac{\partial}{\partial z} = 0 \) and \( \vec{J} = J_x \vec{e}_x \), \( \vec{A} \) has only a \( z \)-component and the Coulomb gauge is automatically fulfilled. Then we get the scalar Poisson differential equation from Eq. (27) or Eq. (29):
\[ \nabla^2 A_z = -\mu_0 J_z. \]  

For current-free regions Eq. (32) reduces to the Laplace equation which reads in cylindrical coordinates
\[ \nabla^2 A_z = r^2 \frac{\partial^2 A_z}{\partial r^2} + r \frac{\partial A_z}{\partial r} + \frac{\partial^2 A_z}{\partial \varphi^2} = 0. \]
2.6 Harmonic fields

A solution of the homogeneous differential equation (33) can be derived with the method of separation and reads: \( A_z(r, \varphi) = \sum_{n=1}^{\infty} (E_n r^n + F_n r^{-n}) (G_n \sin n\varphi + H_n \cos n\varphi) \). These fields are called harmonic. Considering that the field is finite at \( r = 0 \), the \( F_n \) have to be zero for the vector-potential inside the aperture of the magnet while for the solution in the area outside the coil all \( E_n \) vanish. Rearranging Eq. (2.6) yields the vector-potential in the aperture:

\[
A_z(r, \varphi) = \sum_{n=1}^{\infty} r^n (C_n \sin n\varphi - D_n \cos n\varphi),
\]

and the field components can be expressed as

\[
B_r(r, \varphi) = \frac{1}{r} \frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} nr^{n-1} (C_n \cos n\varphi + D_n \sin n\varphi),
\]

\[
B_\varphi(r, \varphi) = -\frac{\partial A_z}{\partial r} = -\sum_{n=1}^{\infty} nr^{n-1} (C_n \sin n\varphi - D_n \cos n\varphi).
\]

The solution in Cartesian coordinates can be obtained from the simple transformations \( B_x = B_r \cos \varphi - B_\varphi \sin \varphi, B_y = B_r \sin \varphi + B_\varphi \cos \varphi \). The coefficients are not known at this stage. They are defined through the (given) boundary conditions on some reference radius or can be calculated from the Fourier series expansion of the numerically calculated (or measured) field in the aperture using the relations

\[
A_n = nr_0^{n-1}C_n \quad \text{and} \quad B_n = nr_0^{n-1}D_n.
\]

We finally get for the field components at a given reference radius \( r_0 \):

\[
B_r(r_0, \varphi) = \sum_{n=1}^{\infty} (B_n \sin n\varphi + A_n \cos n\varphi) = B_N \sum_{n=1}^{\infty} (b_n \sin n\varphi + a_n \cos n\varphi),
\]

\[
B_\varphi(r_0, \varphi) = \sum_{n=1}^{\infty} (B_n \cos n\varphi - A_n \sin n\varphi) = B_N \sum_{n=1}^{\infty} (b_n \cos n\varphi - a_n \sin n\varphi).
\]

The small \( b_n, a_n \) are the multipoles related to the main field \( B_N \) which is \( B_1 \) for the dipole, \( B_2 \) for the quadrupole, etc. \( B_n \) are given in Tesla and \( a_n \) are dimensionless and usually given in units of \( 10^{-4} \) at a reference radius of 17 mm. In some documents, e.g., [41], the field strength of the LHC magnets is defined as

\[
B_n = \frac{B_n}{r_0^{n-1}}
\]

and \( B_n \) are given in \( T, T/m, T/m^2 \) etc. For the scaling of different reference radii we get

\[
A_n(r_1) = \left( \frac{r_1}{r_0} \right)^{n-1} A_n(r_0), \quad B_n(r_1) = \left( \frac{r_1}{r_0} \right)^{n-1} B_n(r_0),
\]

\[
a_n(r_1) = \left( \frac{r_1}{r_0} \right)^{n-N} a_n(r_0), \quad b_n(r_1) = \left( \frac{r_1}{r_0} \right)^{n-N} b_n(r_0).
\]

2.61 Dipole, quadrupole and sextupole flux density distributions

Each value of the integer \( n \) in the solution, Eq. (38), of the Laplace equation corresponds to a different flux distribution generated by different magnet geometries. The three lowest values, \( n=1,2, \) and 3 correspond to a dipole, quadrupole and sextupole flux density distribution. For the dipole field (\( n=1 \)) we
Fig. 4: Field and force distribution inside the aperture of an ideal quadrupole. Left: Magnetic induction of the normal quadrupole ($B_y = gx$, gradient $g$ negative, $B_x = gy$). Middle: x-component of the electromagnetic force field $(\vec{u} \times \vec{B})_x$ on a proton beam parallel to the z-axis into positive z-direction. While this quadrupole is defocusing in the y-plane, it is focusing in the x-plane, see figure on the right where $(\vec{u} \times \vec{B})_y$ is displayed.

get

$$B_r = C_1 \cos \varphi + D_1 \sin \varphi, \quad (43)$$

$$B_\varphi = -C_1 \sin \varphi + D_1 \cos \varphi, \quad (44)$$

$$B_x = C_1, \quad (45)$$

$$B_y = D_1. \quad (46)$$

This is a simple, constant field distribution according to the values of $C_1$ and $D_1$.

**Remark:** Beware that we have not yet studied the conditions necessary to obtain such a field distribution. We will address this issue in the next chapter.

For the pure quadrupole, n=2 (represented in fig. 4) we get from Eq. (35) and (36):

$$B_r = 2 r C_2 \cos 2\varphi + 2 r D_2 \sin 2\varphi, \quad (47)$$

$$B_\varphi = -2 r C_2 \sin 2\varphi + 2 r D_2 \cos 2\varphi, \quad (48)$$

$$B_x = 2(C_2 x + D_2 y), \quad (49)$$

$$B_y = 2(-C_2 y + D_2 x). \quad (50)$$

The amplitudes of the horizontal and vertical components vary linearly with the displacements from the origin, i.e., the gradient is constant. With a zero induction in the origin, the distribution provides linear focusing of the particles. The components of the magnetic fields are coupled, i.e., the distribution in both planes cannot be made independent of each other. Consequently a quadrupole focusing in one plane will defocus in the other. For a normal quadrupole with $C_2 = 0$ we can calculate the field at some position $x' = x + \Delta x$ which yields

$$B_y = 2D_2 x + 2D_2 \Delta x. \quad (51)$$

For a displaced quadrupole, the field contains a constant term, i.e., the second term in Eq. (51). This effect is called feed-down. For the case of the pure sextupole (n=3) we get:

$$B_r = 3 r^2 C_3 \cos 3\varphi + 3 r^2 D_3 \sin 3\varphi, \quad (52)$$

$$B_\varphi = -3 r^2 C_3 \sin 3\varphi + 3 r^2 D_3 \cos 3\varphi, \quad (53)$$

$$B_x = 3C_3(x^2 - y^2) + 6D_3 xy, \quad (54)$$

$$B_y = -6C_3 xy + 3D_3(x^2 - y^2), \quad (55)$$
which is represented in fig. 5. Displacement of a sextupole magnet creates both a quadrupole and a dipole field component

\[ B_y = 3D_3(x^2 - y^2) = 3D_3(x^2 - y^2) + 6D_3x\Delta x + D_3\Delta x^2. \]  

(56)

**Remark:** The treatment of each harmonic separately is a mathematical abstraction. In practical situations many harmonics will be present, i.e., many of the coefficients \(C_n\) and \(D_n\) will be non-vanishing. A successful magnet design will, however, minimize the unwanted terms.

### 2.7 Coil field of superconducting magnets

The problem remains how to calculate the field harmonics from a given current distribution. It is reasonable to focus on the fields generated by line-currents, since the field of any current distribution over an arbitrary cross-section can be approximated by summing the fields of a number of line-currents distributed within the cross-section. The harmonic field, however, cannot account for line currents. We need a special solution of the inhomogeneous (Poisson) differential equation.

#### 2.71 The field of a line current

In two dimensions the particular solution of the Poisson equation (32) is

\[ A_z = \int_A -\frac{\mu_0 J_z}{2\pi} \ln \left(\frac{R}{a}\right) dA' = -\frac{\mu_0 I}{2\pi} \ln \left(\frac{R}{a}\right). \]  

(57)

With the source point \(\vec{r}' = (r_i, \Theta)\), the field point \(\vec{r} = (r_0, \varphi)\), \(|\vec{R}| = |\vec{r} - \vec{r}'|\), and an arbitrary reference radius \(a\), see fig. 6, the cosine law \(R^2 = r_0^2 + r_i^2 - 2r_0r_i\cos(\varphi - \Theta)\) can be rewritten as [28]

\[ R^2 = r_i^2 \left(1 - \frac{r_0}{r_i} e^{i(\varphi - \Theta)}\right) \left(1 - \frac{r_0}{r_i} e^{-i(\varphi - \Theta)}\right). \]

Therefore

\[ \ln \left(\frac{R}{a}\right) = \ln \left(\frac{r_i}{a}\right) + \frac{1}{2} \ln \left(1 - \frac{r_0}{r_i} e^{i(\varphi - \Theta)}\right) + \frac{1}{2} \ln \left(1 - \frac{r_0}{r_i} e^{-i(\varphi - \Theta)}\right). \]  

(58)

With the Taylor series expansion of \(\ln(1 - x)\) which gives for \(|x| < 1\), i.e., for \(r_0 < r_i\) inside the aperture of the magnet, \(\ln(1 - x) = -\sum_{n=1}^{\infty} \frac{1}{n} x^n\), Eq. (57) can be rewritten as

\[ A_z = -\frac{\mu_0 I}{2\pi} \ln \left(\frac{r_i}{a}\right) + \frac{\mu_0 I}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r_0}{r_i}\right)^n \cos(n(\varphi - \Theta)). \]  

(59)

---

Fig. 5: Field and force distribution inside the aperture of an ideal sextupole. Left: Magnetic induction of the normal sextupole \((B_y = g_s(x^2 - y^2)\), gradient \(g_s\) negative, \(B_x = 2g_sxy\)). Middle: \(x\)-component of the electromagnetic force field \(\vec{v} \times \vec{B}\), acting on a beam parallel to the \(z\)-axis into positive \(z\)-direction. Right: \(y\)-component of the electromagnetic force field \(\vec{v} \times \vec{B}\).
The radial component of the magnetic field in the point \((r_0, \varphi)\) is then

\[
B_r = \frac{1}{r} \frac{\partial A_z}{\partial \varphi} = -\frac{\mu_0 I}{2\pi} \sum_{n=1}^{\infty} \left( \frac{r_0^n}{r_i^n} \right) \sin \left( n(\varphi - \Theta) \right)
\]

\[
= -\frac{\mu_0 I}{2\pi} \sum_{n=1}^{\infty} \left( \frac{r_0^n}{r_i^n} \right) \left( \sin n\varphi \cos n\Theta - \cos n\varphi \sin n\Theta \right).
\]

Comparison of the coefficients with equation (38) yields (for \(r_0 < r_i\)):

\[
B_n(r_0) = -\frac{\mu_0 I}{2\pi} \frac{r_0^{n-1}}{r_i^n} \cos n\Theta, \quad A_n(r_0) = \frac{\mu_0 I}{2\pi} \frac{r_0^{n-1}}{r_i^n} \sin n\Theta.
\]

2.72 The imaging method for line currents in a cylinder with constant permeability

The effect of an iron yoke with constant permeability and perfect circular inner shape with radius \(R_{\text{yoke}}\) can be taken into account by means of image currents of the strength and radial position

\[
I' = \frac{\mu_r - 1}{\mu_r + 1} I, \quad r_i' = \frac{R_{\text{yoke}}^2}{r_i},
\]

which are located at the same angular position. Fig. 7 (left) shows the field distribution for a superconducting coil in a (non-saturated) iron yoke of cylindrical inner shape and the representation of the iron yoke with image current (right).

**Remark:** Notice that only the field in the aperture of the magnet can be calculated with the imaging method and that the total current inside the yoke has to be zero. Furthermore the yoke has to be of constant permeability and thus local saturation effects cannot be considered.

Including the effect of the imaging currents, the normal multipole coefficients inside the aperture of the magnet reads for one line current:

\[
B_n(r_0) = -\frac{\mu_0 I}{2\pi} \frac{r_0^{n-1}}{r_i^n} \left( 1 + \frac{\mu_r - 1}{\mu_r + 1} \left( \frac{r_i}{R_{\text{yoke}}} \right)^2 n^2 \right) \cos n\Theta.
\]

This is an important results as it allows to calculate the harmonic content of a field generated by a number of arbitrarily placed current carrying conductors by adding the terms in Eq. (64).
Consider a current shell \( r_i < r < r_e \) with a current density varying with the azimuthal angle \( \Theta \), \( J(\Theta) = J_0 \cos m\Theta \), then we get for the \( B_n \) components

\[
B_n(r_0) = \int_{r_i}^{r_e} \int_{0}^{2\pi} \frac{\mu_0 J_0}{2\pi r^n} \left( 1 + \frac{\mu_r - 1}{\mu_r + 1} \left( \frac{r}{R_{yoke}} \right)^{2n} \right) \cos m\Theta \cos n\Theta r d\Theta dr. \tag{65}
\]

With \( \int_0^{2\pi} \cos m\Theta \cos n\Theta d\Theta = \pi \delta_{m,n} (m, n \neq 0) \) it follows that the current shell produces a pure \( 2m \)-polar field and in the case of the dipole \( (m = n = 1) \) one gets

\[
B_1(r_0) = -\frac{\mu_0 J_0}{2} \left( r_e - r_i \right) + \frac{\mu_r - 1}{\mu_r + 1} \frac{1}{R_{yoke}} \frac{1}{3} \left( r_e^3 - r_i^3 \right). \tag{66}
\]

Obviously, since \( \int_0^{2\pi} \cos m\Theta \sin n\Theta d\Theta = 0 \), all \( A_n \) components vanish. A shell with \( \cos \Theta \) and \( \cos 2\Theta \) dependent current density is displayed in figure 8.

---

**Fig. 7:** Left: Field vectors for a superconducting coil in a (non-saturated) iron yoke of cylindrical inner shape \( (\mu_r=2000, \text{ constant}) \). Right: The representation of the iron yoke with image currents.

**Fig. 8:** Shells with \( \cos \Theta \) (left) and \( \cos 2\Theta \) (right) dependent current density.
2.74 Coil-Block arrangements

As the conductors are keystoned with an insufficient angle to allow for perfect \( \cos \Theta \) geometries and the conductors are connected in series, i.e., carry the same current, coil-blocks separated by copper wedges are designed in order to approximate the ideal current distribution. Fig. 9 shows the contribution of the strand currents in a superconducting dipole coil to the \( B_3 \) and the \( B_5 \) field component as a visualization of Eq. (64). The field errors scale with \( 1/r^n \) where \( n \) is the order of the multipole and \( r \) is the mid radius of the coil.

With equation (64), a semi-analytical method for calculating the fields in superconducting magnets is given. The iron yoke is represented by image currents (the second term in the parentheses). At low field level, when the saturation of the iron yoke is low, this is a sufficient method for optimizing the coil cross-section. Under that assumption some important conclusions can be drawn:

- For a coil without iron yoke the field errors scale with \( 1/r^n \) where \( n \) is the order of the multipole and \( r \) is the mid radius of the coil. It is clear, however, that an increase of coil aperture causes a linear drop in dipole field. Other limitations of the coil size are the beam distance, the electromagnetic forces, yoke size, and the stored energy which results in an increase of the hot-spot temperature during a quench.
- For certain symmetry conditions in the magnet, some of the multipole components vanish, i.e. for an up-down symmetry in a dipole magnet (positive current \( I_0 \) at \((r_0, 0)\) and at \((r_0, -\Theta_0)\)) no \( A_n \) terms occur. If there is an additional left-right symmetry, only the odd \( B_1, B_3, B_5, B_7, \ldots \) components remain.
- The relative contribution of the iron yoke to the total field (coil field plus iron magnetization) is for a non-saturated yoke \((\mu_r \gg 1)\) approximately \( (1 + (B_{\text{yoke}}/r)^{2n})^{-1} \). For the main dipoles with a mean coil radius of \( r = 43.5 \) mm and a yoke radius of \( R_{\text{yoke}} = 89 \) mm we get for the \( B_1 \) component a 19% contribution from the yoke, whereas for the \( B_5 \) component the influence of the yoke is only 0.07%.

It is therefore appropriate to optimize for higher harmonics first using analytical field calculation, and include the effect of iron saturation on the lower-order multipoles only at a later stage.

2.8 Complex Analysis Methods for Magnet Design

With the spread of numerical field computation software now running on desk- and laptop computers, complex analysis methods have become less important for magnet design. However, important applications abound: Complex representation of the field quality (often given as definition of the multipoles), scaling of field measurements and the compensation of the so-called feed down effect due to off-centering.
of the measurement coils (or miss-alignment of magnet chains), and the calculation of fields due to intersecting circles or ellipses as the ideal current distribution for dipole and quadrupole magnets and the basis for a macroscopic model of shielding currents in superconducting filaments.

2.81 Complex potentials
In current and magnetization free, two-dimensional regions both the magnetic scalar-potential as well as the vector-potential can be used to solve Maxwell’s equations, equations (9) and (10). Therefore

\[
\frac{\partial A_z}{\partial y} = -\mu_0 \frac{\partial \Phi}{\partial x} \quad \text{and} \quad \frac{\partial A_z}{\partial x} = \mu_0 \frac{\partial \Phi}{\partial y}
\]  

(67)

which are the Cauchy-Riemann differential equations of the complex potential

\[
W(z) = u(x, y) + iv(x, y) = A_z(x, y) + i\mu_0 \Phi(x, y),
\]  

(68)

which is an analytic function of \(z = x + iy\). Derivatives of \(W\) are also analytic functions of \(z\).

\[
-\frac{dW}{dz} = -\frac{\partial A_z}{\partial x} - i\mu_0 \frac{\partial \Phi}{\partial x} = B_y(x, y) + iB_x(x, y) = \tilde{B}
\]  

(69)

where the field components appear as real functions of \(x\) and \(y\) and the Cauchy-Riemann equations between them express Maxwell’s equations for the two-dimensional case with zero current density. From Eq. (67) follows directly that the Laplace equation holds for both \(A_z\) and \(\Phi\). The complex potential at a point \(z_0 = r_0 e^{i\phi}\) due to a line current at the position \(z = re^{i\theta}\) is given by

\[
W(z_0) = -\frac{\mu_0 I}{2\pi} \ln\left(\frac{z_0 - z}{z_a}\right)
\]  

(70)

where \(z_a\) is an arbitrary complex reference point and therefore

\[
-\frac{dW}{dz} = \tilde{B} = -\frac{\mu_0 I}{2\pi} \frac{1}{z - z_0} = -\frac{\mu_0 I}{2\pi} \frac{1}{(x - x_0) + i(y - y_0)},
\]  

(71)

which yields the well known equations for the magnetic field

\[
B_y = \frac{\mu_0 I}{2\pi} \frac{x - x_0}{(x - x_0)^2 + (y - y_0)^2}, \quad B_x = \frac{\mu_0 I}{2\pi} \frac{y - y_0}{(x - x_0)^2 + (y - y_0)^2}.
\]  

(72)

With the Taylor-series expansion for \(|z_0| < |z|\), i.e., inside the circular aperture of the magnet,

\[
\frac{1}{z - z_0} = \frac{1}{z(1 - \frac{z_0}{z})} = \frac{1}{z} \sum_{n=1}^{\infty} \frac{z_0^n}{z^n} = \sum_{n=1}^{\infty} \frac{z_0^{n-1}}{z^n}
\]  

(73)

we get \(B_y + iB_x = -\frac{\mu_0 I}{2\pi} \sum_{n=1}^{\infty} \frac{z_0^{n-1}}{z^n}\). Bringing together the multipole coefficients, eqns. (38) and (39) yield

\[
\sum_{n=1}^{\infty} (B_n + iA_n) \frac{z_0^n}{r_0^n} = (B_\varphi + iB_r) \frac{z_0}{r_0} = -\frac{\mu_0 I}{2\pi} \sum_{n=1}^{\infty} \frac{z_0^{n-1} z_0}{z^n} \frac{z_0}{r_0}.
\]  

(74)

Therefore \(B_n + iA_n = -\frac{\mu_0 I}{2\pi} \frac{z_0^{n-1}}{z^n}\) which is identical to Eq. (64). From Eq. (74) it follows:

\[
B_y + iB_x = \sum_{n=1}^{\infty} (B_n + iA_n) \frac{z_0}{r_0}^{n-1}
\]  

(75)

which is sometimes given as a definition of the multipole coefficients and which allows an easy reconstruction of the cartesian components of the magnetic flux density from the calculated or measured field components.
2.82 Feed-down

An interesting example for the use of the complex field representation is the calculation of the feed-down effect due to an off-centering of the measurement coil with respect to the magnet axis (or a misalignment of the magnet with respect to the beam axis). The transformation law for the field harmonics \( c_n \rightarrow c_n' \); \( c_n = b_n + i a_n \) can be derived for a translation of the reference frame (center of the measurement coil) \( z \rightarrow z' \); \( z = z' + \Delta z \); \( z = x + iy \) as follows: As the field components in both coordinate systems have to be identical we can write

\[
\sum_{n=1}^{\infty} c_n \left( \frac{z}{r_0} \right)^{n-1} = \sum_{n=1}^{\infty} c_n' \left( \frac{z'}{r_0} \right)^{n-1} = B_y + i B_x .
\]

The transformation law for the field harmonics then reads (for the proof see Appendix A):

\[
c_n' = \sum_{k=n}^{\infty} c_k \frac{(k-1)!}{(k-n)! (n-1)!} \left( \frac{\Delta z}{r_0} \right)^{k-n} .
\]

(76)

For example, off-centering of the measurement coil creates a (measured) quadrupole field component which results from the natural sextupole component feed-down. Equation (77) can also be used to determine the center of the measurement coil, e.g., through the elimination of the measured dipole component in quadrupole magnets. In bending magnets the \( b_{11} \) component is very insensitive to manufacturing errors while \( b_{10} \) is near zero. If \( c_{11}, c_{12} \) and \( c_{13} \) at a reference radius \( r_0 \) are known with sufficient accuracy, the displacement \( \Delta z \) can be calculated from a truncated series of equation (77). Notice, that this holds only for small displacements \( \Delta z \ll r_0 \).

2.83 The Residue theorem and the field of intersecting ellipses

The residue theorem states that if a complex function is meromorphic in a domain \( \Omega \) (i.e., analytic in \( \Omega \) except at a finite number of isolated singularities) and analytic on the boundary \( \partial \Omega \) of \( \Omega \) then it holds:

\[
\oint_{\partial \Omega} f(z) dz = 2\pi i \sum_n R(z_n)
\]

(78)

where the \( R(z_n) \) are the residuals of the poles of the function at the points \( z_n \), i.e., the constant in the numerator of the single pole, if the complex function is given in the form

\[
f(z) = R(z_n) \frac{1}{z - z_n} + g(z_n) .
\]

(79)

For the calculation of the two-dimensional magnetic field generated by intersecting ellipses or circles, Beth [7], uses the complex potential

\[
F(z) = \tilde{H} - \frac{1}{2} J(x,y) z^*
\]

(80)

with \( \tilde{H} = H_y + i H_x \) and the complex conjugate \( z^* = x - iy \). Eq. 80 can be rewritten as

\[
F(z) = \tilde{H} - \frac{1}{2} J z^* = H_y + i H_x - \frac{1}{2} J(x - iy) = H_y - \frac{1}{2} J x + i (H_x + \frac{1}{2} J y) = u(x,y) + iv(x,y) .
\]

(81)

Calculating the partial derivatives of the real valued functions \( u(x,v) \) and \( v(x,y) \) and inserting into the Cauchy Riemann equations yields the two equations

\[
\begin{align*}
\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= J , \\
\frac{\partial H_y}{\partial y} + \frac{\partial H_x}{\partial x} &= 0 .
\end{align*}
\]

(82)
which are nothing but the Poisson and Laplace equations in two-dimensional Cartesian coordinates. It can thus be shown that the complex potential $F(z) = \bar{H} - \frac{1}{2}J(x, y)z^*$ is analytic and that it can be used to solve the field distribution inside current carrying conductors using the residue theorem.

Let $a$ be the minor and $b$ the major half-axis of an infinitely long, elliptic conductor carrying uniform current density $J$. Let the points on the boundary $C'$ of the ellipse be denoted by the small $z = x + iy$ and the field point by $z_0$. The domain inside $C'$ is denoted by the capital $Z_{in}$ and the domain outside the ellipse by $Z_{out}$. This gives the following expression for $\bar{H}_{in}(z_0)$ for $z_0 = x_0 + iy_0 \in Z_{in}$ [7]:

$$\bar{H}_{in}(x_0, y_0) = \frac{J}{a + b}(bx_0 - iay_0)$$  \hspace{1cm} (83)

and the components of the magnetic field within an elliptic conductor with constant current density $J$ are:

$$H_x(x_0, y_0) = -J \frac{a}{a + b} y_0, \quad H_y(x_0, y_0) = J \frac{b}{a + b} x_0.$$  \hspace{1cm} (84)

For the round conductor $a = b$ we get the well known relations $H_x(x_0, y_0) = -\frac{1}{2}Jy_0$ and $H_y(x_0, y_0) = \frac{1}{2}Jx_0$.

![Fig. 10: Intersecting circles and ellipses that create an ideal dipolar field inside the aperture.](image)

As an application of the above result it is now straightforward to show that in the center of two intersecting ellipses or circles with opposite current density a homogeneous field is generated. With the two intersecting circles shifted by $c$ and with the two local coordinate systems $x_1 = x + c/2$, $x_2 = x - c/2$, $y_1 = y$, $y_2 = y$ (positive current in circle 1) it follows:

$$B_x = B_x^{(1)} + B_x^{(2)} = -\mu_0 \frac{1}{2} J(y_1 - y_2) = 0,$$

$$B_y = B_y^{(1)} + B_y^{(2)} = \mu_0 \frac{1}{2} J \left((x + \frac{c}{2}) - (x - \frac{c}{2})\right) = \mu_0 \frac{1}{2} Jc.$$  \hspace{1cm} (86)

For the intersecting ellipses the $B_x$ component is zero due to the same reasoning, and the $y$-component takes the value:

$$B_y = \mu_0 Jc \frac{b}{a + b}.$$  \hspace{1cm} (87)

**Remark:** This important result has two implications: First it paves the way for designing block coil magnets with constant current density in the conductors and secondly it can reproduce shielding current densities in hard superconductors subjected to changing excitation field, i.e., a macroscopic way to model persistent currents, chapter 5.
3. Numerical Field Computation

Magnets for particle accelerators have always been a key application of numerical methods in electromagnetism. Hornby [19], in 1963, developed a code based on the finite difference method for the solving of elliptic partial differential equations and applied it to the design of magnets. Winslow [44] created the computer code TRIM (Triangular Mesh) with a discretization scheme based on an irregular grid of plane triangles by using a generalized finite difference scheme. He also introduced a variational principle and showed that the two approaches lead to the same result. In this respect, the work can be viewed as one of the earliest examples of the finite element method applied to the design of magnets. The POISSON code which was developed by Halbach and Holsinger [18] was the successor of this code and was still applied for the optimization of the superconducting magnets for the LHC during the early design stages. In the early 1970's a general purpose program (GFUN) for static fields had been developed by Newman, Turner and Trowbridge that was based on the magnetization integral equation and was applied to magnet design.

When the LHC magnets are ramped to their nominal field of 8.33 T in the aperture, the yoke is highly saturated, and numerical methods have to be used to replace the imaging method. In this case it is advantageous to use numerical methods that allow a distinction between the coil-field and the iron magnetization effects, to confine both modeling problems on the coils and FEM-related numerical errors to the 20% of field contribution from the iron magnetization. The integral equation method of GFUN would be appropriate, however, it leads to a very large, fully populated matrix of the linear equation system.

The method of coupled boundary-elements/finite-elements (BEM-FEM), developed by Fetzer, Haas and Kurz at the University of Stuttgart, Germany, combines a finite-element description using incomplete quadratic (20-node) elements and a gauged total vector-potential formulation for the interior of the magnetic parts, and a boundary element formulation for the coupling of these parts to the exterior, which includes excitational coil fields. This implies that the air regions need not to be meshed at all. Of course the method is not limited to the accelerator magnet design. In order to explain the principles of numerical field computation the principle steps of the finite element and boundary element techniques are outlined here under:

- Formulation of the physical laws by means of partial differential equations.
- Transformation of these equations into an integral equation with the weighted residual method.
- Integration by parts (using Green’s theorems) in order to obtain the so-called weak integral form. Consideration of the natural boundary conditions.
- Discretization of the domain into (higher-order) finite elements.
- Approximation of the solution as a linear-combination of so-called shape functions.
- In case of the finite element (Galerkin) method, using the shape functions as the weighting functions of the weak integral form.
- In case of the boundary-element method, another partial integration of the weak integral form results in an integral equation. Using the fundamental solution of the Laplace operator as weighting functions yields an algebraic system of equations for the unknowns on the domain boundary. In case of the coupled boundary-element/finite-element method, the two domains are coupled through the normal derivatives of the vector-potential on their common boundary.
- Consideration of the essential boundary conditions in the resulting equation system.
- Numerical solution of the algebraic equations. A direct solver with Newton iteration is used in the 2D case; the domain decomposition method with a $M(B)$ iteration [15] is applied in the 3D case.

In order to understand the special properties of the numerical methods and the reasoning which leads to their application in the design and optimization of superconducting magnets, it is sufficient to concentrate on some aspects of the formulations. We will therefore omit the treatment of the linear equation system.
3.1 Green’s identities

The mathematical foundations of numerical field calculation require some knowledge about vector-analysis. We shall recall Green’s first and second identity

\[ \int_{\Omega} \left( \text{grad} \phi \cdot \text{grad} \psi + \phi \nabla^2 \psi \right) d\Omega = \int_{\Gamma} \phi \text{grad} \psi \cdot \vec{n} d\Gamma \]  
(88)

\[ \int_{\Omega} \left( \phi \nabla^2 \psi - \psi \nabla^2 \phi \right) d\Omega = \int_{\Gamma} \left( \phi \text{grad} \psi - \psi \text{grad} \phi \right) \cdot \vec{n} d\Gamma \]  
(89)

with \( \Omega \) being a 3 dimensional domain with boundary surface \( \Gamma \). Green’s identities are generalizations of the integration by parts rules

\[ \int_{x_1}^{x_2} \left( \phi \phi'' + \phi'^2 \right) dx = [\phi \phi']_{x_1}^{x_2} \]  
(90)

\[ \int_{x_1}^{x_2} \left( \phi \psi'' - \psi \phi'' \right) dx = [\phi \psi' - \psi \phi']_{x_1}^{x_2} \]  
(91)

to three dimensions. Green’s identities play a vital role in numerical field computation as they constitute the junction between the finite element (FEM) and the boundary element (BEM) method as shown in fig. 11.

3.2 Finite element method with total vector-potential formulation

Consider the elementary model problem for a superconducting magnet consisting of two different domains: \( \Omega_i \) the iron region with permeability \( \mu \) and \( \Omega_a \) the air region with the permeability \( \mu_0 \). The regions are connected to each other at the interface \( \Gamma_{ai} \). The non-conductive air region \( \Omega_a \) may also contain a certain number of conductor sources \( \vec{J} \) which do not intersect the iron region \( \Omega_i \).

3.2.1 The weighted residual

Let us start with the total vector potential formulation

\[ \text{curl} \left( \frac{1}{\mu} \text{curl} \vec{A} \right) - \text{grad} \left( \frac{1}{\mu} \text{div} \vec{A} \right) = \vec{J} \quad \text{in} \ \Omega. \]  
(92)

The domain \( \Omega = \Omega_a \cup \Omega_i \) is discretized into finite-elements in order to solve this problem numerically. For the approximate solution of \( \vec{A} \) as an interpolation of its values on the nodes of the finite element mesh, the differential equation (92) is only approximately fulfilled:

\[ \text{curl} \left( \frac{1}{\mu} \text{curl} \vec{A} \right) - \text{grad} \left( \frac{1}{\mu} \text{div} \vec{A} \right) = \vec{J} = \vec{R} \]  
(93)
with a residual (error) vector $\vec{R}$. A linear equation system for the unknown nodal values of the vector potential $\vec{A}^{(k)}$ can be obtained by minimizing the weighted residuals $\vec{R}$ in an average sense over the domain $\Omega$, i.e.,

$$\int_{\Omega} \vec{w}_a \cdot \vec{R} \, d\Omega = 0, \quad a = 1, 2, 3 \quad (94)$$

with the vector weighting functions $\vec{w}_1 = (w_1, 0, 0)^T$, $\vec{w}_2 = (0, w_2, 0)^T$, $\vec{w}_3 = (0, 0, w_3)^T$, where $w_1, w_2, w_3$ are arbitrary (but known) weighting functions. Forcing the weighted residual to zero yields

$$\int_{\Omega} \vec{w}_a \cdot \left( \frac{1}{\mu} \text{curl} \vec{A} - \text{grad} \frac{1}{\mu} \text{div} \vec{A} \right) \, d\Omega = \int_{\Omega} \vec{w}_a \cdot \vec{J} \, d\Omega, \quad a = 1, 2, 3. \quad (95)$$

### 3.22 The weak form

With the following generalizations of the integration by parts rule

$$\int_{\Omega} \left( \frac{1}{\mu} \text{curl} \vec{A} \right) \cdot \vec{w}_a \, d\Omega = \int_{\Omega} \frac{1}{\mu} \text{curl} \vec{A} \cdot \text{curl} \vec{w}_a \, d\Omega - \int_{\Gamma} \frac{1}{\mu} \left( \text{curl} \vec{A} \times \vec{n} \right) \cdot \vec{w}_a \, d\Gamma, \quad (96)$$

$$\int_{\Omega} \left( - \frac{1}{\mu} \text{div} \vec{A} \right) \cdot \vec{w}_a \, d\Omega = \int_{\Omega} \frac{1}{\mu} \text{div} \vec{A} \text{div} \vec{w}_a \, d\Omega - \int_{\Gamma} \frac{1}{\mu} \text{div} \vec{A} (\vec{n} \cdot \vec{w}_a) \, d\Gamma, \quad (97)$$

and the consideration of the boundary conditions on $\Gamma$, the weighted residual can be transformed to

$$\int_{\Omega} \frac{1}{\mu} \text{curl} \vec{w}_a \cdot \text{curl} \vec{A} \, d\Omega + \int_{\Omega} \frac{1}{\mu} \text{div} \vec{w}_a \text{div} \vec{A} \, d\Omega = \int_{\Omega} \vec{w}_a \cdot \vec{J} \, d\Omega \quad (98)$$

with $a = 1, 2, 3$. In two dimensions, where $\frac{\partial}{\partial z} = 0$ and the Coulomb gauge is automatically fulfilled, Eq. (98) further reduces to

$$\int_{\Omega} \frac{1}{\mu} \text{curl} \vec{w}_3 \cdot \text{curl} \vec{A}_z \, d\Omega = \int_{\Omega} \vec{w}_3 \cdot \vec{J}_z \, d\Omega. \quad (99)$$

Eq. (98) is called the weak form of the vector-potential formulation because the second derivatives have been removed and the continuity requirements on $\vec{A}$ have been relaxed at the expense of an increase in the continuity conditions of the weighting functions. Only this makes possible the use of elements with linear shape functions. Inside these elements the first derivative of the shape functions is a constant and the second derivative vanishes. On the element boundary we find a jump in the first derivative and a Dirac $\delta$-function for the second. Thus there would be a problem in Eq. (98) if the second derivative was present. This level of continuity is termed as $C_0$ continuous.

**Remark:** The current density $\vec{J}$ appears on the right hand side of eq. (98). In consequence, when using the FE-method for the solution of this problem the relatively complicated shape of the coils must be modeled in the FE-mesh.

### 3.23 Finite element shape functions

The domain $\Omega = \Omega_1 \cup \Omega_2$ is discretized into finite-elements $\Omega_j$, it is “meshed” in common parlance. The method is explained by means of the most simple (2D, linear, triangular) finite elements. Within a particular element $j$ the $z$ component of the vector potential $A_j = A_z(x, y)$ is approximated by

$$A_j = \alpha_1 + \alpha_2 x + \alpha_3 y. \quad (100)$$

The above functions (with finite support) are called *basis* functions and are defined on the element $j$ only. The approximated potentials $A^{(k)}$ at the three local nodes $k = 1, 2, 3$ of the element $j$ can then
be expressed as \( A^{(k)} = \alpha_1 + \alpha_2 x_k + \alpha_3 y_k \) and so forth. This set of 3 equations can be solved for the unknowns \( \alpha_1, \alpha_2, \alpha_3 \) using Cramer’s rule for instance. For a general point \( \vec{x} \) within the element \( j \) the potential can then be approximated with

\[
A_j(\vec{x}) = \alpha_1 + \alpha_2 x + \alpha_3 y = \sum_{k=1}^{3} N_k(\vec{x})A^{(k)}
\]

with the so-called element shape functions (or trial functions) \( N_k \)

\[
N_k(\vec{x}) = N_k(x, y) = \frac{a_k + b_k x + c_k y}{2S},
\]

where \( S \) is the surface of the element and the \( a_k, b_k, c_k \) are nothing else than linear combinations of the coordinates of the nodal points. The \( N_k \) are the barycentric coordinates of a point \( \vec{x} \) with respect to the node \( k \). As we shall see, in the FEM formulation only the element shape functions appear but not the basis functions (100). Obviously, the element shape function has to equal one when the point \( \vec{x} \) is joined with the node \( k \); it equals zero when the point is outside the element. Figure 12 shows the element-wise defined (linear) basis functions \( A_j = \alpha_1 + \alpha_2 x + \alpha_3 y \) and the shape function \( N_k \) in the local node \( k = 3 \) of element \( j = 6 \).

3.24 Assembling the matrix

So far the basis functions were defined to be scalars. In 3 dimensions the solution is approximated element-wise by the potential vectors \( \vec{A}_j \) (containing the three scalar components of the vector potential) which are expanded with respect to the element shape functions (or trial functions) \( N_k(\vec{x}) \) and their nodal values \( \vec{A}^{(k)} \) which are to be determined in the solution:

\[
\vec{A}_j = \sum_{k=1}^{K} N_k(\vec{x})\vec{A}^{(k)} \quad \text{in } \Omega_j,
\]

where \( K \) is the number of nodes of the element \( \Omega_j \) and \( \vec{A} = (A_x, A_y, A_z)^T \). A linear equation system for the unknowns \( \vec{A}^{(k)} \) can be obtained by forcing the weighted residual to zero in an average sense over the element \( \Omega_j \) and integration by parts, which yields the weak integral form for one element. The weighting
functions are chosen as the element shape functions $N_l, l = 1, K$ (Galerkin method) and we get:

$$
\int_{\Omega} \frac{1}{\mu} \text{curl} \vec{N}_{la} \cdot \text{curl} \left( \sum_{k=1}^{K} N_k(\vec{x}) \vec{A}^{(k)} \right) d\Omega + \\
\int_{\Omega} \frac{1}{\mu} \text{div} \vec{N}_{la} \text{div} \left( \sum_{k=1}^{K} N_k(\vec{x}) \vec{A}^{(k)} \right) d\Omega = \int_{\Omega} \vec{N}_{la} \cdot \vec{J} d\Omega
$$

(104)

for $l = 1, K$ and $a = 1, 2, 3$, with $\vec{N}_{k,1,1} = (N_{k,l}, 0, 0)^T$, $\vec{N}_{k,2,2} = (0, N_{k,l}, 0)^T$ and $\vec{N}_{k,3,3} = (0, 0, N_{k,l})^T$. The equation system (104) can be written in a matrix representation as

$$
[k] \{A\} = \{f\}. 
$$

(105)

The matrix $[k]$ is often called the stiffness matrix (by reference to elasticity problems), and $\{f\}$ is the element force vector. $\{A\}$ is the vector of the nodal potential function. The stiffness matrix has the form

$$
[k] = \begin{bmatrix}
  k_{11} & \cdots & k_{1K} \\
  \vdots & \ddots & \vdots \\
  k_{K1} & \cdots & k_{KK}
\end{bmatrix}
$$

(106)

for a element with $K$ nodes (three degrees of freedom per node). The coefficients $k_{lm}(l, m = 1, \ldots, K)$ in this matrix are $3 \times 3$ matrixes and can be calculated by means of Gauss-integration. The Galerkin method has the important property that the stiffness matrix is sparse and symmetric. The equation system for the entire domain $\Omega$ can be assembled through the merging of the element (sub)-systems in the resulting equation system

$$
[K] \{A\} = \{F\}
$$

(107)

with the introduction of the so-called nodal functions $N_{G,n}(\vec{x})$ with $N_{G,n}(\vec{x}) = N_k(\vec{x})$ for $\vec{x} \in \Omega_j$, where $n$ is the global node number in the domain $\Omega$.

3.3 Numerical field computation for superconducting magnets

3.3.1 Quadrilateral higher order elements

The so-called simplex elements (triangular in 2D, tetrahedral in 3D) have the disadvantage that curved shapes can only be modeled by polygonal approximations. The advantage of a higher order approximation of the potentials may be lost due to a rather rough geometric approximation. An alternative are the isoparametric elements which have curved sides. They can therefore be found in most commercial software packages. It avoids numerically unfavorable prisms when the geometry is simply extruded into the third dimension.

A mesh generator based on geometrical domain decomposition, which was developed at the University of Stuttgart, Germany [30] has been implemented in the ROXIE program package. The following extensions have been added.

1) Extension of the method to 8 noded (higher order) quadrilateral elements. 2) Parametric input for the definition of design variables for mathematical optimization. 3) Implementation of design features for the definition of material boundaries. 4) Modular magnet geometry input by means of the GNU m4 macro language. 5) A morphing algorithm for optimization and sensitivity studies which avoids re-meshing and changing mesh topologies.

The quadrilateral mesh generator relies on the method of geometrical domain decomposition [30]. In a first step, the input geometry is decomposed into areas that are topologically equivalent to disks, i.e.,
holes are eliminated. The shape of the sub-domains is optimized by minimizing a function that depends on the ratios \( \frac{l^2}{A} \), \( \frac{l}{U} \), and the angles of the newly created areas; \( l \) is the length of the cut, \( A \) is the area, and \( U \) is the circumference of the sub-domain. This decomposition is continued until the remaining areas are regarded as simple (when they are similar to triangles or rectangles). These areas are then filled with quadrilateral elements using a modified paving strategy [9]. In this approach an area is filled from the outside to the inside by adding full rows of quadrilateral elements. Finally, a smoothing algorithm is applied. A linear combination of Laplace-smoothing, edge-smoothing and angle-smoothing is used to enlarge small angles, to reduce large angles, and to increase short distances between mesh points while leaving the topology unchanged. As an example, figure 13 shows the meshed iron yoke, insert, and collar geometry of the LHC main dipole.

![Fig. 13: Quadrilateral higher order finite element mesh of the LHC main dipole iron yoke, insert, and stainless steel collar.](image)

### 3.32 Coupling Method between Boundary Elements (BEM) and Finite Elements (FEM)

The disadvantage of the finite element method is that only a finite domain can be discretized, and therefore the field calculation in the magnet coil-ends with their large fringe-fields requires a large number of elements in the air region. The relatively new boundary-element method is defined on an infinite domain and can therefore solve open boundary problems without approximation with far-field boundaries. The disadvantage is that non-homogeneous materials are difficult to consider. The BEM-FEM method couples the finite element method inside magnetic bodies \( \Omega_i = \Omega_{\text{FEM}} \) with the boundary-element method in the domain outside the magnetic material \( \Omega_a = \Omega_{\text{BEM}} \), by means of the normal derivative of the vector-potential on the interface \( \Gamma_{ai} \) between iron and air. The application of the BEM-FEM method to magnet design has the following intrinsic advantages:

- The coil field can be taken into account in terms of its source vector potential \( \vec{A}_s \), which can be obtained easily from the filamentary currents \( I_s \) by means of Biot-Savart type integrals without the meshing of the coil.
- The BEM-FEM coupling method allows for the direct computation of the reduced vector potential \( \vec{A}_r \) instead of the total vector potential \( \vec{A} \). Consequently, errors do not influence the dominating contribution \( \vec{A}_s \) due to the superconducting coil.
- Because the field in the aperture is calculated through the integration over all the BEM elements, local field errors in the iron yoke cancel out and the calculated multipole content is sufficiently accurate even for very sparse meshes.
• The surrounding air region need not be meshed at all. This simplifies the preprocessing and avoids artificial boundary conditions at some far-field-boundaries. Moreover, the geometry of the permeable parts can be modified without regard to the mesh in the surrounding air region, which strongly supports the feature based, parametric geometry modeling that is required for mathematical optimization.

• The method can be applied to both 2D and 3D field problems.

3.33 The FEM part

Inside the magnetic domain $\Omega_i$ a gauged vector-potential formulation is applied:

$$
-\frac{1}{\mu_0} \nabla^2 \vec{A} = \vec{J} + \text{curl} \vec{M} \quad \text{in} \quad \Omega_i.
$$

Considering that the free current density in the iron yoke is zero, and forcing the weighted residual to zero yields

$$
-\int_{\Omega_i} \frac{1}{\mu_0} \nabla^2 \vec{A} \cdot \vec{w}_a \, d\Omega_i = \int_{\Omega_i} (\text{curl} \vec{M}) \cdot \vec{w}_a \, d\Omega_i \quad a = 1, 2, 3.
$$

with $\vec{w}_{1,2,3}$. With Green’s first theorem

$$
\int_{\Omega_i} \nabla^2 \vec{A} \cdot \vec{w}_a \, d\Omega_i = -\int_{\Omega_i} \text{grad} (\vec{A} \cdot \vec{e}_a) \cdot \text{grad} w_a \, d\Omega_i + \oint_{\Gamma} \frac{\partial \vec{A}}{\partial n_i} \cdot \vec{w}_a \, d\Gamma
$$

and the identity

$$
\int_{\Omega_i} \text{curl} \vec{M} \cdot \vec{w}_a \, d\Omega_i = \int_{\Omega_i} \vec{M} \cdot \text{curl} \vec{w}_a \, d\Omega_i - \oint_{\Gamma} (\vec{M} \times \vec{n}_i) \cdot \vec{w}_a \, d\Gamma
$$

we get for the weak form

$$
\frac{1}{\mu_0} \int_{\Omega_i} \text{grad} (\vec{A} \cdot \vec{e}_a) \cdot \text{grad} w_a \, d\Omega_i - \frac{1}{\mu_0} \oint_{\Gamma_{ai}} \left( \frac{\partial \vec{A}}{\partial n_i} - (\mu_0 \vec{M} \times \vec{n}_i) \right) \cdot \vec{w}_a \, d\Gamma_{ai} =
$$

$$
\int_{\Omega_i} \vec{M} \cdot \text{curl} \vec{w}_a \, d\Omega_i
$$

with $a = 1, 2, 3$. The continuity condition of the tangential component of the magnetic field at the interface between iron (FEM domain) and air is equivalent to

$$
\frac{\partial \vec{A}_i}{\partial n_i} - (\mu_0 \vec{M} \times \vec{n}_i) + \frac{\partial \vec{A}_a}{\partial n_a} = 0,
$$

where $\vec{n}_i$ is the normal vector on $\Gamma_{ai}$ pointing out of the FEM domain $\Omega_i$, and $\vec{n}_a$ is the normal vector on $\Gamma_{ai}$ pointing out of the BEM domain $\Omega_a$. The boundary integral term on the boundary between iron and air $\Gamma_{ai}$ in (112) serves as the coupling term between the BEM and the FEM domain. Let us for the moment assume that the normal derivative on $\Gamma_{ai}$

$$
\vec{Q}_{\Gamma_{ai}} = -\frac{\partial \vec{A}_{\text{BEM}}}{\partial n_a}
$$

is given. If the domain $\Omega_i$ is discretized into finite elements $\Omega_j$ ($C^0$-continuous, isoparametric 20-noded hexahedron elements are used) and the Galerkin method is applied to the weak formulation, then a nonlinear system of equations is obtained

$$
\begin{pmatrix}
[K_{\Omega_i\Omega_i}] & [K_{\Omega_i\Gamma_{ai}}] & 0 \\
[K_{\Gamma_{ai}\Omega_i}] & [K_{\Gamma_{ai}\Gamma_{ai}}] & [T] \\
\end{pmatrix}
\begin{pmatrix}
\{\vec{A}_i\} \\
\{\vec{A}_{\Gamma_{ai}}\} \\
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
\end{pmatrix}
$$

(115)
with all nodal values of $\vec{A}, \vec{A}_i, \vec{Q}_a$ grouped in arrays

$$
\begin{align*}
\{ \vec{A}_i \} = (A_i^{(1)}, A_i^{(2)}, \ldots), \\
\{ \vec{A}_a \} = (A_a^{(1)}, A_a^{(2)}, \ldots), \\
\{ \vec{Q}_a \} = (Q_a^{(1)}, Q_a^{(2)}, \ldots).
\end{align*}
$$

(116)

The subscripts $\Gamma_a$ and $\Omega_a$ refer to nodes on the boundary and in the interior of the domain, respectively. The domain and boundary integrals in the weak formulation yield the stiffness matrices $[K]$ and the boundary matrix $[T]$. The stiffness matrices depend on the local permeability distribution in the nonlinear material. All the matrices in (115) are sparse.

3.34 The BEM part

By definition, the BEM domain $\Omega_a$ contains no iron, and therefore $\vec{M} = 0$ and $\mu = \mu_0$. The governing equation

$$
\nabla^2 \vec{A} = -\mu_0 \vec{J}
$$

(117)

decomposes in Cartesian coordinates into three scalar Poisson equations to be solved. For an approximate solution of these equations and the weighted residual forced to zero yields:

$$
\int_{\Omega_a} \nabla^2 A w d\Omega_a = - \int_{\Omega_a} \mu_0 J w d\Omega_a.
$$

(118)

Employing Green’s second theorem yields

$$
\int_{\Omega_a} A \nabla^2 w d\Omega_a = - \int_{\Omega_a} \mu_0 J w d\Omega_a + \int_{\Gamma_a} A \frac{\partial w}{\partial n_a} d\Gamma_a - \int_{\Gamma_a} \frac{\partial A}{\partial n_a} w d\Gamma_a.
$$

(119)

In Eq. (119) it is already considered that all the boundary integrals on the far field boundary $\Gamma_{BEM,\infty}$ vanish. Now the weighting function is chosen as the fundamental solution of the Laplace equation, which is in 3D

$$
w = u^* = \frac{1}{4\pi R}, \quad \frac{\partial w}{\partial n_a} = q^* = -\frac{1}{4\pi R^2}, \quad \nabla^2 w = -\delta(R)
$$

(120)

we get the Fredholm integral equation of the second kind:

$$
\frac{\Theta}{4\pi} A + \int_{\Gamma_a} Q_{\Gamma_a} u^* d\Gamma_a + \int_{\Gamma_a} A_{\Gamma_a} q^* d\Gamma_a = \int_{\Omega_a} \mu_0 J u^* d\Omega_a.
$$

(121)

The right hand side of Eq. (121) (the last remaining domain integral) is a Biot-Savart-type integral for the source vector potential $A_s$.

The components of the vector potential $\vec{A}$ at arbitrary points $\vec{r}_0 \in \Omega_a$ (e.g. on the reference radius for the field harmonics) have to be computed from (121) as soon as the components of the vector potential $\vec{A}_{\Gamma_a}$ and their normal derivatives $\vec{Q}_{\Gamma_a}$ on the boundary $\Gamma_{\Gamma_a}$ are known. $\Theta$ is the solid angle enclosed by the domain $\Omega_a$ in the vicinity of $\vec{r}_0$.

For the discretization of the boundary $\Gamma_{\Gamma_a}$ into individual boundary elements $\Gamma_{ai,j}$, again $C^0$-continuous, isoparametric 8-noded quadrilateral boundary elements (in 3D) are used. In 2D 3-noded line elements are used. They have to be consistent with the elements from the FEM domain touching this boundary. The components of $\vec{A}_{\Gamma_a}$ and $\vec{Q}_{\Gamma_a}$ are expanded with respect to the element shape functions, and (121) can be rewritten in terms of the nodal data of the discrete model,

$$
\frac{\Theta}{4\pi} \vec{A} = \vec{A}_s - \{ \vec{Q}_{\Gamma_a} \} \cdot \{ g \} - \{ \vec{A}_{\Gamma_a} \} \cdot \{ h \}.
$$

(122)
In (122), \( g \) results from the boundary integral with the kernel \( u^* \), and \( h \) results from the boundary integral with the kernel \( q^* \). The discrete analogue of the Fredholm integral equation can be obtained from (122) by successively putting the evaluation point \( \vec{r}_0 \) at the location of each nodal point \( \vec{r}_j \). This procedure is called point-wise collocation and yields a linear system of equations,

\[
[G] \{ \vec{Q}_{\Gamma_{ai}} \} + [H] \{ \vec{A}_{\Gamma_{ai}} \} = \{ \vec{A}_s \}.
\]

(123)

In (123), \( \{ \vec{A}_s \} \) contains the values of the source vector potential at the nodal points \( \vec{r}_j \), \( j = 1, 2, \ldots \). The matrices \([G]\) and \([H]\) are asymmetric and fully populated. Equation (123) gives exactly the missing relationship between the Dirichlet data \( \{ \vec{A}_{\Gamma_{ai}} \} \) and the Neumann data \( \{ \vec{Q}_{\Gamma_{ai}} \} \) on the boundary \( \Gamma_{ai} \).

An overall numerical description of the field problem can be obtained by complementing the FEM description (115) with the BEM description (123) which results in

\[
\begin{pmatrix}
[K_{\Omega,\Omega} & K_{\Omega,\Gamma_{ai}} & 0 & 0 \\
K_{\Gamma_{ai},\Omega} & K_{\Gamma_{ai},\Gamma_{ai}} & [T] & 0 \\
0 & [H] & [G] & 0 \\
0 & 0 & [Q_{\Gamma_{ai}}] & \{ \vec{A}_s \}
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0 \\
\{ \vec{A}_s \}
\end{pmatrix}.
\]

(124)

It can be shown [21] that this procedure yields the correct physical interface conditions, the continuity of \( \vec{n} \cdot \vec{B} \) and \( \vec{n} \times \vec{H} \) across \( \Gamma_{ai} \).

3.4 Numerical examples

3.4.1 Field quality in collared coils

Using double aperture stainless-steel collars with a relative permeability of \( \mu_r = 1.0025 \) creates asymmetries in the magnetic field in the case of warm measurements of the collared coil assembly where only one aperture is powered. As the BEM-FEM method does not require the meshing of the coil (which can therefore be modeled with the required accuracy) and does not require a “far field” boundary condition which would influence considerably the results of this unbounded field problem the BEM-FEM method is specially qualified for the calculations.

The additional field errors generated by the asymmetric stainless steel collar are a \( \Delta b_2 \) of -0.239 units in \( 10^{-4} \) (at 17 mm reference radius), \( \Delta b_3 = -1.173 \), \( \Delta b_4 = -0.012 \) and \( \Delta b_5 = 0.305 \).

![Fig. 14: Geometric model of one dipole coil powered for warm measurement in the combined collar structure assuming a constant relative permeability of \( \mu_r = 1.0025 \). The figure displays the magnetic flux density in the collars.](image-url)
3.42 End fields in the dipole short models

As was stated in the introduction, the LHC requires a major effort to guarantee that the superconducting dipoles will perform to specifications. For this reason an R&D program for the development of the superconducting dipoles was started in 1995. The aim of the program is to study the influence of individual coil parameters, pre-stress in the coil, collar material and yoke structure on a series of otherwise identical model dipoles. A maximum of turn-around and testing efficiency is achieved by reducing the length of the models from 14 m to approximately 1 m and by manufacturing single aperture models with only one coil in a re-usable iron-yoke. The short models have a coil length of 1.05 m and a magnetic yoke of only 402 mm approximately centered in the magnet in order to leave the configuration in the end and connection region unchanged with respect to the long prototypes.

The magnetic field homogeneity of the models is systematically measured in a vertical test set-up where the magnet is suspended inside a cryostat. A drawback of the short length of the dipole models is that end effects influence the magnetic field quality in the center of the magnet. As the rotating pick-up coils deliver multipoles averaged over their length, and the pick-up in the center of the magnet has a length of 200 mm (half the length of the magnetic yoke) the interpretation of measurements becomes difficult. In order to study systematic effects in the field quality, it is necessary to calculate, with a high precision, the 3D multipole field errors in these magnets as a function of the $z$-position. Fig. 15 shows the geometric model of the coil-test facility (CTF). To prove the reliability of the method, the results of the 3D field calculation are compared to the measurements on the single aperture model MBSMS21. The multipoles are computed as a function of the excitation current over a length of 550 mm along the magnet bore. The computed dipole field $B_1(z)$ and the multipoles $b_n(z)$ are used to compute the average multipoles over the length spanned by the measurement pick-up coils. This averaging step is necessary to obtain quantities comparable to the measured values. The results of the measured and simulated normal dipole and normal sextupole component are shown in fig. 16 together with the 2D calculations for the long magnet. It can be seen that the short models show a globally different saturation behavior compared to the long dipole prototypes.

3.43 End field in long dipole prototype magnets

In order to reduce the peak field in the coil-end and thus increase the quench margin in the region with a weaker mechanical structure, the magnetic iron yoke in LHC prototype magnets was replaced by non-
magnetic stainless steel laminations approximately 100 mm from the onset of the ends. The BEM-FEM coupling method was therefore used for the calculation of the end-fields. The computing time for the 3D calculation is in the order of 5 hours on a DEC Alpha 5/333 workstation. The iterative solution of the linear equation system converges better in the case of a high excitational field than in the case of the injection field with its non-saturated iron yoke. It is therefore still impossible to apply mathematical optimization techniques to the 3D field calculation with iron yoke. However, as the additional effect from the fringe field on the field quality is low, it is sufficient to calculate the additional effect and then partially compensate with the coil design, if necessary. It has already been explained that the BEM-FEM coupling method allows the distinction between the coil field and the reduced field from the iron magnetization. Fig. 17 shows the field components along a line in the end-region of the twin-aperture dipole prototype magnet (MBP2), 43.6 mm above the beam-axis in aperture 2 (on a radius between the inner and outer layer coil) from \( z = -200 \) mm inside the magnet yoke to \( z = 200 \) mm outside the yoke. The iron yoke ends at \( z = -80 \) mm, the onset of the coil-end is at \( z = 0 \).

4. Integral Quantities of the Field Solutions

The calculation of stored energy and inductance is vital for the calculation of the quench behavior of superconducting magnets. The integration of the magnetic energy density in the entire domain is not practical as with the BEM-FEM coupling method the air region is not meshed at all. We will therefore derive more practical equations both for the linear and the nonlinear case. We will show that for the calculation of the voltage drop across the terminal during a quench the differential inductance has to be considered if the iron yoke is saturated at high field. This is not important for the LHC main dipoles but can become dominant for insertion quadrupoles.

4.1 Linear circuits

4.11 Stored energy

In the linear case, the stored magnetic energy in a volume \( V \) is given by the integral

\[
W = \frac{1}{2} \int_V \vec{H} \cdot \vec{B} \, dV.
\]  

(125)

Fig. 16: Measured and computed field components \( B_1 \) (left) and \( b_3 \) (right) as a function of the excitation (between injection and nominal field) averaged over the length of the measurement pick-up coil (200 mm).
Because of \( \text{div} (\vec{A} \times \vec{H}) = \vec{H} \cdot \text{curl} \vec{A} - \vec{A} \cdot (\text{curl} \vec{H}) \), Eq. (125) can be rewritten as:

\[
W = \frac{1}{2} \int_V \vec{H} \cdot \text{curl} \vec{A} \, dV = \frac{1}{2} \int_V \text{div} (\vec{A} \times \vec{H}) \, dV + \frac{1}{2} \int_V \vec{A} \cdot \text{curl} \vec{H} \, dV
\]

\[= \frac{1}{2} \oint (\vec{A} \times \vec{H}) \cdot d\vec{a} + \frac{1}{2} \int_V \vec{A} \cdot \text{curl} \vec{H} \, dV. \tag{126}\]

The term \( \frac{1}{2} \oint (\vec{A} \times \vec{H}) \cdot d\vec{a} \) vanishes on the far-field boundary as \( A \propto 1/r, \vec{H} \propto 1/r^2, \, d\vec{a} \propto r^2 \).

Inner surfaces (which have to be considered from both sides) are not contributing to the total energy as long as they carry no surface currents (\( \vec{J}_a = 0 \), which is true in case of finite conductivity and excitation without jump discontinuity). Proof:

\[
W = \frac{1}{2} \oint (\vec{A} \times \vec{H}) \cdot d\vec{a} = \frac{1}{2} \sum_i \int_{a_i} [\vec{A} \times (\vec{H}_2 - \vec{H}_1)] \cdot \vec{n}_2 \, da_i
\]

\[= \frac{1}{2} \sum_i \oint_{a_i} \vec{A} \cdot [(\vec{H}_2 - \vec{H}_1) \times \vec{n}_2] \, da_i = \frac{1}{2} \sum_i \oint_{a_i} \vec{A} \cdot \vec{J}_a \, da_i \tag{127}\]

which is zero for \( \vec{J}_a = 0 \). The magnetic energy can then be calculated with

\[
W = \frac{1}{2} \int_V \vec{A} \cdot \text{curl} \vec{H} \, dV = \frac{1}{2} \oint \vec{A} \cdot \vec{J} \, dV. \tag{128}\]

For the 2-dimensional calculation of coil cross-sections made of \( k \) multi-filamentary strands, the stored energy per unit length can be calculated as

\[
W/l = \frac{1}{2} \sum_k A_{z_k} I_k. \tag{129}\]

In this expression \( A_{z_k} \) refers to the vector potential due to the currents (other than the \( I_k \) in the strand) which produce the field \( \vec{B}_k \) in that strand and therefore neglects the magnetic energy in the strand.
4.12 The energy inside a strand

From Ampère’s law \( \oint \mathbf{H} \cdot d\mathbf{s} = \int \mathbf{J} \cdot d\mathbf{a} \) we get for a round wire with radius \( r_0 \) and current \( I \) the equation \( H = \frac{I}{2\pi r_0} \). For magnetically neutral materials of constant permeability, the energy in a cylinder of radius \( r \), thickness \( dr \) and length \( l \) is \( dW = \frac{1}{2} BH 2\pi r l dr = \frac{1}{2}\mu_0 H^2 2\pi r l dr = \frac{\mu_0 I^2}{4\pi r_0} r^3 dr \). Therefore the total energy in the strand is

\[
W = \int_0^{r_0} \frac{\mu_0 I^2}{4\pi r_0} r^3 dr = \frac{\mu_0 I^2}{16\pi}
\]

which is independent of the strand radius. For one aperture of the LHC main dipole the stored energy at 8.33 T (linear calculation with equivalent inner radius of the iron yoke of 98 mm) is 237 KJ/m. The inner layer consists of 15 turns (have to be considered \( \times 4 \) for one aperture), with each turn containing 2\( \times 18 \) strands with a current of 320 A each. The outer layer consists of 25 turns, with each turn containing 2\( \times 14 \) strands with a current of 411.5 A each. The energy stored in the strands can then be calculated to 4.3 J/m and can, indeed, be neglected.

4.13 Self and mutual inductance

With Eq. (128) it yields

\[
W = \frac{\mu_0}{8\pi} \int_V \int_{V'} \frac{\mathbf{J}(\mathbf{r}) \cdot \mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' dV.
\]

For a set of \( n \) closed current loops with current densities \( \mathbf{J}_i(\mathbf{r}), (i = 1, 2, \ldots, n) \), \( \mathbf{J}(\mathbf{r}) = \sum_{i=1}^{n} \mathbf{J}_i(\mathbf{r}) \) we get

\[
W = \sum_{i=1}^{n} \sum_{j=1}^{n} W_{ij} = \frac{\mu_0}{8\pi} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_V \int_{V'} \frac{\mathbf{J}_i(\mathbf{r}) \cdot \mathbf{J}_j(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' dV
\]

\[
= \frac{\mu_0}{8\pi} \sum_{i=1}^{n} \sum_{j=1}^{n} I_i I_j \int_V \int_{V'} \frac{\mathbf{J}_i(\mathbf{r}) \cdot \mathbf{J}_j(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' dV.
\]

With the mutual inductances defined as

\[
L_{ij} = \frac{\mu_0}{4\pi I_i I_j} \int_V \int_{V'} \frac{\mathbf{J}_i(\mathbf{r}) \cdot \mathbf{J}_j(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV' dV,
\]

Eq. (132) can be rewritten as

\[
W = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} L_{ij} I_i I_j.
\]

We see from Eq. (133) that the inductance only depends on the coil geometry. It also shows the symmetry \( L_{ij} = L_{ji} \). For \( i = j \) the coefficients (133) are called self-inductance. The inductances can be calculated directly using Eq. (133) or by calculating the stored energy and comparing with (134).

The self and mutual inductances can be derived by powering one single coil at a time (e.g., coil \( i \), containing \( k \) individual wires and total current \( I \)) and calculating the stored magnetic energy according to Eq. (129). This gives \( L_{ii} = \frac{2W_{\text{single}}}{I^2} \) (\( W_{\text{single}} = W_{ii} \)). Subsequently, powering any two coils \( i \) and \( j \) with the same current \( I \) yields (because of the symmetry of the mutual inductances) \( L_{ij} = \frac{1}{2} \left( \frac{2W_{\text{double}}}{I^2} - L_{ii} - L_{jj} \right) \) with \( W_{\text{double}} = W_{ij} + W_{ji} \).
4.14 The magnetic flux

In section 2.81 we showed that outside any current-carrying conductor the field can be represented through a magnetic scalar potential \( \vec{H} = -\nabla \Phi \). If we now assume one single closed loop of a wire with negligible cross-section (i.e., \( dV \) does not contain the conductor and hence the inner energy of it), the stored magnetic energy can be calculated by integrating

\[
W = \frac{\mu_0}{2} \int_V \vec{H}^2 dV = \frac{\mu_0}{2} \int_V (\nabla \Phi)^2 dV,
\]

where \( V \) is the (current free) external volume. Using the identity \( \text{div} (\Phi \nabla \Phi) = \Phi \nabla^2 \Phi + (\nabla \Phi)^2 \), considering that \( \nabla^2 \Phi = 0 \), and applying Gauss' theorem \( \int_V \text{div} \vec{g} dV = \int_A \vec{g} \cdot d\vec{a} \) yields:

\[
W = \frac{\mu_0}{2} \int_V (\nabla \Phi)^2 dV = \frac{\mu_0}{2} \int_V \Phi \nabla \Phi \cdot d\vec{a}.
\]

Now the surface is split up into a surface at infinite distance and the surface of the current loop \( a_s \) which, itself, consists of an upper (direction of \( \vec{n}_1 \)) and a lower surface (direction of \( \vec{n}_2 \)):

\[
W = \frac{\mu_0}{2} \int_{a_\infty} \Phi \nabla \Phi \cdot d\vec{a} + \frac{\mu_0}{2} \int_{a_s} \Phi \nabla \Phi \cdot d\vec{a}
\]

\[
= 0 + \frac{\mu_0}{2} \int_{a_s} [\Phi_1 \nabla \Phi \cdot \vec{n}_1 - \Phi_2 \nabla \Phi \cdot \vec{n}_2] da.
\]

Because of the continuity of \( B_n \) we get \( \mu_0 \nabla \Phi \cdot \vec{n}_1 = \mu_0 \nabla \Phi \cdot \vec{n}_2 = \mu_0 \nabla \Phi \cdot \vec{n} = -B_n \) and therefore

\[
W = \frac{\mu_0}{2} \int_{a_s} \nabla \Phi \cdot \vec{n} da = \frac{1}{2} \int_{a_s} (\Phi_2 - \Phi_1) B_n da.
\]

The surface of the current loop \( a_s \) can be regarded as a double layer of fictitious magnetic charges, on which \( \Phi_2 - \Phi_1 \) is constant, i.e., \( \Phi_2 - \Phi_1 = I \) and therefore

\[
W = \frac{1}{2} \int_{a_s} (\Phi_2 - \Phi_1) B_n da = \frac{1}{2} \int_{a_s} IB_n da = \frac{1}{2} I \psi.
\]

Comparing the result (139) with \( W = \frac{1}{2} L_1 I^2 = \frac{1}{2} (L_{11} I) I \) yields \( \psi = L_{11} I \). For multiple conductors we get from (139):

\[
W = \frac{1}{2} \sum_{i=1}^n I_i \psi_i.
\]

Because of Eq. (134) it follows that \( \psi_i = \sum_{j=1}^n L_{ij} I_j \). With Eq. (128) and Eq. (139) we also find for a single coil:

\[
\psi = \int_V \vec{A} \cdot \vec{J} dV.
\]

If the currents are time dependent, the induced voltage will be

\[
U_i = -\frac{d\psi_i}{dt} = -\sum_{j=1}^n L_{ij} \frac{dI_j}{dt}
\]

and for a single coil \( U = -L \frac{dI}{dt} \) which is also called a “back emf”.
4.2 Non-linear circuits

4.21 Magnetic energy

In the non-linear case, the increment of the magnetic energy in a volume \( V \) is given by
\[
\delta W = \int_V \vec{H} \cdot \delta \vec{B} \, dV.
\]
Because of \( \text{div} (\delta \vec{A} \times \vec{H}) = \vec{H} \cdot \text{curl} \delta \vec{A} - \delta \vec{A} \cdot \text{curl} \vec{H} \), Eq. (4.21) can be rewritten as
\[
\delta W = \int_V \delta \vec{A} \cdot \text{curl} \vec{H} \, dV.
\]

The term \( \int (\delta \vec{A} \times \vec{H}) \cdot d\vec{a} \) vanishes on the far-field boundary as \( A_1 / r \propto 1/r^2 \), \( H_1 / r \propto 1/r^2 \), \( d\vec{a} / r \propto r^2 \). Therefore
\[
W = \int_V \text{J} \cdot \delta \vec{A} \, dV.
\]

4.22 Self inductance in non-linear circuits

In the presence of iron parts with saturation dependent magnetization, we can still define the self inductance as
\[
L(I) = \frac{\psi(I, t)}{I(t)},
\]
with the flux linkage depending on \( I, t \) and the current \( I \) being a function of \( t \). Therefore the back emf. is
\[
U(t) = -\frac{d\psi(I, t)}{dt}.
\]

and the stored energy is
\[
W = \int_0^t I \, d\psi.
\]

With the total differential of \( \psi: d\psi = \frac{\partial \psi}{\partial t} \, dt + \frac{\partial \psi}{\partial I} \, dI \) it follows:
\[
|U(t)| = \frac{d\psi(I, t)}{dt} = \frac{\partial \psi(I, t)}{\partial t} + \frac{\partial \psi(I, t)}{\partial I} \frac{dI(t)}{dt} = L(I) \frac{dI(t)}{dt} + I(t) \frac{dL(I)}{dt} \frac{dI(t)}{dt} \frac{dI(t)}{dt}.
\]

If the induced voltage is measured during the ramping of the magnet, then the so-called differential inductance
\[
L_d(I) = L(I) + I \frac{dL(I)}{dI}
\]
is obtained and can be listed for the range of current values. The stored energy can then be calculated from the differential inductance with
\[
W = \int_0^{I_0} I L_d(I) \, dI.
\]
Fig. 18: Left: Magnetic vector potential in the iron yoke of the LHC insertion quadrupole (MQXA). Right: Self inductance in (mH/m) and differential inductance for the insertion quadrupole as a function of the excitational current.

Fig. 18 shows the vector potential in the iron yoke of the LHC insertion quadrupole MQXA and the self and differential inductance as a function of the excitational current. From Eq. (145), using the chain rule, one can easily verify that

\[ L_d(I) = \frac{d\psi(I)}{dI} . \]  

(151)

5. Superconductor Magnetization

In NbTi conductors, normal conducting deposits of a titanium-rich phase, dislocations and grain boundaries serve as pinning centers for the flux tubes. The pinning of flux tubes is instrumental in achieving high critical current densities. However, the very phenomenon that makes type II (hard) superconductors useful in high field magnets is responsible for magnetic hysteresis and thus for field dependent multipole errors. For the calculation of field errors in superconducting magnets due to filament magnetization a couple of “ingredients” are necessary which will be described in this section.

- A phenomenological model for filament magnetization including hysteresis modeling and taking into consideration the field dependence of the critical current density.
- The combination of these models (on the strand level) with methods of numerical field computation, for the consideration of iron saturation effects, and
- an iteration scheme to calculate the feedback of the magnetization on the field distribution within the coil.

5.1 The critical state model

According to the critical state model, Bean [5], a hard superconductor tries to expel a varying external field by generating a bipolar current distribution of the critical density \( J_c \), which depends on the local field level and the temperature. This phenomenological model takes into account that the maximum current density in the conductor is directly related to the maximum pinning force. Although the critical current
density decreases with field in all real superconductors, the Bean model assumes a field-independent critical current density to simplify the mathematical treatment of the magnetization problem. The limitations of the Bean model stem from the idealization of the electrical field versus current $\vec{E}(\vec{J})$ relation and from the fact that the explicit solution of the Maxwell equations are only possible for simple shapes of the superconductor.

Consider a superconducting slab (infinitely long in y and z-direction) which is supposed to be previously unexposed to a magnetic field and which is therefore said to be in the virgin state. An external field applied parallel to the slab creates a field inside the slab according to Ampère’s law (in one dimension)

$$\text{curl} \vec{H}_s = \frac{\partial}{\partial x} e_z = J_c e_z = \vec{J}_c.$$  \hspace{1cm} (152)

The slope of the field inside the slab is therefore equal to $J_c$, positive where $J_c$ is positive (directed out of the paper plane) and negative where $J_c$ is negative. At $H_p = J_c d/2$, where $d$ is the thickness of the slab, the entire slab is in the critical state and $H_p$ is called the penetration field.

The critical state model was originally derived for a slab of superconducting material and experimentally confirmed by Coffey [11] who mapped the field distribution in a test sample (with a little gap) of hard superconductors using a Hall probe. The slab model has been modified for cylindrical filaments by M. Wilson [42], c.f. fig. 19. Recall that a current distribution of intersecting ellipses creates an ideal dipole (shielding) field inside. Following Wilson [42] and Mess [28], and assuming a constant (shielding) current density, this current distribution is approximated by a shell with an elliptic inner boundary, see fig. 19. This boundary has the half axis $a$ equivalent to the filament radius and the small half axis

---

Fig. 19: The Bean model as modified by Wilson for cylindrical conductors. Notice the scaling of the critical current density with the strength of the external field $B$. To simplify the mathematical treatment, the current density is assumed to be constant within the cylinder but dependent on the external field, resulting in different slopes for the penetrating and trapped field inside the filament.
The contribution of two area elements \( dxdy \) at the locations \((x,y)\) and \((-x,y)\) to the shielding field is

\[
dB_s = -2\frac{\mu_0 J_c}{2\pi\sqrt{x^2 + y^2}} \cos \Theta dx dy
\]  

(153)

with \( \cos \Theta = x/\sqrt{x^2 + y^2} \) we get

\[
B_s = -\frac{\mu_0 J_c}{\pi} \int_{-a}^{a} \left[ \int_{v(y)}^{u(y)} \frac{x}{x^2 + y^2} dx \right] dy
\]  

(154)

where \( u(y) = b\sqrt{1 - \frac{y^2}{a^2}} = \frac{b}{a}\sqrt{a^2 - y^2} \) resulting from the equation of the ellipse and \( v(y) = \sqrt{a^2 - y^2} \) resulting from the equation of the circle. Integration yields:

\[
B_s = -\frac{2\mu_0 J_c a}{\pi} \left( 1 - \frac{b}{a} \arcsin \sqrt{1 - \frac{b^2}{a^2}} \right) - \sqrt{1 - \frac{b^2}{a^2}}
\]  

(155)

The maximum field that can be shielded from the center of the filament is called the **penetration field** where the current distribution resembles two half cylinders (small half axis of the ellipse \( b = 0 \)), see fig. 19 c.

\[
B_p(B,T) = \frac{\mu_0 J_c(B,T) d}{\pi},
\]  

(156)

with the filament diameter \( d \). Wilson proposes to use the following dependency of \( J_c \) on the (external) field at constant temperature which can be expressed by the relation [20]

\[
J_c(B) = \frac{J_0 B_0}{B_0 + |B|}
\]  

(157)

where \( J_0 \) and \( B_0 \) are fit constants determined by magnetization measurements \( (J_0 \approx 50kA/mm^2, B_0 \approx 0.29T) \). For the calculation of the resulting magnetic moment \( m \) \([A\cdot m^2]\) the above exercise can be repeated considering again the area elements and the magnetic moment generated by the resulting current loop:\n
\[
dm = -J_c 2x l \, dx \, dy \text{ and therefore}
\]

\[
m = -2J_c l \int_{-a}^{a} \left[ \int_{v(y)}^{u(y)} x \, dx \right] dy = -\frac{4}{3} J_c (1 - \frac{b^2}{a^2}) a^2 l
\]  

(158)

The magnetization (magnetic moment per unit volume) \([A/m]\) is \( M = \frac{m}{\pi a l} = -\frac{4}{3\pi} J_c (1 - \frac{b^2}{a^2}) a \). The magnetization of a strand containing only fully penetrated filaments can then be calculated to

\[
M_p(B,T) = -\frac{2}{3\pi} J_c(B,T) d
\]  

(159)

and has the same direction as the applied field ( \( \vec{M} = M \vec{B}/|B| \)), which varies over the cross-section of the magnet winding. With \( \lambda = 0.29 \) (the filling factor for the filamentary superconductor) and a filament diameter \( d = 6\mu m \), the magnetization \( \mu_0 M \) is about 9 mT at 0.53 T injection field level, where the critical current density of NbTi is 19000 A/mm². If the field is raised above the penetration field level, the shielding current distribution is maintained but with field penetrating into the filament and therefore \( M_p \) decreases proportionally to the critical current density.
5.2 Magnetization model with varying current density

The Wilson model has been modified, to allow for varying current densities within the filament. The filament is described as a set of nested intersecting ellipses with varying current density. Each layer produces a dipole screening field which screens the field inside the superconductor and therefore increases the critical current density in the inner layers, c.f. fig. 21.

5.21 The $J_c(B,T)$ dependence

The dependence of the critical current density on the modulus of the magnetic induction $B = |\vec{B}|$ is given by the following fit [4], where $B_c = B_{c0}(1 - (T/T_{c0})^{1.7})$:

$$ J_c(B,T) = \frac{J_{c0} B^{\alpha - 1}}{(B_c)^\alpha} \left( 1 - \frac{B}{B_c} \right)^\beta \left( 1 - \left( \frac{T}{T_{c0}} \right)^{1.7} \right)^\gamma. \tag{160} $$

The fit parameters for the LHC main-magnet cables are a critical current density at $4.2$ K and $5$ T of $J_{c0} = 3 \cdot 10^9$ A/m², an upper critical field $B_{c0} = 14.5$ T, a critical temperature of $T_{c0} = 9.2$ K, a normalization constant $C_0 = 27.04$ T and the fit parameters $\alpha = 0.57$, $\beta = 0.9$ and $\gamma = 2.32$.

For small magnetic inductions $B$, where the persistent currents influence the field quality most, Eq. (160) strives for infinity with $B^\alpha - 1 = B^{-0.43} \sim 1/\sqrt{B}$ for $B \rightarrow 0$. For the computation of the induction inside the filament, Eq. (160) is approximated around the actual value of the applied field $B_{out} = |\vec{B}_{out}|$ with the following function ($T$ is constant):

$$ J_c(B) \sim J_c(B_{out}) \frac{\sqrt{B_{out}}}{\sqrt{B}} \equiv \mathcal{F}(B_{out}) \frac{\sqrt{B}}{\sqrt{B}}. \tag{161} $$

5.22 The geometry and screening field of nested intersecting circles

Fig. 22 shows on the left the geometry of nested intersecting circles. For illustration the number of intersecting circles is $n = 3$. On the right the geometry of nested intersecting ellipses with $n = 5$ is displayed.

---

Fig. 20: Measured (blue) and calculated (red) hysteresis of a LHC strand subjected to a varying field with patterns of the persistent currents according to the critical state model as applied by Wilson to cylindrical filaments. It can be seen that the Wilson model fails to explain the "peak shifting" of the maximal magnetization at low field values.
The thickness $c$ of the current carrying shell is then constant for all circles $c = \frac{U}{n}$ and the radius of the $i$-th circle is $a_i = U - (2i - 1)\frac{c}{2}$. In case of intersecting ellipses the other semi-axis can be calculated by $b_i = a_i\frac{V}{U}$. The shielding field (denoted $t$) of two intersecting ellipses is constant in the aperture and can be calculated to $t = \mu_0 J_c c \frac{V}{2(U + V)}$, where $U$ and $V$ are the half-axes of the ellipses and $c$ is the displacement between them (which is identical with the thickness of the current layer on the $x$-axis). Easier relations result for the intersecting circles of radius $r$ where the shielding field is given by

$$t = 0.5\mu_0 J_c c r.$$  (162)
Let now \( q \) be the relative penetration parameter which is zero at the surface of the filament and equals one in the center. The uniform dipole field produced by two intersecting circles with opposite current densities shifted by the relative distance \( \Delta q = q_2 - q_1 \), can then be expressed as

\[
|\Delta \vec{r}| = \frac{\mu_0 r}{2} \int_{q_1}^{q_2} J_c(B(q)) \, dq, \tag{163}
\]

where \( \Delta \vec{r} \) is the shielding field, \( r \) is the filament radius and \( q_1, q_2 \) are the relative penetration parameters that limit the shielding current layer. Such pairs of circles are nested inside concentric circles. This equation will now be used to find a differential equation for the differential shielding \( d\vec{r}(dq) \). We get from (163) and (161):

\[
dB(q) = \xi \mu_0 H J_c(B(q)) \, dq = \frac{\xi \mu_0 H \mathcal{F}(B_{out})}{\sqrt{B(q)}} \, dq, \tag{164}
\]

where \( r \) is the filament radius. The geometry factor \( H = r/2 \) corresponds to the ideal screening field of two intersecting circles. As a refinement \( H = r(2 - 2 \ln 2) = 0.614 r \), correcting for the little spaces that are left when a round filament is filled with a series of intersecting circles (see fig. 22). The parameter \( \xi \) equals \(-1\) in case of ramping up and \( \xi = 1 \) for ramping down. In the first case, the orientation of the magnetic moment of the screening current is opposite to the orientation of the outside field \( B_{out} \) and \( B \) decreases inside the filament.

Equation (164) is a differential equation for \( B(q) \), considering the dependence of \( J_c \) on \( B(q) \), that can be solved with the known boundary condition \( B(0) = B_{out} \) in a closed analytical form, yielding:

\[
B(q) = \left( B_{out}^{3/2} + \frac{3}{2} \xi H \mathcal{F}(B_{out}) \mu_0 q \right)^{2/3} \tag{165}
\]

Figure 23 shows \( B(q) \) according to Eq. (165) and the dependence of the critical current density \( J_c(B(q)) \) on the penetration depth \( q \). The dotted line shows the magnetic induction for a constant current density \( J_c(B_{out}) \) and demonstrates the importance of using Eq. (165) instead. The magnetic induction at \( q = 0 \) equals the external field \( B_{out} \). The shown field distribution is a fully penetrated state, on the shielding branch reached after increasing the external field from negative field values to \( B_{out} = 0.08 \, \text{T} \) (\( \xi = -1 \)). As is shown in Figure 23, this results in a decreasing field \( B(q) \) along the penetration depth which produces an increase of \( J_c(B(q)) \) along \( q \). At \( B(q) = 0 \), the critical current density reaches its maximum value. There the strong increase of \( J_c \) produces a sharp decline of \( B(q) \). The course of \( J_c(q) \) shows the importance of expressing \( J_c \) as a function of \( q \) rather than assuming a constant value.

\subsection{The peak-shifting in the hysteresis curve}

From the expression for the magnetic induction \( B(q) \) inside the filament, the magnetization due to the radial slice of current \( J_c(q) \) between the penetrations \( q_i \) and \( q_{i+1} \) is derived. Individual slice magnetizations are needed to describe the hysteresis after changes of the ramp direction. Such a change (\( dB_{out}/dt \) changes sign) will produce a new layer of screening currents with opposite polarity (\( \xi \) switches sign). For small changes the new current layer will penetrate the filament only from \( q_1 = 0 \) to \( q_2 \leq 1 \) while the currents inside persist. The values for \( q_i \) are calculated using Eq. (165). For minor excitation loops, the magnetization is obtained as the superposition of \( n \) different layers,

\[
M = \sum_{i=1}^{n} M_i = \sum_{i=1}^{n} \int_{q_i}^{q_{i+1}} M_i(q) \, dq. \tag{166}
\]
In Eq. (166) \( M_i \) denotes the modulus of the magnetization \( \bar{M}_i \); it has negative values if the orientation is opposite to the external field. \( M \) is the magnetization density upon \( q \).

\[
M_i = \frac{4r\xi}{\mu_0\pi} \int_{q_i}^{q_{i+1}} J_c(B(q))(1 - q)^2 dq = \frac{4r\xi F}{\mu_0\pi} \int_{q_i}^{q_{i+1}} \frac{(1 - q)^2}{\sqrt{B(q)}} dq. 
\]  

Equation (167) can be solved analytically and (together with Eq. (165)) yields a closed expression for the filament magnetization:

\[
M_i = \frac{4r B(q)}{5\pi F^2 \mu_0^3 H^3} \left[ B_{out}^3 + \xi H F \mu_0 \left( 5 - 4q + \frac{5}{4}q^2 \right) \xi H F \mu_0 - (q - 4) B_{out}^3 \right]^{q = q_{i+1}}_{q = q_i} \]  

From the expressions for the magnetic induction \( B(q) \), Eq. (165), and the magnetization \( M \), eqs. (167) and (166), both parameters can be computed as a function of the penetration depth \( q \). Figure 24 (upper plot left), shows the values for increasing external fields \( B_{out} \) (\( \xi = -1 \)) for the virgin curve creating one layer extending from \( q_1 = 0 \) to \( q_2(B_{out}) \). Depending on \( B_{out} \), the field decreases until a certain penetration depth, where complete screening of the external field is obtained. The remaining part of the filament stays field free. In the lower plot the contribution \( m(q) \) of a slice \( dq \) to the total magnetization \( M \) is presented. The value of \( M \) can be obtained by integrating the presented curves (see indication on the plot).

Figure 24 upper right, illustrates the same quantities as on the left hand side, but for a filament that has already experienced a negative outside field before (different history) and hence is fully penetrated. Since the currents inside the superconductor persist, there is a remaining negative field \( B(q) \) inside, whereas in the case of the virgin curve the field remains zero for \( q > q_2(B_{out}) \). The lower plot in fig. 24 right, also explains why the maximum magnetization does not occur at \( B_{out} = 0 \): The magnetization is given by the integrated area \( M \) under the \( m(q) \) curve, which is biggest for small values of \( B_{out} \neq 0 \). This characteristic behavior has also been observed in measurements (see fig. 25), and is in good agreement with the calculations.

5.24 Calculation of \( B_{p1} \)

The magnetic induction \( B_{p1} \) denotes the value of the outside magnetic induction where the modulus of the filament magnetization passes through its first maximum when ramping up on the virgin curve (see

![Fig. 23: Magnetic induction B(q) as a function of the penetration depth q (continuous line). The dashed line denotes the current density Jc(q). The dotted line shows the magnetic induction for a calculation assuming a constant current density.](image-url)
fig. 25). Since the magnetization has been calculated in a closed analytical form (Eq. (168)), we can now derive the maximum of the virgin curve by solving $dM(B_{out})/dB_{out} = 0$. For the virgin curve the magnetization consists of only one layer, $n = 1$, that penetrates from $q_1 = 0$ to

$$q_2 = 2B_{out}^{3/2}/(\mathcal{H}\mathcal{F}(B_{out})\mu_0),$$

(169)

obtained by solving $B(q_2) = 0$. In this region of $B_{out} = B_{p1}$ we find that $\mathcal{F}'(B_{out}) \cong 0$ and hence we obtain

$$B_{p1} \cong (\mathcal{H}\mathcal{F}(B_{p1})\mu_0)^{2/3}(15 - 5\sqrt{5})^{1/3}/2,$$

(170)

$$q_2 \cong \sqrt{5/6} - 5\sqrt{5}/18 \cong 0.46.$$  

(171)

The recursive Eq. (170) yields a good estimate for $B_{p1}$ after few iterations. From Eq. (171) can be seen that the maximum modulus of the magnetization does occur at a penetration depth of $q_2 \cong 0.46$ rather than in the fully penetrated state. This fact is illustrated by the lower plot of fig. 24 right, where the area will be maximal for $q \rightarrow 0.46$ (solid line). Note, that the value of $q_2$ is independent of the critical current fit, provided $d\mathcal{F}(B_{out})/dB_{out} \cong 0$ (i.e. the critical current diverges with $1/\sqrt{B_{out}}$, for $B_{out} \rightarrow 0$, see Eq. (161)).

![Fig. 24](image)

Fig. 24: Left (top and bottom): Magnetic induction $B(q)$ and the magnetization contribution $m(q)$ as a function of the penetration depth $q$ for the virgin curve. Right (top and bottom): Course of the magnetic induction $B(q)$ and the magnetization contribution $m(q)$ as a function of the penetration depth $q$ for a filament already been exposed to a magnetic field before (non-virgin curve).
5.25 The hysteresis loop for a multifilamentary strand

Figure 25 presents computations of the filament magnetization according to Eq. (168) multiplied with the filling factor $\lambda$. The virgin curve and several hysteresis loops are displayed. The comparison of the calculation and the measured magnetization of a superconducting strand (dashed line) shows good agreement apart from the region of $B$ close to zero. There the difference between the magnetizations of one filament and a whole strand becomes significant. Since the outside field for each filament varies slightly due to the position in the strand cross-section and is additionally influenced by the field arising from the screening currents in the neighboring filaments, $B_{\text{out}}$ will be different for each filament according to its exact position. This results in a spread of filament magnetizations and hence in a smoothening of the region of $B$ close to zero. Since the Rutherford cables used in LHC magnets consist of many individual strands, this region will be smoothened out automatically due to the differences of $B_{\text{out}}$ at the individual strand positions within the coil.

![Computed magnetization curve for filament diameter of $r = 3.5 \mu m$, filling factor $\lambda = 1/2.95$, and temperature $T = 1.9 K$ compared with measurements from a superconducting strand [45].](image)

5.26 Strand magnetization in the LHC main dipole coil

The external field, seen by individual filaments, depends on their position in the coil geometry. Filaments in the outer layer of the coil (close to abscissa) experience low fields (dark blue regions in fig. 26, left), but high variations in the field if saturation effects cause a movement of this low field region within the coil. Filaments in the inner coil layer experience higher field (red and purple regions in fig. 26, left, which is increasing with inverse dependency to the positioning angle. The modulus of the superconducting filament magnetization in the coil cross-section is shown in fig. 26, right. Even at nominal field there are filaments in the coil cross-section remaining still non-fully penetrated.

5.3 Field errors due to the filament magnetization

The magnetic moment per unit length of a strand with cross-section $A$, $m/l = A \vec{M}$ can be represented by a small dipole of line-currents with the intensity $-I_s$ and $I_s$ spaced by a distance $S$ apart and located perpendicular to the field direction. The magnetic moment of such a single dipole $m/l = I_s \cdot S$ must equal the magnetic moment arising from persistent currents, i.e., $I_s = \frac{A}{s}$ where $S$ can be chosen as...
the strand diameter. For a strand surface of approx. \( A = 0.5 \text{ mm}^2 \) we get about 5 A, which compares to about 20 A of transport current in each strand at injection field level.

A more elegant method is to calculate the vector potential at a point \( \mathbf{r} \) from a magnetic moment at point \( \mathbf{r}' \) (for the coordinate system see fig. 6) using the identity

\[
A_z(\mathbf{r}) = \frac{\mu_0 m}{2\pi} \times \text{grad} \mathbf{r}' \times \ln \frac{|\mathbf{r} - \mathbf{r}'|}{a}
\]  

(172)

which holds for two-dimensional problems. With Eq. (57) and Eq. (59) we get

\[
\ln \frac{|\mathbf{r} - \mathbf{r}'|}{a} = \ln \left( \frac{r_i}{a} \right) - \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{r_0}{r_i} \right)^n \cos(n(\varphi - \theta)) .
\]  

(173)

and with the gradient in two dimensional cylindrical coordinates \( \text{grad} \mathbf{r}' = \frac{\partial}{\partial r_i} \mathbf{e}_{r'} + \frac{1}{r_i} \frac{\partial}{\partial \theta} \mathbf{e}_{\theta} \) it follows that

\[
\text{grad} \mathbf{r}' \times \ln \frac{|\mathbf{r} - \mathbf{r}'|}{a} = \frac{1}{r_i} \left[ (1 + \sum_{n=1}^{\infty} \left( \frac{r_0}{r_i} \right)^n \cos(n(\varphi - \theta)) \right) \mathbf{e}_{r'} + \sum_{n=1}^{\infty} \left( \frac{r_0}{r_i} \right)^n \sin(n(\varphi - \theta)) \mathbf{e}_{\theta} \right] .
\]  

(174)

Introducing this result in Eq. (172) and calculating the cross-product yields

\[
A_z = \frac{\mu_0}{2\pi r_i} \mathbf{e}_z \left[ m_{\mathbf{r}} \sum_{n=1}^{\infty} \left( \frac{r_0}{r_i} \right)^n \sin(n(\varphi - \theta)) - m_{\mathbf{\theta}} (1 + \sum_{n=1}^{\infty} \left( \frac{r_0}{r_i} \right)^n \cos(n(\varphi - \theta))) \right] .
\]  

(175)

With \( B_r(r_0, \varphi) = \frac{1}{r_0} \frac{\partial A_z}{\partial \varphi} \) and \( \sin(n\varphi - n\theta) = \sin n\varphi \cos n\theta - \cos n\varphi \sin n\theta \), \( \cos(n\varphi - n\theta) =

![Fig. 26: Left: Modulus of magnetic field in the coil. Right: Modulus of superconducting filament magnetization, both at about 1.3 T field level ramped from zero on the virgin curve.](image)
\[\cos n\phi \cos n\theta + \sin n\phi \sin n\theta,\] it follows that

\[B_r(r_0, \varphi) = \frac{\mu_0}{2\pi r_0 r_i} \left[ \sum_{n=1}^{\infty} \left( \frac{r_0}{r_i} \right)^n m \left( \cos n\phi \cos n\theta + \sin n\phi \sin n\theta \right) \right.\]

\[-m \theta \sum_{n=1}^{\infty} \left( \frac{r_0}{r_i} \right)^n \left( -\sin n\phi \cos n\theta + \cos n\phi \sin n\theta \right) \left. \right] \]

\[= \frac{\mu_0}{2\pi r_0 r_i} \left[ \sum_{n=1}^{\infty} n \left( \frac{r_0}{r_i} \right)^n (m_n \cos n\phi - m \theta \sin n\theta) \cos n\phi \right.\]

\[+ \left. \sum_{n=1}^{\infty} n \left( \frac{r_0}{r_i} \right)^n (m_n \sin n\phi + m \theta \cos n\theta) \sin n\phi \right].\]  

(176)

For the multipole coefficients we finally obtain

\[A_n = \frac{\mu_0}{2\pi} \frac{r_0^{n-1}}{r_i^n} n (m_n \cos n\phi - m \theta \sin n\theta), \quad (177)\]

\[B_n = \frac{\mu_0}{2\pi} \frac{r_0^{n-1}}{r_i^n} n (m_n \sin n\phi + m \theta \cos n\theta). \quad (178)\]

The contribution of the strand magnetization to the \(B_3\) field component is displayed in fig. 27.

Fig. 27: Left: Contribution of the strand current to the \(B_3\) field component. Right: Contribution of the strand magnetization to the \(B_3\) field component (both at about 1.3 T field level ramped from zero on the virgin curve and at 17 mm reference radius).

6. Summary

The design and optimization of superconducting accelerator magnets requires numerical methods which allow the accurate modeling (both in two and three dimensions) of the coil with its keystoned cables, wedges and insulation, and which limit the numerical errors to the effect from the iron yoke. Semi-analytical methods for the calculation of the excitational field from the coil combined with the BEM-FEM coupling method and higher order quadrilateral finite elements provide the required accuracy of the field solution. Although this report cannot provide in depth treatment of the fundamentals in field theory and numerical field computation, we have motivated the development of the CERN field-computation program ROXIE which is used as an approach towards an integrated design of superconducting magnets.
7. Appendix A: Feed-down of multipole components

In order to derive the transformation law for the field harmonics \( c_n \rightarrow c'_n \) \((c_n = b_n + i a_n)\) for a translation of the reference frame \( z \rightarrow z' : z = z' + \Delta z \) \((z = x + iy)\), we consider the field components to be identical in both coordinate systems.

\[
\sum_{n=1}^{\infty} c_n \left( \frac{z}{r_0} \right)^{n-1} = \sum_{n=1}^{\infty} c'_n \left( \frac{z'}{r_0} \right)^{n-1} = B_y + i B_x . \tag{179}
\]

Using the binomial series

\[
(\Delta z + z')^{n-1} = \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)! (k-1)!} (z')^{k-1} \Delta z^{n-k} , \tag{180}
\]

the left hand side of equation (179) becomes

\[
\sum_{n=1}^{\infty} c_n \left( \frac{z}{r_0} \right)^{n-1} = \sum_{n=1}^{\infty} \frac{c_n}{r_0^{n-1}} (z' + \Delta z)^{n-1}
\]

\[
= \sum_{n=1}^{\infty} \frac{c_n}{r_0^{n-1}} \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)! (k-1)!} (z')^{k-1} \Delta z^{n-k} . \tag{181}
\]

For the coefficients \( C_{k_0} \) of one particular power of \( z' \), e.g., \((z')^{k_0-1}\) we get

\[
C_{k_0} = \sum_{n=k_0}^{\infty} \frac{c_n}{r_0^{n-1}} \frac{(n-1)!}{(n-k_0)! (k_0-1)!} \Delta z^{n-k_0} \tag{182}
\]

\[
= \frac{1}{r_0^{k_0-1}} \sum_{n=k_0}^{\infty} c_n \frac{(n-1)!}{(n-k_0)! (k_0-1)!} \left( \frac{\Delta z}{r_0} \right)^{n-k_0} . \tag{183}
\]

Merging equations (182) and (181) yields (indices changed: \( n \rightarrow k \) and \( k_0 \rightarrow n \))

\[
\sum_{n=1}^{\infty} c_n \left( \frac{z}{r_0} \right)^{n-1} = \sum_{n=1}^{\infty} (z')^{n-1} C_n
\]

\[
= \sum_{n=1}^{\infty} (z')^{n-1} \frac{1}{r_0^{n-1}} \sum_{k=n}^{\infty} c_k \frac{(k-1)!}{(k-n)! (n-1)!} \left( \frac{\Delta z}{r_0} \right)^{k-n}
\]

\[
= \sum_{n=1}^{\infty} c'_n \left( \frac{z'}{r_0} \right)^{n-1} . \tag{184}
\]

Comparing the coefficients of \( \left( \frac{z'}{r_0} \right)^{n-1} \) leads to the transformation law for the complex field harmonics

\[
c'_n = \sum_{k=n}^{\infty} c_k \frac{(k-1)!}{(k-n)! (n-1)!} \left( \frac{\Delta z}{r_0} \right)^{k-n} . \tag{185}
\]

References


[27] Mathematica 4.1 is a trademark by Wolfram Research Inc.


[40] The LHC study group, Large Hadron Collider, The accelerator project, CERN/AC/93-03

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