Confinement versus Bose-Einstein condensation

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Abstract. The deconfinement phase transition at high baryon densities and low temperatures evades a direct investigation by means of lattice gauge calculations. In order to make this regime of QCD accessible by computer simulations, two proposals are made: (i) A Lattice Effective Theory (LET) is designed which incorporates gluon and diquark fields. The deconfinement transition takes place when the diquark fields undergo Bose-Einstein condensation. (ii) Rather than using eigenstates of the particle number operator, I propose to perform simulations for a fixed expectation value of the baryonic Noether current. This approach changes the view onto the finite density regime, but evades the sign and overlap problems. The latter proposal is exemplified for the LET: Although the transition from the confinement to the condensate phase is first order in the coupling constant space at zero baryon densities, the transition at finite densities appears to be a crossover.

INTRODUCTION

Lattice gauge simulations leave no doubt that Quantum Chromodynamics (QCD) exhibits a transition from the baryonic regime to the Quark Gluon Plasma (QGP) at high temperatures and small baryon densities. It is this high temperature regime which is currently under investigation at RHIC, Brookhaven [1] and which will be a major target of LHC, Cern. It is believed that a transition to the QGP also appears at high baryon densities and small temperatures. Very little is known about the latter transition from first principle simulations: lattice simulations at finite values of the baryonic chemical potential $\mu$ encounter a severe sign/overlap problem which limits their scope to the range of small $\mu$ [2,3].

At asymptotic baryon densities it is assumed that the quarks form a Fermi surface. In this case, perturbative gluon interactions support the existence of a diquark BCS state known as color superconductor [4,5]. At the present stage of research, no first principle results are available for the region of the QCD phase diagram where the transition from the baryonic phase to the QGP occurs at small temperatures and intermediate densities. Here, I will argue that the transition is driven by the Bose-Einstein condensation of diquarks. At high densities, the Bose-Einstein condensate gradually develops to a diquark BCS state.

In the present paper, two proposals are put forward to provide access to the finite density transition of QCD: (i) It is argued that the transition is within the reach of a Lattice Effective Theory (LET) which incorporates gluons and diquarks as dynamical degrees of freedom. At the stage of the present model, the baryonic current is entirely supported by the diquarks. (ii) As in the case of QCD, the LET also suffers from a sign problem at finite baryon chemical potential. In order to get first insights, I propose
to change the point of view: Rather than considering only eigenstates of the particle density operator, simulations are performed for a given expectation value of the baryonic Noether density. In the case of the LET, we will find that the finite density transition is a crossover rather than of first order.

LATTICE EFFECTIVE THEORY

Model building

The central assumption for describing the finite density transition is that only gluon and diquark degrees of freedom are relevant for the intermediate density region of the phase diagram. In the hadronic phase, diquarks are confined to a length scale of \( \approx 1 \) fm. Even if the transition is first order, the correlation length might become much larger than \( \approx 1 \) fm before the system is disordered by bubble nucleation. Therefore, the working hypothesis of the present approach is that the degrees of freedom relevant at the transition are gluons and point-like scalar (diquark) fields, i.e.,

\[
\phi^a(x) = \varepsilon^{abc} \varepsilon_{AB} q^b_{cA}(x) \gamma_5 q^c_B(x),
\]

where \( q^b_{cA}(x) \) are the charge conjugated quark fields, \( A, B = 1 \ldots 2 \) are flavor- and \( a, b, c = 1 \ldots 3 \) are color indices, respectively. The Effective Action, which should describe physics at the transition scale, is a SU(3) pure gauge theory supplemented with a scalar (Higgs, diquark) field which belongs to the the fundamental representation of the gauge group. The Lattice Effective Theory is modeled by the action

\[
S = -\frac{\beta}{3} \sum_{\mu < \nu, x} \text{tr} U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x) + \sum_x \phi^a(x)\phi^a(x) - \kappa \sum_{\mu, x} \phi(x)U_\mu(x)\phi^\dagger(x+\mu) - \text{h.c.} + \sum_x \lambda [\phi^2(x)]^2.
\]

Thereby, the gluon degrees of freedom are encoded by the link fields \( U_\mu(x) \), \( \beta \) is the usual prefactor of the Wilson action, which largely describes the gluon-dynamics, and \( \kappa \) is the Higgs hopping parameter which is related to the (bare) Higgs mass \( m \) by

\[
m^2 a^2 = 1 - 8 \kappa,
\]

where \( a \) is the lattice spacing. Turning off the gluon interaction (\( U_\mu = 1 \)), a Bose-Einstein condensate (BEC) is formed for \( \kappa > 1/8 \). Thereby, the BEC regime is stabilized by the Higgs quartic term (\( \lambda > 0 \)).

The \( SU(3) \) Higgs mechanism and residual confinement

Confinement effects in the gauged SU(2) Higgs model were extensively studied in [6, 7, 8]. Here, the SU(3) Higgs system will be investigated for the first time.
For a sufficiently large Higgs hopping parameter $\kappa$, we expect that the system passes into the phase of condensed diquarks. In order to realize the formation of a scalar expectation value, we are forced to fix the gauge degree of freedom: Since a gauge transformation $\Omega(x) \in SU(3)$ acts on the fields as

$$U^{\Omega}_{\mu}(x) = \Omega(x) U_{\mu}(x) \Omega^\dagger(x + \mu), \quad \phi^{\Omega}(x) = \Omega(x) \phi(x),$$

any residual gauge degree of freedom would wipe out the expectation value $\langle \phi(x) \rangle$. Here we choose the Minimal Landau Gauge, i.e.,

$$\sum_{\mu,x} \text{tr} U^{\Omega}_{\mu}(x) \xrightarrow{\Omega} \text{maximal}. \quad (4)$$

Note that the gauge constraint (4) leaves a global gauge transformation $\Omega(x)$ unfixed. The spontaneous breaking of this residual global gauge symmetry is signaled by a non-vanishing value $\langle \phi^{\Omega}(x) \rangle$ and marks the occurrence of the Higgs phase [7].

Which fields gain a mass due to the formation of the BEC of diquarks? What is the fate of the Would-Be Goldstone bosons? What is different for $SU(3)$ compared with the familiar $SU(2)$ Higgs mechanism? In order to answer these questions, let us invoke a semi-classical approach for the moment. Thereby, the scalar field is decomposed into a classical part and fluctuations, $\phi^{\Omega}(x) = \phi_c + \phi(x)$. After (minimal) Landau gauge fixing, we may choose without a loss of generality

$$\phi_c = v (1, 0, 0)^T; \quad SU(3) \rightarrow SU(2). \quad (5)$$

This implies that only a part of the global $SU(3)$ color group is broken by the condensate, and that a $SU(2)$ color symmetry remains intact. From the Higgs kinetic term, we detect the masses of the gluon fields $A^a_{\mu}, a = 1 \ldots 8$, i.e.,

$$[D_{\mu} \phi(x)]^\dagger D_{\mu} \phi(x) = \ldots + A^a_{\mu}(x) M^{ab} A^b_{\mu}(x), \quad M^{ab} = \frac{1}{2} \phi_c^\dagger \{t^a, t^b\} \phi_c,$$

where $D_{\mu}$ is the gauge covariant derivative, and $t^a, a = 1 \ldots 8$ are the generators of the $SU(3)$ algebra. Using (5), a direct calculation of the mass matrix $M^{ab}$ reveals that

$$A^1_{\mu}, A^2_{\mu}, A^3_{\mu} + \sqrt{3} A^8_{\mu}, A^4_{\mu}, A^5_{\mu} : \text{massive}; \quad A^6_{\mu}, A^7_{\mu}, A^3_{\mu} - \sqrt{3} A^8_{\mu} : \text{massless}.$$ 

Unless in the case of $SU(2)$, there are not enough Higgs fields to give a mass to all gluons. As expected, the gluons corresponding to the unbroken global $SU(2)$ color symmetry remain massless. The interesting question which solely arises in the context of $SU(3)$ (and which will be partially answered below) is whether color charges which transform under the invariant $SU(2)$ subgroup are still confined. Since the diquark field $\phi^1$ is built up from quarks of color 2 and 3 (see [1]), the integrity of $\phi^1$ as point-like particle would be preserved by confining effects throughout the BEC transition.

In order to explore the phase diagram of the Lattice Effective Theory of gluons and diquarks as function of $\beta$ and $\kappa$ ($\lambda$ will be a given number in the studies below), we need
an order parameter which detects the BEC phase. After installing the gauge condition \( (4) \), one might think to use \( \langle \phi_c \rangle \). The point is that in an ergodic lattice simulation each lattice configurations would generate a different direction for \( \phi_c \) implying that \( \langle \phi_c \rangle = 0 \) by virtue of the residual global gauge degree of freedom. Let us define

\[
v^a = \frac{1}{N_x} \sum_x \phi^a_\Omega(x),
\]

where \( N_x \) is the number of space-time points. The crucial observation is that in the BEC phase the fields \( \phi^\Omega(x) \) are (almost) uniquely oriented throughout space-time, i.e., \( v^2 = \mathcal{O}(1) \), while in the color unbroken phase \( v^2 \approx 0 \). This suggests to use

\[
\Phi^2 = \left\langle \frac{1}{N_x^2} \sum_a \left[ \sum_x \phi^\Omega_a(x) \right] \left[ \sum_y \phi^\Omega_a(y) \right] \right\rangle
\]

(7)

as the Litmus paper for the BEC transition. In the (global) color unbroken phase, the correlation length \( \xi \) of the gauged scalar fields is defined from the (disconnected) Green function by

\[
\langle \phi^\Omega_a(x) \phi^\Omega_a(y) \rangle \propto \exp \left\{ -|x-y|/\xi \right\} , \quad \xi \text{ finite}.
\]

(8)

We therefore find that \( \Phi^2 \) vanishes in the infinite volume limit, i.e.,

\[
\lim_{N_x \to \infty} \Phi^2 \approx \lim_{N_x \to \infty} \frac{\xi}{N_x} = 0,
\]

while \( \Phi^2 = \mathcal{O}(1) \) in the BEC phase.

**Numerical results**

The Lattice Effective Theory, corresponding to the action \( (2) \), can be simulated on a computer using a generalized version of the Cabibbo Marinari algorithm. Micro-canonical reflections concerning both, the scalar fields and the link fields, are employed to reduce autocorrelations. In this first investigation, neither a scaling analysis nor a study of the line of constant physics is pursued. The aim of the simulations was to reveal the underlying physics at a qualitative level. The numerical results were obtained for \( \lambda = 0.1 \) on \( 8^4 \) and \( 12^4 \) lattices. Landau gauge, see \( (4) \), is implemented using a standard iteration over-relaxation procedure.

The findings for the “diquark condensate” \( \Phi \) \( (7) \) are summarized in figure \( \textbf{1} \). The open symbols have been obtained on a \( 8^4 \) lattice, while the full symbols correspond to a \( 12^4 \) lattice.

We find that diquark condensation sets in if \( \kappa \) exceeds a critical value. This finding is a highly non-trivial result for the following reason: Landau gauge fixing is performed at the level of the link fields and implies the maximization of the gauge functional \( (4) \). In particular the iterative procedure used here only locates a local maximum, and repeating
the maximization on the same link configurations generically yields different gauge transformations \( \Omega(x) \), each of which generates a different Gribov copy of the links and scalar fields, respectively. This implies that an average over the Gribov copies within the first Gribov horizon is performed when the expectation value \( \Phi \) is calculated. Since the scalar field transforms homogeneously, the first guess would be that the correlation of the gauged scalar fields is destroyed by the average over the first Gribov horizon. The non-trivial result is that this is not the case: The scalar correlation length \( \xi \) is insensitive to the Gribov noise.

Having in mind that in the case without gluonic interactions Bose-Einstein condensation sets in for \( \kappa > 0.125 \), see (3), we here observe that the onset of the condensation is postponed to much larger values of \( \kappa \) due to gluonic interactions. A coexistence of the hadronic phase and the diquark BEC phase is not observed.

For \( \beta \) values as large as 5.8 (and for a \( 8^4 \) lattice), the system is in the deconfinement phase by virtue of temperature and volume effects. Figure 1 shows that the BEC transition changes form first order to second order (or higher) when \( \beta \) is increased. A possible explanation for this observation is: If the gluons are deconfined due to volume effects, only five of them acquire a mass by virtue of the SU(3) Higgs mechanism. Three gluons remain massless and give rise to the critical phenomenon with infinite correlation length. At small \( \beta \) values, the system is in the “hadronic” phase and the gluons possess a mass gap. If confinement persists for color states corresponding to the unbroken SU(2) subgroup, the three SU(2) gluons possess a mass gap due to confinement. The remaining five gluons are massive because of the Higgs mechanism. This would imply that there is no massless excitation which could give rise to a second order transition. This line of arguments favors the picture that color states of the residual SU(2) subgroup are still confined after the BEC transition.

**FIGURE 1.** The diquark condensate defined in (7) as function of the Higgs hopping parameter \( \kappa \) for several values \( \beta \) (for a definition of the parameters see (2)). The colored symbols indicate the positions in parameter space where finite density simulations will be carried out.
FINITE DENSITY DECONFINEMENT TRANSITION

Lattice gauge theory results

The generic approach to Yang-Mills thermodynamics at finite baryon densities is based on the introduction of a non-zero chemical potential. In the case of a SU(2) gauge group, the fermion determinant is real and can be included in the probabilistic measure. Numerical simulations can be performed by using standard algorithms, although this numerical approach consumes a lot of computer time due to the non-local nature of the action [9]. In the case of a SU(3) gauge group, the fermion determinant acquires imaginary parts for a non-vanishing chemical potential and cannot be considered to be part of the probabilistic measure. The most prominent example to circumvent this conceptual difficulty considers the fermion determinant as part of the correlation function to be calculated. Thereby, the probabilistic measure of zero-density Yang-Mills theory is used to generate the gauge field configurations. However, it turns out that this approach suffers from the so-called “overlap” problem implying that for realistic lattice sizes an unrealistic number of Monte-Carlo steps is necessary to achieve reliable results [10].

At the present stage, the scope of lattice QCD simulations is limited to the regime of small baryon densities. Two approaches have been proven to be fruitful: (i) the approach based upon a Taylor expansion with respect to the chemical potential $\mu$ around $\mu = 0$ [11]; (ii) the method employing simulations at imaginary chemical potential and finally seeking a continuation to real chemical potential [2]. Finally, I would like to mention a recently proposed technique where multi-parameter re-weighting is used in order to reduce the severeness of the overlap problem [12].

It is fair to say that a direct lattice study of the QCD phase transition at intermediate baryon densities (and small temperature) is not feasible at the present level of investigations.

A change of view

The solution of the sign/overlap problem in finite density lattice QCD probably requires new type of algorithms such as cluster algorithms or D-theory [13, 14]. Here, I would like to suggest to change the question of interest in a way which makes the problem solvable and which nevertheless sheds light onto the region of the QCD phase diagram where the finite density deconfinement transition takes place.

In order to put my proposal into the proper context, let me briefly review the origin of the sign problem. The standard question which we used to ask is: what is the ground state energy of the system if we only consider field configurations which are eigen states of the particle number operator $\hat{N}$:

$$E(B) = \langle \phi | \hat{H} | \phi \rangle, \quad \hat{N} \ | \phi \rangle = B \ | \phi \rangle,$$

(9)

where $\hat{H}$ is the Hamilton operator. A conversion of the latter formulation to a functional
integral setup usually involves the introduction of a chemical potential and generically leads to the sign problem.

Here, I propose to consider, instead of (9), the ground state energy where the fields on average possess a given particle number, i.e.,

\[ E(B) = \langle \phi | \hat{H} | \phi \rangle, \quad \langle \phi | \hat{N} | \phi \rangle = B. \] (10)

This approach certainly disregards certain features of the multi-fermion system; the hope is, however, that the approach sketches the deconfinement transition at high densities at least qualitatively. It is well known how to formulate the approach (10) in the functional integral language [15]. The quantity of interest is the effective action \( \Gamma \), which originates from the partition function by means of a Legendre transformation:

\[ Z[\mu] = \int \mathcal{D}\phi \exp \left\{ -S + \int d^4x \mu(x) \rho(x) \right\}, \] (11)

\[ \Gamma[\rho_c] = -\ln Z[\mu] + \int d^4x \mu(x) \rho_c(x), \quad \rho_c(x) = \frac{\delta \ln Z[\mu]}{\delta \mu(x)}, \] (12)

where \( \rho(x) \) is the zeroth component of \( U(1) \) Noether current corresponding to conserved baryon charge. Using a constant external source, i.e., \( \mu(x) = \mu \), the effective action is turned into the effective potential for a constant baryon density \( \rho_c \). The technical advantage of the approach to systems of finite (classical) density is that the additional factor \( \exp \int d^4x \mu(x) \rho(x) \) in (11) can be included to the action during the Monte-Carlo Updates leading to significant overlap with the finite density configurations.

**LET study of the finite density transition**

Let me stress that the classical density approach, outlined in the previous subsection, is applicable to lattice QCD. Nevertheless, such simulations involve the inclusion of dynamical quarks and are very time consuming. For this reason, we will here explore the method resorting to the Lattice Effective Theory (LET) discussed in the previous section.

In the present case, the baryon density is solely supported by the diquarks. The corresponding Noether density \( \rho(x) \) is given by the zeroth component of the current

\[ j_{\mu}(x) = \frac{1}{2i} \left[ \phi^\dagger(x) U_{\mu}(x) \phi(x + \mu) - \phi^\dagger(x + \mu) U^\dagger_{\mu}(x) \phi(x) \right]. \]

The action of the LET is then given by

\[ S_{\text{den}} = S - \mu \sum_x \rho(x), \] (13)

where the zero density part \( S \) is defined in (2). It is still feasible to modify the Cabibbo Marinari algorithm to include the finite \( \mu \) term making the lattice simulation straightforward.
FIGURE 2. Left Panel: The expectation value of the baryonic Noether current as function of the external source $\mu$. The blue crosses indicate the parameters with which the probability distribution of $\rho$ will be studied below. Right panel: Illustration of the baryonic matter contributing to $\rho_c$.

For the actual simulation, two regimes are of particular interest: (i) The large volume regime, where the gluons which belong to the residual unbroken SU(2) subgroup are confined. (ii) The small volume regime, where the Higgs transition (as function of the Higgs hopping parameter $\kappa$) is 2nd order. Here, I used the parameters (see figure 1, right panel for an illustration)

$$\beta = 5.5, \kappa = 0.24 \quad \text{(large volume regime)}$$

$$\beta = 6.0, \kappa = 0.18 \quad \text{(small volume regime)}$$

The results are presented in figure 2. We observe that the behavior of the expectation value of the baryonic Noether current, $\rho_c = \langle \rho \rangle$, as function of the overlap enhancing factor $\mu$ is qualitatively the same for both scenarios: $\langle \rho \rangle$ increases linearly with $\mu$ until a critical value $\mu_c$ is reached. For $\mu > \mu_c$, the linear dependence continues with a bigger slope. There is an intuitive understanding for this behavior (for an illustration see figure 2 right panel): At small $\mu$, we are in a “confined phase” where small color electric flux tubes connect static color sources until string breaking occurs at large distances. In this phase, the diquarks are bound to color singlet states which are in the present theory 3-diquark states or, equivalently, sexta-quark states. For $\mu > \mu_c$, the theory loses its confining capabilities; diquarks are no longer bound to color singlet sexta-quarks. The baryon number susceptibility has increased due to deconfinement. It is interesting to note that even the Higgs transition is of first order when the Higgs hopping term is varied, the finite density transition is more like a crossover which describes sexta-quarks dissolving into liberated diquarks.

Let us discuss how a particular expectation value $\rho_c = \langle \rho \rangle$ is realized in a lattice simulation for a given external parameter $\mu$. For this purpose, we ask: How big is the probability, $P(\rho) \ d\rho$, of finding $\rho$ in the interval $[\rho, \rho + d\rho]$ for an actual lattice configuration? The normalized distribution $dP/d\rho$ is shown in figure 3 for $\mu = 0.05$ (confined phase) and for $\mu = 0.12$ (deconfined phase). It is intuitive that the distribution is broader in the deconfined regime. Finally, I point out that simulations for a given value
of $\rho_c$ can be done without too much of a loss of statics: In this case, we would confine us to configurations with $\rho$ belonging to a bin where $dP/d\rho$ peaks.

**CONCLUSIONS**

In this paper, the proposal [7] to describe the deconfinement phase transition at high baryon density by an effective theory of diquarks and gluons is thoroughly investigated by means of lattice simulations. The dynamics of the Lattice Effective Theory (LET) is dictated by the action of a gauged SU(3) Higgs model where the scalar Higgs field simulates the point-like diquark.

In a first step the regime of vanishing baryon density is explored. For this purpose, a detailed lattice simulation of the SU(3) Yang-Mills theory with a scalar field in fundamental representation was performed for the first time. In the large volume limit a first order deconfinement transition occurs, if the Higgs hopping parameter exceeds a critical strength. The diquarks undergo Bose-Einstein condensation. It turns out that only five of the eight gluons acquire a mass by the SU(3) Higgs mechanism, and preliminary evidence was found that confinement with respect to the unbroken SU(2) subgroup is intact.

In order to gain first insights into the regime of intermediate baryon densities, I here propose to perform lattice simulations at a fixed expectation value $\rho_c$ of the baryonic Noether current. This procedure is outlined for the LET designed above. I stress, however,
that this approach is applicable to full QCD simulations. The practical benefit of this change of view onto the density regime is that severe overlap and sign problems are avoided. Overlap is ensured by an external parameter $\mu$ which couples to the Noether density.

At small $\mu$, the linear rise of $\rho_c$ with increasing $\mu$ is due to the population of the vacuum with color singlet bound states consisting of three diquarks (sexta-quarks). Above a critical value, deconfinement occurs, and the vacuum is populated by liberated diquarks. The transition appears to be a crossover. A significant signal of deconfinement is only visible in the baryon number susceptibility which rapidly increases at the transition.

More realistic LETs also incorporate the “valence quark” and give rise to baryons as bound states of diquarks and valence quarks. The investigation of these LETs are left to future work.

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