We investigate the stability of higher dimensional rotating black holes against scalar perturbations. In particular, we make a thorough numerical and analytical analysis of six-dimensional black holes, not only in the low rotation regime but in the high rotation regime as well. Our results suggest that higher dimensional Kerr black holes are stable against scalar perturbations, even in the ultra-spinning regime.

PACS numbers:

I. INTRODUCTION

Exact solutions to Einstein equations are extremely useful, specially if they describe simple yet physically attainable systems. Indeed, take for example the famous Schwarzschild metric: with this exact solution at hand, describing the geometry outside a spherically symmetric distribution of matter, one was able to compute the deflection of light as it passes near the Sun (and to match the theoretical prediction against the observational data), thereby giving a strong support to Einstein’s theory. We now know that the outside geometry of many astrophysical objects is well described by the Schwarzschild metric, and we can start studying them by investigating the properties of this metric. One of the most important things that one should study first is the classical stability of a given solution. In fact, if a solution is not stable, then it will most certainly not be found in nature, unless the instability timescale is much larger than the age of our universe. What does one mean by stability? In this classical context, stability means that a given initially bounded perturbation of the spacetime remains bounded for all times. For example, the Schwarzschild spacetime is stable against all kinds of perturbations, massive or massless. Thus, the Schwarzschild geometry is indeed appropriate to study astrophysical objects. On the other hand the Kerr spacetime, describing a rotating black hole, is stable against massless field perturbations but not against massive bosonic fields (although the instability timescale is typically much larger than the age of the Universe, thus presenting no real danger). In four dimensional (asymptotically flat) spacetime, the Kerr-Newman family is the most general black hole solution to Einstein equations, and, if we exclude massive bosons, they are all stable. However, unstable solutions seem to be more common than previously thought. Take for example anti-de Sitter (AdS) spacetimes. This has become a very popular background spacetime since it was conjectured that there is a duality between the gravitational degrees of freedom in the bulk of AdS space and a Conformal Field Theory formulated on the boundary of that space; this is the AdS/CFT correspondence conjecture. A black hole in this spacetime corresponds to a thermal state on the CFT. Are AdS black holes stable? Not all, as shown recently by Cardoso and Dias, who have proved that small Kerr-AdS black holes are classically unstable. This instability is due to a “black hole bomb” effect, whereby waves are successively amplified near the black hole event horizon and reflected at the boundary of the AdS spacetime (the special thing about AdS spacetime is that its boundaries, spatial infinity, works as a wall). If we now consider higher dimensional spacetimes, which are of interest to string theory and/or extra dimensional scenarios, instabilities seem to be much more common. For instance, even though higher dimensional Schwarzschild black holes are stable, their rotating counterparts seem not to be, at least for large rotation. Indeed, it was proved by Gregory and Laflamme that black branes are classically unstable against a sector of gravitational perturbations (the tensorial sector), and this result was used recently by Emparan and Myers to argue that ultra-spinning higher dimensional black holes should be similarly unstable (recall that for spacetime dimensions $D$ greater than 5, $D > 5$, there is no limit on the rotation parameter). Recently Cardoso and Lemos have uncovered a new universal instability for rotating black branes and strings, which holds for any massless field perturbation. The gist of their argument is that transverse dimensions in a black brane geometry act as an effective mass for the fields, which simulates a mirror enclosing a rotating black hole, thereby creating
a black hole bomb 17,18.

For other types of instabilities see for example 12,13.

Here, we shall investigate the stability against scalar perturbations of ultra-spinning black holes. In four and five dimensions, there is an upper bound for the rotation parameter \( a \) of a Kerr black hole 8, and when the black hole saturates that bound we say it is an extremal black hole. Now, it is known 14,15,16 that the characteristic frequencies (quasinormal frequencies, or QN frequencies) of four or five dimensional Kerr black holes always have a negative imaginary part (the field is decomposed according to \( \Psi \sim e^{-i\omega t}\Phi(r, \text{angles}) \)) so the Kerr spacetime is stable. However, as the black hole approaches extremality, the imaginary part of the QN frequency tends to zero, thus raising the possibility that if there was no upper bound on \( a \) the QN frequencies could have a positive imaginary part, or in other words, the spacetime could be unstable. This will be our main motivation for this study. We are not studying tensorial perturbations, so we shall not be dealing with Gregory-Laflamme type of instabilities. Instead, we are more interested in finding out what are the consequences, if any, of having arbitrarily large angular momentum for a black hole. We shall focus, for concreteness, on six-dimensional rotating black holes, but we suspect that the general features born out of this study are valid for any spacetime dimension greater than five. Previous work on related subject includes that of Ida et al 14, and Berti et al 17 who studied five dimensional Kerr black holes, in large and compact extra dimensions respectively. As we remarked, in five dimensions there is a bound on the rotation of the black hole, and thus these works are not suitable for studying possibly new phenomena appearing for unbound rotation parameter.

### II. FORMULATION OF THE PROBLEM AND BASIC EQUATIONS

#### A. The background metric

Here we adopt the notation of Ida et al 14, and we also correct some typos appearing in their equations. In four dimensions, there is only one possible rotation axis for a cylindrically symmetric spacetime, and there is therefore only one angular momentum parameter. In higher dimensions there are several choices of rotation axis and there is a multitude of angular momentum parameters, each referring to a particular rotation axis 7. Here we shall concentrate on the simplest case, for which there is only one angular momentum parameter, which we shall denote by \( a \). The metric of a \((4 + n)\)-dimensional Kerr black hole with only one non-zero angular momentum parameter is given in Boyer-Lindquist-type coordinates by 7

\[
g = \frac{\Delta - a^2 \sin^2 \vartheta}{\Sigma} dt^2 - \frac{2a(r^2 + a^2 - \Delta)}{\Sigma} \sin^2 \vartheta dtd\varphi + \left( r^2 + a^2 \right) \left( r^2 + a^2 - \mu r^2 \right) \sin^2 \vartheta d\vartheta d\Omega^2 \]

where

\[
\Delta = \omega \vartheta \sin^2 \vartheta, \quad \Delta = r^2 + a^2 - \mu r^2, \quad \Sigma = r^2 + a^2 \cos^2 \vartheta, \quad \Omega = r^2 + a^2 - \mu r^2, \quad \omega = \sqrt{r^2 + a^2 - \mu r^2}, \quad \mu = \frac{m}{2},
\]

and \( d\Omega^2 \) denotes the standard metric of the unit \( n \)-sphere. This metric describes a rotating black hole in asymptotically flat, vacuum space-time with mass and angular momentum proportional to \( \mu \) and \( \mu a \), respectively. Hereafter, \( \mu, a > 0 \) are assumed.

The event horizon is located at \( r = r_H \), such that \( \Delta|_{r=r_H} = 0 \), which is homeomorphic to \( S^{2+n} \). For \( n = 0 \), the standard 4-dimensional case, an event horizon exists only for \( a < \mu/2 \). When \( n = 1 \), an event horizon exists only when \( a < \sqrt{\mu} \), and the event horizon shrinks to zero-area in the extreme limit \( a \to \sqrt{\mu} \). On the other hand, when \( n \geq 2 \), which is the part of the parameter space which we shall focus on, \( \Delta = 0 \) has exactly one positive root for arbitrary \( a > 0 \). This means there is no bound on \( a \), and thus there are no extreme Kerr black holes in higher dimensions.

#### B. Separation of variables and boundary conditions

Consider now the evolution of a massless scalar field \( \Psi \) in the background described by 11. The evolution is governed by the curved space Klein-Gordon equation

\[
\frac{\partial}{\partial x^{\mu}} \left( \sqrt{-g} g^{\mu\nu} \frac{\partial}{\partial x^{\nu}} \Psi \right) = 0,
\]

where \( g \) is the determinant of the metric. The metric appearing in 11 should describe the geometry referring to both the black hole and the scalar field, but if we consider that the amplitude of \( \Psi \) is so small that its contribution to the energy content can be neglected, than the Kerr metric 11 should be a good approximation to \( g_{\mu\nu} \) in 4. We shall thus work in this perturbative approach. It turns out that it is possible to simplify considerably equation 4 if we separate the angular variables from the radial and time variables, as is done in four dimensions 13. This separation was accomplished, for higher dimensions, in 19 for five dimensional Kerr holes (who work with two spin parameters) and also in 20 for a general \( 4+n \)-dimensional Kerr hole. Since we are considering only one angular momentum parameter, the separation is somewhat simplified, and we can follow 19. In the end our results agree with the results in 19,20, if we consider only one angular momentum parameter in their equations.

We consider the ansatz \( \phi = e^{i\omega t - im\vartheta} R(r) S(y) Y(\Omega) \), and substitute this form in 11, where \( Y(\Omega) \) are hyperspherical harmonics on the \( n \)-sphere, with eigenvalues
given by \(-j(j + n - 1)\) (\(j = 0, 1, 2, \cdots\)). Then we obtain the separated equations

\[
\frac{1}{\sin \vartheta \cos^n \vartheta} \left( \frac{d}{d\vartheta} \sin \vartheta \cos^n \vartheta \frac{dS}{d\vartheta} \right) + \left[ \omega^2 a^2 \cos^2 \vartheta - m^2 \cos^2 \vartheta - j(j + n - 1) \sec^2 \vartheta + A \right] S = 0, \tag{5}
\]

and

\[
r^{-n} \frac{d}{dr} \left( r^n \Delta \frac{dR}{dr} \right) + \left\{ \frac{[\omega(r^2 + a^2) - ma]^2}{\Delta} - \frac{j(j + n - 1)a^2}{r^2} - \lambda \right\} R = 0, \tag{6}
\]

where \(\lambda := A - 2m\omega a + \omega^2 a^2\).

The equations (5) and (6) must be supplemented by appropriate boundary conditions, which are given by

\[
R \sim \begin{cases} (r - r_H)^i \sigma & \text{as } r \to r_H, \\ r^{-(n+2)/2} e^{-i\sigma} & \text{as } r \to \infty. \end{cases}
\]

(7)

where

\[
\sigma := \frac{(r_H^2 + a^2)\omega - ma}{(n-1)(r_H^2 + a^2) + 2r_H}, \tag{8}
\]

has been determined by the asymptotic behavior of the Eq. (5). In other words, the waves must be purely ingoing at the horizon and purely outgoing at the infinity. For assigned values of the rotational parameter \(a\) and of the angular index \(l, j, m\) there is a discrete (and infinite) set of frequencies called quasinormal frequencies, QN frequencies or \(\omega_{QN}\), satisfying the wave equation (5) with the boundary conditions just specified by Eq. (6). The QN frequencies are in general complex numbers, the imaginary part describing the decay or growth of the perturbation, because the time dependence is given by \(e^{-i\omega t}\). We expect the black hole to be stable against small perturbations, and therefore \(\omega_{QN}\) is expected to have a positive imaginary part, so that the perturbation decays exponentially as time goes by (recall that the time dependence of the wave function is \(e^{i\omega t}\)). As usual, we will order the QN frequencies \(\omega_{QN}\) according to the absolute value of their imaginary part: the fundamental mode (labeled by an integer \(n = 0\)) will have the smallest imaginary part (in modulus), and so on. We refer the reader to [21] and [22] for further details on QN frequencies.

III. NUMERICAL COMPUTATION

Now that we have a well posed problem, we have to solve for the characteristic QN frequencies. The most powerful method to date is that of Leaver [14], which makes use of a continued fraction representation, and which can determine the resonant frequency \(\omega\) and the separation constant \(A\) with very high accuracy. We assume the following series expansion for \(S\)

\[
S = (\sin \vartheta)^{|m|} (\cos \vartheta)^j \sum_{k=0}^{\infty} a_k (\cos^2 \vartheta)^k, \tag{9}
\]

which automatically satisfies the regular boundary conditions at \(\vartheta = 0, \pi/2\) whenever converges. Substituting this into Eq. (5), we obtain the three-term recurrence relations

\[
\alpha_0 \alpha_1 + \beta_0 \alpha_0 = 0, \quad \alpha_k \alpha_{k+1} + \beta_k \alpha_k + \gamma_k \alpha_{k-1} = 0, \quad (k = 1, 2, \cdots) \tag{10}
\]

where

\[
\alpha_k = -2(k + 1)(2j + n + 2k + 1), \quad \beta_k = (j + |m| + 2k)(j + n + |m| + 2k + 1) - A, \quad \gamma_k = -\omega_{QN}^2. \tag{11}
\]

Here, we have defined the dimensionless quantity \(\omega_{*} := \omega r_H\) and \(a_{*} := a/r_H\), since the behavior of the system depends only on \(a_{*}\). When \(a_{*} = 0\), the eigenvalue \(A\) is explicitly determined from the requirement that the series expansion ends within finite terms, since otherwise divergent. Thus we have

\[
A = (2\ell + j + |m|)(2\ell + j + |m| + n + 1) + O(\omega_{*} a_{*}), \quad (\ell = 0, 1, 2, \cdots) \tag{12}
\]

and the 0th-order eigenfunctions are given in terms of the Jacobi polynomials:

\[
P_{\ell jm} = (\sin \vartheta)^{|m|} (\cos \vartheta)^j \times F \left( -\ell, \ell + j + |m| + n + \frac{1}{2}, j + n + \frac{1}{2}; \cos^2 \vartheta \right). \tag{13}
\]

In a similar way, for the \(n > 1\) case, we expand the radial function \(R\) into the form

\[
R = e^{-\omega r} \left( \frac{r - r_H}{r_H} \right)^{i\sigma} \left( \frac{r}{r_H} \right)^{-\left(n+2\right)/2-i\sigma} \times \sum_{k=0}^{\infty} b_k \left( \frac{r - r_H}{r} \right)^k, \tag{14}
\]

where \(b_0\) is taken to be \(b_0 = 1\). If \(n = 2\), the expansion coefficients \(b_k\) in equation (14) are determined via the seven-term recurrence relation (it’s just a matter of substituting expression (12) in the wave equation (5)), given by

\[
\tilde{\alpha}_0 b_1 + \tilde{\beta}_0 b_0 = 0, \quad \tilde{\alpha}_1 b_2 + \tilde{\beta}_1 b_1 + \tilde{\gamma}_1 b_0 = 0, \quad \tilde{\alpha}_2 b_3 + \tilde{\beta}_2 b_2 + \tilde{\gamma}_2 b_1 + \tilde{\delta}_2 b_0 = 0, \quad \tilde{\alpha}_3 b_4 + \tilde{\beta}_3 b_3 + \tilde{\gamma}_3 b_2 + \tilde{\delta}_3 b_1 + \tilde{\varepsilon}_3 b_0 = 0, \quad \tilde{\alpha}_4 b_5 + \tilde{\beta}_4 b_4 + \tilde{\gamma}_4 b_3 + \tilde{\delta}_4 b_2 + \tilde{\varepsilon}_4 b_1 + \tilde{\zeta}_4 b_0 = 0, \quad \tilde{\alpha}_k b_{k+1} + \tilde{\beta}_k b_k + \tilde{\gamma}_k b_{k-1} + \tilde{\delta}_k b_{k-2} + \tilde{\varepsilon}_k b_{k-3} + \tilde{\zeta}_k b_{k-4} + \tilde{\eta}_k b_{k-5} = 0, \quad (k = 5, 6, \cdots) \tag{15}
\]

where

\[
\tilde{\alpha}_k = (1 + k)(1 + k + 2i\sigma)(3 + a_{*}^2)^2, \tag{16}
\]
\[ \tilde{\beta}_k = -18 - 36k^2 - 3\lambda - 27i\sigma + 36\sigma^2 + 18\omega_\ast + 2\omega_\ast^2 \]
\[ -9k(3 + 8i\sigma + 2i\omega_\ast) - 9\omega_\ast + 4\omega_\ast^2 \]
\[ -a_2^2(12 + 3j(j + 1) + 30k^2 + 2m^2 + \lambda + 21i\sigma) - 30\sigma^2 + 3k(20 + 4i\omega_\ast + 6\omega_\ast - 12\sigma\omega_\ast) \]
\[ -a_4^2(2 + j(j + 1) + 6k^2 + 4i\sigma - 6\sigma^2) + 2k(2 + 6i\sigma + i\omega_\ast) + i\omega_\ast - 2\sigma\omega_\ast + 2\omega_\ast^2 \]
\[ \tilde{\gamma}_k = -2m\omega_\ast a_3 + a_1^2(7 + 4j(j + 1) + 15k^2 - 10i\sigma - 15\sigma^2 - 2i\omega_\ast - 8\sigma\omega_\ast + 2k(5i + 15i + 4\omega_\ast)) \]
\[ + 3\{20k^2 + \lambda - 11i\sigma - 20\sigma^2 - 3i\omega_\ast - 12\omega_\ast - 2\sigma_\ast^2 + ik(11\lambda + 4\sigma + 12\omega_\ast) \}
\[ + a_2^2(30 + 9j(j + 1) + 62k^2 + m^2 + 2\lambda - 38i\sigma - 62\sigma^2 - 9i\omega_\ast - 30\sigma\omega_\ast - 6\omega_\ast^2 \]
\[ + 2ik(19i + 62i + 18\omega_\ast) \}, \]
\[ \tilde{\delta}_k = -66 - 54k^2 - \lambda + 10i\sigma + 54\sigma^2 \]
\[ + 6k(17 - 18i\sigma - 5i\omega_\ast) + 30i\omega_\ast + 30\sigma\omega_\ast + 4\omega_\ast^2 \]
\[ -2a_2^2\{14 + 3j(j + 1) + 10k^2 - 20i\sigma - 10\sigma^2 - 6i\omega_\ast - 6\sigma\omega_\ast - 6\omega_\ast^2 \]
\[ -2i(10i + 10i + 10\sigma + 3\omega_\ast) \}
\[ - a_2^2(90 + 10j(j + 1) + 68k^2 - \lambda + 132i\sigma - 68\sigma^2 - 40i\omega_\ast - 40\sigma\omega_\ast - 6\omega_\ast^2 \]
\[ + 4ik(33\lambda + 34\sigma + 10\omega_\ast) \}, \]
\[ \tilde{\epsilon}_k = 81 + 28k^2 - 9i\sigma - 28\sigma^2 - 21i\omega_\ast - 12\sigma\omega_\ast \]
\[ - \omega_\ast^2 + ik(91i + 56\sigma + 12\omega_\ast) \]
\[ + a_2^2(47 + 4j(j + 1) + 15k^2 - 50i\sigma - 15\sigma^2 \]
\[ - 14i\omega_\ast - 8\sigma\omega_\ast - 6\omega_\ast^2 + 2ik(21 + 15\sigma + 4\omega_\ast) \}
\[ + a_2^2(126 + 5j(j + 1) + 42k^2 - 138i\sigma - 42\sigma^2 \]
\[ - 35i\omega_\ast - 20\sigma\omega_\ast - 2\omega_\ast^2 \]
\[ + 2ik(60i + 42i + 10\omega_\ast) \}, \]
\[ \tilde{\zeta}_k = -(1 + a_2^2)\{44 + 8k^2 - 37\sigma^2 - 8\sigma^2 - 5i\omega_\ast - 2\sigma\omega_\ast \]
\[ + ik(37i + 16\sigma + 2\omega_\ast) \]
\[ + a_2^2\{34 + j(j + 1) + 6k^2 - 28i\sigma - 6\omega_\ast^2 \]
\[ - 5i\omega_\ast - 2\sigma\omega_\ast + 2ik(14i + 6\sigma + \omega_\ast) \}, \]
\[ \tilde{\eta}_k = (1 + a_2^2)^2(-3 + k + i\sigma)^2 \].

By making a Gaussian elimination four times, one can reduce the seven-term recurrence relations to the three-term recurrence relations, which is given by
\[ \tilde{\alpha}_0 b_1 + \tilde{\beta}_0 b_0 = 0, \]
\[ \tilde{\alpha}_k b_{k+1} + \tilde{\beta}_k b_k + \tilde{\gamma}_k b_{k-1}, \quad k = 1, 2, \ldots, \]
\[ \tilde{\alpha}_3 b_0 + \tilde{\beta}_3 b_0 = \tilde{\alpha}_4 b_1 + \tilde{\beta}_4 b_1 + \tilde{\gamma}_4 b_0. \]

For more details on how to obtain the coefficients of the three-term recurrence relation, we refer the reader to [23].

An eigenfunction satisfying the QN mode (QN) boundary conditions behaves at the event horizon and infinity as Eq. (9). Therefore, one can see that the expanded wave function satisfies the QNM boundary conditions if the expansion in Eq. (13) converges at spatial infinity. This convergence condition for the expansion, namely the QNM conditions, can be written in terms of the continued fraction as [14, 24]
\[ \tilde{\alpha}_0 \tilde{\beta}_0 \tilde{\alpha}_1 \tilde{\gamma}_1 \tilde{\alpha}_2 \tilde{\gamma}_2 \tilde{\alpha}_3 \tilde{\gamma}_3 \ldots \equiv \tilde{\beta}_0 - \frac{\tilde{\alpha}_0 \tilde{\gamma}_1}{\tilde{\beta}_1 - \frac{\tilde{\alpha}_1 \tilde{\gamma}_2}{\tilde{\beta}_2 - \frac{\tilde{\alpha}_2 \tilde{\gamma}_3}{\tilde{\beta}_3 - \ldots}}} = 0. \] (16)

where the first equality is a notational definition commonly used in the literature for infinite continued fractions. Here, we shall adopt such a convention. As for the determination of the separation constant \( A \), exactly the same technique of the continued fraction can be applied. The continued fraction equation for the separation constant is then given by
\[ \beta_0 - \frac{\alpha_0\gamma_1\alpha_1\gamma_2\alpha_2\gamma_3}{\beta_1 - \frac{\alpha_1\gamma_2\alpha_2\gamma_3}{\beta_2 - \frac{\alpha_2\gamma_3}{\beta_3 - \ldots}}} = 0. \] (17)

In order to obtain the QNMs, one has to solve numerically the two coupled algebraic equations, following for example the procedure in [27].

### IV. NUMERICAL RESULTS

Using the method described above, we have made an extensive search for the QN frequencies of six-dimensional rotating black holes, for several values of the parameters \( l, j, m \). The numerical results are summarized in Table I and in Figs. 12. To check our code we have first computed the QN frequencies of six-dimensional Schwarzschild black holes, and compared them with analytic WKB results [22]. The results are shown in Table I along with a computation of the error involved in the WKB calculation. Although for \( l = j = m = 0 \) the error is quite large (14% for the real part) it quickly decreases as \( l, j, m \) increase. We can thus say that the code is tested (or then that the WKB approximation yields good results...). The full numerical results for six-dimensional rotating black holes is presented in Figs. 12. In Fig. 1 we present the results referring to the real part of the fundamental QN frequency, as a function of \( a \), for several \( l, j, m \) values. The real part of \( \omega_{QN} \) seems to decrease monotonically as \( a/r_H \) increases, and for very large \( a/r_H \), it asymptotes to zero.

<table>
<thead>
<tr>
<th>( l )</th>
<th>( j )</th>
<th>( m )</th>
<th>( \omega_{QN} )</th>
<th>( \omega_{QN} )</th>
<th>( % ) Re</th>
<th>( % ) Im</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.8894+0.5331i</td>
<td>0.7862+0.5265i</td>
<td>13.6</td>
<td>1.2</td>
</tr>
<tr>
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<td>1</td>
<td>1</td>
<td>1.4465+0.5093i</td>
<td>1.3846+0.4933i</td>
<td>4.3</td>
<td>3.1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2.5791+0.4989i</td>
<td>2.5455+0.4942i</td>
<td>1.3</td>
<td>0.9</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3.1478+0.4973i</td>
<td>3.1205+0.4944i</td>
<td>0.9</td>
<td>0.6</td>
</tr>
</tbody>
</table>
 FIG. 1: Real part of the fundamental QN frequency as a function of the rotation parameter $a$ for some $l, j, m$ values. The maximum is reached at zero rotation, and as $a$ increases the real part of $\omega_{QN}$ decreases monotonically.

 FIG. 2: Imaginary part of the fundamental QN frequency as a function of the rotation parameter $a$ for some $l, j, m$ values. Notice that, for all values of $a$ the imaginary part is always positive, which means that even ultra-spinning black holes are stable.

In Fig. 2 we present the results referring to the imaginary part of the fundamental QN frequency, as a function of $a$, for several $l, j, m$ values. Although the pattern is more complex now, one can see that up to $a/r_H = 40$ the imaginary part of $\omega_{QN}$ is still positive, and thus the modes are stable. It proves very difficult to get numerical results for higher values of $a/r_H$, but if the trend continues, and we have no reason to believe otherwise, it looks like an instability will never set in. Results for higher overtones, both for the real and imaginary part follow a similar trend.

V. ANALYTICAL RESULTS

We concentrate on $n > 1$ (six and higher dimensions). Let us first re-write the radial wave equation in terms of dimensionless variables,

$$y^{-n}(y^n \Delta R')' + \left[ (\omega_s(y^2 + a^2) - m\omega_s)^2 - \frac{j(j + n - 1)a^2}{y^2} \right] R = 0,$$

(18)

where $y = r/r_H$ and $\Delta = \Delta_a = y^2 + a^2 - (1 + a^2)y^{1-n}$. The horizon is at $y = 1$ which is the real root of $\Delta = 0$.

In order to bring the radial wave equation into a Schrödinger-like form, it is convenient to introduce

$$\Psi(y) = y^{n/2}(g(y))^{1/4} \Delta R(y), g(y) = (y^2 + a^2)^2 - a^2 \Delta.$$

(19)

In terms of $\Psi$, the wave equation reads

$$-h(y)(h(y) \Psi')' + V(y)\Psi = (\omega_s - m\omega_s)^2 \Psi,$$

(20)

where

$$h(y) \equiv \frac{\Delta}{\sqrt{g(y)}}.$$ 

(21)

The potential is

$$\frac{g(y)}{\Delta} V(y) = A - 2m\omega_s \omega_s - m^2 a^2 \omega_s^2 + \frac{n(n + 2)}{4} + \frac{j(j + n - 1) + n/2(n/2 - 1)f}{y^2} + \frac{n^2(1 + a^2)}{4y^{n+1}} + m\omega_s \frac{y^2 - 1}{\Delta}[(m\omega_s - 2\omega_s)(y^2 + a^2) + ma] + \frac{1}{4} \left( \frac{5g''}{4g^2} \Delta + \frac{g'''}{g} \Delta + \frac{g'}{g} \Delta' \right),$$

(22)

and we have introduced the angular velocity of the horizon,

$$\Omega_H = \frac{\omega_s}{r_H} = \frac{a}{r_H + a^2}.$$ 

(23)
The potential vanishes at the horizon \((y = 1)\) and approaches a constant at infinity \((V \to m\Omega, (m\Omega - 2\omega_*)\) as \(y \to \infty\)). In general, it depends on the frequency \(\omega\). However, in the two extremal limits \(a \to 0\) (Schwarzschild limit) and \(a \to \infty\) (where one expects an instability to develop), the dependence on \(\omega\) drops out due to the vanishing of the angular velocity \((\Omega \to 0)\) in these limits.

In terms of the tortoise coordinate \(y_*\) defined by
\[
\frac{dy}{dy_*} = h(y),
\] (24)
the wave equation may be written as
\[
-\frac{d^2\Psi}{dy_*^2} + V[y(y_*)] \Psi = (\omega_* - m\Omega_*)^2 \Psi.
\] (25)
Let us consider the two extremal limits separately. In the limit \(a \to 0\), we obtain the Schwarzschild wave equation
\[
-\frac{d^2\Psi}{dy_*^2} + V_0[y(y_*)] \Psi = \omega_*^2 \Psi,
\] (26)
where
\[
V_0(y) = \left(1 - \frac{1}{y^{n+1}}\right) \left\{ \frac{L^2 - \frac{1}{4}}{y^2} + \frac{(n + 2)^2}{4ny^{n+3}} \right\},
\] (27)
and \(L = 2\ell + |m| + \frac{n+1}{2}\). To estimate the eigenfrequencies, expand around the maximum, \(y_{\text{max}}\) of the potential. Setting \(V_0'(y_{\text{max}}) = 0\), we obtain
\[
y_{\text{max}} = \left(\frac{n + 3}{2}\right)^{1/(n+1)} + o(1/L).
\] (28)
Expanding around the maximum, we may approximate the potential by
\[
V_0[y(y_*)] \approx \alpha^2 - \beta^2(y_* - y_*(y_{\text{max}}))^2,
\] (29)
where
\[
\alpha^2 = V_0(y_{\text{max}}),
\] (30)
and
\[
\beta^2 = -\frac{1}{2} \left. \frac{d^2V_0}{dy_*^2} \right|_{y_* = y_*(y_{\text{max}})} = -\frac{1}{2}(h(y_{\text{max}}))^2 V_0''(y_{\text{max}}).
\] (31)
Explicitly,
\[
\alpha^2 = \frac{n + 1}{n + 3} \left(\frac{2}{n + 3}\right)^{2/(n+1)} L^2 + o(1),
\] (32)
\[
\beta^2 = \frac{(n + 1)^3}{(n + 3)^2} \left(\frac{2}{n + 3}\right)^{4/(n+1)} L^2 + o(1).
\] (33)
The wave equation becomes
\[
-\Psi'' - \beta^2 x^2 \Psi = (\omega_*^2 - \alpha^2) \Psi, \quad x = y_* - y_*(y_{\text{max}}).
\] (34)
The solutions obeying the right boundary conditions at \(x \to \pm \infty\) are
\[
\Psi_N = H_N(\sqrt{i\beta x}) e^{i\beta x^2/2}, \quad N = 0, 1, 2, \ldots
\] (35)
where \(H_N\) are Hermite polynomials, with corresponding eigenvalues
\[
\omega_*^2 = \alpha^2 + 2i\beta(N + \frac{1}{2}).
\] (36)
Explicitly,
\[
\omega_* = C(n) \left\{ L + i\sqrt{n + 1}(N + \frac{1}{2}) \right\} + o(1/L),
\] (37)
with
\[
C(n) = \sqrt{\frac{n + 1}{n + 3} \left(\frac{2}{n + 3}\right)^{n+1}}.
\] (38)
This result is exactly what one gets by using a standard WKB approach.

Turning to the other extreme, \(a \to \infty\), we have
\[
\hat{\Delta} \approx \alpha^2 \left(1 - \frac{1}{y^{n-1}}\right),
\] (39)
\[
g(y) \approx \frac{\alpha^2}{y^{n-1}},
\] (40)
\[
h(y) \approx \frac{1}{y^{(n-1)/2}}
\] (41)
and the potential becomes to leading order in \(1/a\),
\[
V_{\infty}(y) = y^{n-3} \left(1 - \frac{1}{y^{n-1}}\right) \times \left\{ \left(\frac{j + n - \frac{1}{2}}{2}\right)^2 - \left(\frac{n - 3}{4}\right)^2 + \frac{(n + 1)^2}{16y^{n-1}} \right\},
\] (42)
where we assumed \(A \lesssim o(a^2)\). The wave equation has a well-defined limit as \(a \to \infty\). However, the potential is positive and diverges as \(y \to \infty\) for \(n > 3\), so subleading terms are needed to estimate the eigenfrequencies. For \(n \leq 3\) (six and seven dimensions), \(\omega\) approaches a constant value independent of \(a\) which is easily found by solving the Schrödinger equation. This asymptotic value only depends on \(j\).

In six dimensions \((n = 2)\), the potential exhibits a maximum and may be approximated by an inverted harmonic oscillator potential, as in the Schwarzschild limit. The frequencies can be found explicitly as functions of \(j\) taking advantage of the fact that the equation for the maximum \(V_\infty(y_{\text{max}}) = 0\) is quadratic. In Table [we list the QN frequencies as a function of \(j\), obtained using this analytical scheme for \(a \to \infty\).

\section{VI. CONCLUSIONS}

We have investigated numerically the stability of six-dimensional rotating Kerr black holes, with one rotation
TABLE II: In this Table we show the results of an analytical WKB type scheme for computing the QN frequencies in the ultra-spinning regime, $a \to \infty$. The results depend only on $j$. This scheme shows that $\omega_{QN}$ asymptotes to a constant value, which is consistent both qualitatively and quantitatively with the numerical results shown in Figures 11 and 12.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$\omega_{QN}/r_H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$0 + 0.162i$</td>
</tr>
<tr>
<td>1</td>
<td>$0.576 + 0i$</td>
</tr>
<tr>
<td>2</td>
<td>$1.078 + 0i$</td>
</tr>
</tbody>
</table>

parameter. Our results suggest that this geometry is stable against scalar field perturbations, even if the black hole is ultra-spinning. We thus rule out the possible existence of a new kind of instability for higher dimensional, ultra-spinning black holes. It would be interesting to check numerically or analytically the conjecture in [10], stating that ultra-spinning black holes should be unstable against gravitational perturbations (more specifically, they suggest that the Gregory-Laflamme [9] instability should be the cause). This, for the moment, is a major challenge specially because there is no known formalism to handle gravitational perturbations of higher dimensional Kerr black holes. Such a formalism could also prove useful in studying at depth the recently discovered instability for rotating black branes and strings [11].

**Acknowledgements**

V.C. acknowledges financial support from FCT through grant SFRH/BPD/2003. G.S. is supported in part by the US Department of Energy under grant DE-FG05-91ER40627. S.Y. is supported by the Grant-in-Aid for the 21st Century COE “Holistic Research and Education Center for Physics of Self-organization Systems” from the ministry of Education, Science, Sports, Technology, and Culture of Japan.