Three Dimensional Topological Field Theory induced from Generalized Complex Structure

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Abstract

We construct a three-dimensional topological sigma model which is induced from a generalized complex structure on a target generalized complex manifold. This model is constructed from maps from a three-dimensional manifold $X$ to an arbitrary generalized complex manifold $M$. The theory is invariant under the diffeomorphism on the world volume and the $b$-transformation on the generalized complex structure. Moreover the model is manifestly invariant under the mirror symmetry.

We derive from this model the Zucchini’s two dimensional topological sigma model with a generalized complex structure as a boundary action on $\partial X$. As a special case, we obtain three dimensional realization of a WZ-Poisson manifold.

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1 Introduction

A generalized complex structure is introduced by Hitchin [1] to unify the complex and symplectic geometry, and studied in detail by Gualtieri in [2].

It is found that Geometry of the $N = (2, 2)$ supersymmetric sigma model [3] has been formulated by a generalized Kähler geometry, which is a generalization of Kähler geometry to a generalized complex geometry. Noncommutative deformation of $N = 2$ supersymmetric sigma model on a Calabi-Yau manifold and its topological twisted sigma model can be described in terms of a generalized complex structure [4][5]. [6][7][8][9] have studied a realization of a generalized complex structure by a $N = (2, 2)$ supersymmetric sigma model and a topological sigma model. In [10][11][12], it has been investigated that the supersymmetric $SU(3)$ manifolds are generalized Calabi-Yau manifolds. Since mirror symmetry is correspondence of complex and symplectic manifolds, the generalized complex geometry is considered as a natural framework of mirror symmetry. Mirror symmetry on a generalized complex geometry is investigated in [13][14][15]. The relation of the generalized complex structure with the supersymmetric Poisson sigma model has been discussed in [16].

In this paper, we propose a realization of a generalized complex structure by a three dimensional topological sigma model (a topological membrane).

We consider that a three dimensional topological field theory is a natural framework to analyze a generalized complex structure. Since mirror symmetry exchange $A$ model and $B$ model, it seems to be natural that the model with a generalized complex structure has $M$-theoretic, or membrane theoretic realization. Integrability condition of a generalized complex structure is defined by a Courant bracket [17]. On the other hand, three dimensional topological sigma model naturally has a Courant algebroid structure [18][19]. We will find that a 3-form $H$ in a twisted generalized complex structure is naturally introduced in three dimensional topological sigma model.

Recently, Zucchini has constructed a two-dimensional topological sigma model induced from a target generalized complex structure as a generalization of a Poisson sigma model [20]. However a generalized complex structure is a sufficient but not necessary condition in his model. Our model solves this nonequivalence. Moreover we consider a direct relation of our model with the Zucchini’s model. If we consider a three dimensional world volume $X$ with a two dimensional boundary $\Sigma = \partial X$, we can derive the Zucchini action as a boundary action.
on $\Sigma$ of our model on $X$. As a special case, we can realize a WZ-Poisson sigma model [21] as a boundary action of a three dimensional topological sigma model.

The paper is organized as follows. In section 2, we summarize about a generalized complex structure. In section 3, we define a model induced from a generalized complex structure. We propose an action of a three dimensional topological sigma model with a generalized complex structure. We also consider the action with a twisted generalized complex structure. In section 4, we analyze the relation with the 3D nonlinear gauge theory. In section 5, we derive the Zucchi’s two dimensional model with a generalized complex structure as a boundary action of our model. Section 6 is conclusion and discussion.

2 Generalized Complex Structure

In this section, we summarize a generalized complex structure, based on description of section 3 in [7] and section 2 in [20].

Let $M$ be a manifold of even dimension $d$ with a local coordinate $\{\phi^i\}$. We consider the vector bundle $TM \oplus T^*M$. We consider a section $X + \xi \in C^\infty(TM \oplus T^*M)$ where $X \in C^\infty(TM)$ and $\xi \in C^\infty(T^*M)$.

$TM \oplus T^*M$ is equipped with a natural indefinite metric of signature $(d, d)$ defined by

$$
\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(i_X \eta + i_Y \xi),
$$

for $X + \xi, Y + \eta \in C^\infty(TM \oplus T^*M)$, where $i_V$ is an interior product with a vector field $V$. In the Cartesian coordinate $(\partial/\partial \phi^i, d\phi^i)$, we can write the metric as follows

$$
\mathcal{I} = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix},
$$

with $X + \xi, Y + \eta \in C^\infty(TM \oplus T^*M)$, where $\mathcal{L}_V$ denotes Lie derivation with respect a vector field $V$ and $d_M$ is the exterior differential of $M$. This bracket is antisymmetric but do not satisfy the Jacobi identity. We may define a so called Dorfman bracket as follows

$$
(X + \xi) \circ (Y + \eta) = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2}d_M(i_X \eta - i_Y \xi),
$$

for $X + \xi, Y + \eta \in C^\infty(TM \oplus T^*M)$. We define a Courant bracket on $TM \oplus T^*M$,

$$
[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2}d_M(i_X \eta - i_Y \xi),
$$

with $X + \xi, Y + \eta \in C^\infty(TM \oplus T^*M)$, where $\mathcal{L}_V$ denotes Lie derivation with respect a vector field $V$ and $d_M$ is the exterior differential of $M$. This bracket is antisymmetric but do not satisfy the Jacobi identity. We may define a so called Dorfman bracket as follows

$$
(X + \xi) \circ (Y + \eta) = [X, Y] + \mathcal{L}_X \eta - i_Y d\xi,
$$

for $X + \xi, Y + \eta \in C^\infty(TM \oplus T^*M)$. We define a Courant bracket on $TM \oplus T^*M$,
which satisfies the Jacobi identity but is not antisymmetric. Antisymmetrization of a Dorfman bracket coincides with a Courant bracket.

A generalized almost complex structure $\mathcal{J}$ is a section of $C^\infty(\text{End}(TM \oplus T^*M))$, which is an isometry of the metric $\langle \cdot, \cdot \rangle$, $\mathcal{J}^*\mathcal{J} = \mathcal{I}$, and satisfies

$$\mathcal{J}^2 = -1. \quad (5)$$

A $b$-transformation is an isometry defined by

$$\exp(b)(X + \xi) = X + \xi + i_X b, \quad (6)$$

where $b \in C^\infty(\wedge^2 T^*M)$ is a 2–form. A Courant bracket is covariant under the $b$-transformation

$$[\exp(b)(X + \xi), \exp(b)(Y + \eta)] = \exp(b)[X + \xi, Y + \eta], \quad (7)$$

if the 2–form $b$ is closed. The $b$-transform of $\mathcal{J}$ is defined by

$$\hat{\mathcal{J}} = \exp(-b)\mathcal{J} \exp(b). \quad (8)$$

$\mathcal{J}$ has the $\pm \sqrt{-1}$ eigenbundles. Therefore we need complexification of $TM \oplus T^*M$, $(TM \oplus T^*M) \otimes \mathbb{C}$. The projectors on the eigenbundles are constructed by

$$\Pi_\pm = \frac{1}{2}(1 \mp \sqrt{-1}\mathcal{J}). \quad (9)$$

The generalized almost complex structure $\mathcal{J}$ is integrable if

$$\Pi_\pm[\Pi_\pm(X + \xi), \Pi_\pm(Y + \eta)] = 0, \quad (10)$$

for any $(X + \xi), (Y + \eta) \in C^\infty(TM \oplus T^*M)$, where the bracket is the Courant bracket. Then $\mathcal{J}$ is called a generalized complex structure. Integrability is equivalent to the single statement

$$N(X + \xi, Y + \eta) = 0, \quad (11)$$

for all $X + \xi, Y + \eta \in C^\infty(TM \oplus T^*M)$, where $N$ is the generalized Nijenhuis tensor defined by

$$N(X + \xi, Y + \eta) = [X + \xi, Y + \eta] - [\mathcal{J}(X + \xi), \mathcal{J}(Y + \eta)] + \mathcal{J}[\mathcal{J}(X + \xi), Y + \eta] + \mathcal{J}[X + \xi, \mathcal{J}(Y + \eta)]. \quad (12)$$
The $b$-transform $\hat{J}$ of a generalized complex structure $J$ is a generalized complex structure if the 2–form $b$ is closed.

We decompose a generalized almost complex structure $J$ in coordinate form as follows

$$J = \begin{pmatrix} J & P \\ Q & -J^* \end{pmatrix},$$

where $J \in C^\infty(TM \otimes T^*M)$, $P \in C^\infty(\wedge^2 TM)$, $Q \in C^\infty(\wedge^2 T^*M)$.

Then the condition $J^2 = -1$ is as follows:

$$J^i_k J^k_j + P^{ik} Q_{kj} + \delta^i_j = 0, \quad J^i_k P^{kj} + J^j_k P^{ki} = 0, \quad Q_{ik} J^k_j + Q_{jk} J^k_i = 0,$$

where

$$P^{ij} + P^{ji} = 0, \quad Q_{ij} + Q_{ji} = 0.$$  \hfill (15)

The integrability condition (10) is equivalent to the following condition

$$A^{ijk} = B^{ijk} = C_{ijk} = D_{ijk} = 0,$$  \hfill (16)

where

$$A^{ijk} = P^{kl} \partial_l P^{jk} + P^{jl} \partial_l P^{ki} + P^{kl} \partial_l P^{ij},$$
$$B^{ijk} = J^l_i \partial_l P^{jk} + P^{lk} (\partial_l J^k_i - \partial_l J^k_j) + P^{kl} \partial_l J^j_i - \partial_l J^j_k P^{lk},$$
$$C^{ijk} = J^l_i \partial_l J^k_j - J^l_j \partial_l J^k_i - J^k_i \partial_l J^l_j + J^k_j \partial_l J^l_i + P^{kl} (\partial_l Q_{ij} + \partial_l Q_{ji} + \partial_l Q_{li}),$$
$$D_{ijk} = J^l_i (\partial_l Q_{jk} + \partial_l Q_{lj}) + J^l_j (\partial_l Q_{ki} + \partial_l Q_{lk}) + J^l_k (\partial_l Q_{ij} + \partial_l Q_{lj}) - Q_{ji} \partial_l J^l_k - Q_{kj} \partial_l J^l_i - Q_{li} \partial_l J^l_j.$$  \hfill (17)

Here $\partial_i$ is a differentiation with respect to $\phi^i$. The $b$–transform is

$$\hat{J}^i_j = J^i_j - P^{ik} b_{kj},$$
$$\hat{P}^{ij} = P^{ij},$$
$$\hat{Q}_{ij} = Q_{ij} + b_{ik} J^k_j - b_{jk} J^k_i + P^{kl} b_{ki} b_{lj}.$$  \hfill (18)
where \( b_{ij} + b_{ji} = 0 \).

The usual complex structures \( J \) is embedded in generalized complex structures as the special form

\[
J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}.
\]  

(19)

Indeed, one can check this form satisfies conditions, (14) and (16) if \( J \) is a complex structure. Similarly, the usual symplectic structures \( Q \) is obtained as the special form of generalized complex structures

\[
J = \begin{pmatrix} 0 & -Q^{-1} \\ Q & 0 \end{pmatrix}.
\]  

(20)

This satisfies (14) and (16) when \( Q \) is a symplectic structure, i.e. it is closed. Other exotic examples exist. There exists manifolds which cannot support any complex or symplectic structure, but admit generalized complex structures.

The Courant bracket on \( TM \oplus T^*M \) can be modified by a closed 3–form. Let \( H \in C^\infty(\wedge^3 T^*M) \) be a closed 3–form. We define the \( H \) twisted Courant brackets by

\[
[X + \xi, Y + \eta]_H = [X + \xi, Y + \eta] + i_X i_Y H,
\]  

(21)

where \( X + \xi, Y + \eta \in C^\infty(TM \oplus T^*M) \). Under the \( b \)-transform with \( b \) a closed 2–form,

\[
[\exp(b)(X + \xi), \exp(b)(Y + \eta)] = \exp(b)[X + \xi, Y + \eta],
\]  

(22)

holds with the brackets \([,] \) replaced by \([,]_H \). For a non closed \( b \), one has

\[
[\exp(b)(X + \xi), \exp(b)(Y + \eta)]_{H - d_M b} = \exp(b)[X + \xi, Y + \eta]_H.
\]  

(23)

So, the \( b \)-transformation shifts \( H \) by the exact 3–form \( d_M b \):

\[
\tilde{H} = H - d_M b.
\]  

(24)

One can define an \( H \) twisted generalized Nijenhuis tensor \( N_H \) as follows

\[
N(X + \xi, Y + \eta) = [X + \xi, Y + \eta]_H - [\mathcal{J}(X + \xi), \mathcal{J}(Y + \eta)]_H + \mathcal{J}[\mathcal{J}(X + \xi), Y + \eta]_H + \mathcal{J}[X + \xi, \mathcal{J}(Y + \eta)]_H,
\]  

(25)
by using the brackets $[,]_H$ instead of $[,]$. A generalized almost complex structure $\mathcal{J}$ is $H$ integrable if

$$N_H(X + \xi, Y + \eta) = 0,$$

(26)

for all $X + \xi, Y + \eta \in C^\infty(TM \oplus T^*M)$. Then we call $\mathcal{J}$ an $H$ twisted generalized complex structure.

The $H$ integrability conditions is obtained as

$$A_H^{ijk} = B_H^{ijk} = C_H^{ijk} = D_H^{ijk} = 0,$$

(27)

where

$$A_H^{ijk} = A^{ijk},$$
$$B_H^{ijk} = B_i^{jk} + P^j P^{km} H_{ilm},$$
$$C_H^{ijk} = C_{ij}^k - J^l_i P^{km} H_{jlm} + J^l_j P^{km} H_{ilm},$$
$$D_H^{ijk} = D_{ijk} - H_{ijk} + J^l_i J^m_j H_{klm} + J^l_i J^m_k H_{ilm} + J^l_k J^m_i H_{jlm}. $$

(28)

3 Action and Symmetry

Let $X$ be a three-dimensional manifold and $M$ be a target manifold of a smooth map $\phi : X \to M$ with local coordinate expression $\{\phi^i\}$. We also have a vector bundle $TM \oplus T^*M$ over $M$. We introduce $A^i$ as a section of $T^*X \otimes \phi^*(TM)$, and $B_{1i}$ as a section of $T^*X \otimes \phi^*(T^*M)$. We further introduce $B_{2i}$ as a section of $\wedge^2 T^*X \otimes \phi^*(T^*M)$.

We propose the following action induced from a generalized complex structure:

$$S = S_0 + S_1;$$
$$S_0 = \int_X -B_{2i} d\phi^i + B_{1i} dA^i,$$
$$S_1 = \int_X -J^l_i B_{2i} A^i - P^{ij} B_{2i} B_{1j} + \frac{1}{2} \frac{\partial Q_{jk}}{\partial \phi^i} A^i A^j A^k$$
$$+ \frac{1}{2} \left( - \frac{\partial J^k_i}{\partial \phi^j} + \frac{\partial J^k_j}{\partial \phi^i} \right) A^i A^j B_{1k} + \frac{1}{2} \frac{\partial P^{jk}}{\partial \phi^i} A^i B_{1j} B_{1k}. $$

(29)

(29) is represented as

$$S_0 = \int_X -\langle 0 + B_2, d(\phi + 0) \rangle + \frac{1}{2} \langle A + B_1, d(A + B_1) \rangle + \text{total derivative},$$
$$S_1 = \int_X -\langle 0 + B_2, \mathcal{J}(A + B_1) \rangle - \frac{1}{2} \langle A + B_1, A^i \frac{\partial \mathcal{J}}{\partial \phi^i} (A + B_1) \rangle. $$

(30)
We can confirm that if $J$, $P$ and $Q$ are components of a generalized complex structure (13), the action (29) is invariant under the following gauge transformation

$$
\delta \phi^i = J^j_i c^j + P^{ij} t_{ij},
\delta A^i = dc^i - P^{ij} t_{2j} + \left( -\frac{\partial J^i_k}{\partial \phi^j} + \frac{\partial J^j_k}{\partial \phi^i} \right) A^i c^k + \frac{\partial P^{ki}}{\partial \phi^j} (A^j t_{1k} - c^j B_{1k}),
\delta B_{1i} = dt_{1i} - J^j_i t_{2j} + \left( \frac{\partial Q_{jk}}{\partial \phi^i} + \frac{\partial Q_{ki}}{\partial \phi^j} + \frac{\partial Q_{ij}}{\partial \phi^k} \right) A^j c^k + \frac{\partial P^{jk}}{\partial \phi^i} (A^i t_{1k} - c^i B_{1k}),
\delta B_{2i} = dt_{2i} - \frac{\partial J^k_j}{\partial \phi^i} (B_{2j} c^k + t_{2j} A^k) - \frac{\partial P^{jk}}{\partial \phi^i} (B_{2j} t_{1k} + t_{2j} B_{1k})
+ \frac{\partial}{\partial \phi^i} \left( \frac{\partial Q_{kl}}{\partial \phi^j} + \frac{\partial Q_{lj}}{\partial \phi^k} + \frac{\partial Q_{jk}}{\partial \phi^l} \right) A^j A^k c^l + \frac{1}{2} \frac{\partial}{\partial \phi^j} \left( -\frac{\partial J^k_m}{\partial \phi^i} + \frac{\partial J^m_k}{\partial \phi^i} \right) (A^j B_{1k} t_{1m} + 2 A^j c^k B_{1m})
+ \frac{1}{2} \frac{\partial^2 P^{kl}}{\partial \phi^j \partial \phi^i} (2 A^j B_{1k} t_{1l} + c^j B_{1k} B_{1l}),
$$

where $c^i$ and $t_{1i}$ are 0-form gauge parameters, and $t_{2i}$ is a 1-form gauge parameter. We call this model the three dimensional generalized complex sigma model. Precisely speaking, the action (29) is gauge invariant if and only if the condition

$$
\mathcal{A}^{ijk} = B_{i}^{jk} = C_{ij}^{k} = 0,
\frac{\partial \mathcal{D}^{ijk}}{\partial \phi^i} + (ijkl \text{ cyclic}) = 0,
$$

is satisfied.

We can extend the model in case of a twisted generalized complex structure $H \neq 0$. We define the following action as a three dimensional topological field theory induced from a twisted generalized complex structure:

$$
S_H = S_{H0} + S_{H1};
S_{H0} = \int_X -B_2 d\phi^i + B_{1i} dA^i,
S_{H1} = \int_X -J^i_j B_{2i} A^j - P^{ij} B_{2i} B_{1j} + \frac{1}{2} \left( J^j_i H_{jkl} + \frac{\partial Q_{jk}}{\partial \phi^i} \right) A^i A^j A^k
+ \frac{1}{2} \left( -P^{kl} H_{ijkl} - \frac{\partial J^k_j}{\partial \phi^i} + \frac{\partial J^{ij}_k}{\partial \phi^i} \right) A^i A^j B_{1k} + \frac{1}{2} \frac{\partial P^{jk}}{\partial \phi^i} A^i B_{1j} B_{1k}.
$$
We call this model a three dimensional twisted generalized complex sigma model. (33) is written as

\[ S_0 = \int_X -\langle 0 + B_2, d(\phi + 0) \rangle + \frac{1}{2}(A + B_1, d(A + B_1)) + \text{total derivative}, \]

\[ S_1 = \int_X -\langle 0 + B_2, J(A + B_1) \rangle - \frac{1}{2}(A + B_1, \left[ \begin{array}{cc} H_{ijk}^k & 0 \\ 0 & 0 \end{array} \right] J + A^{i} \frac{\partial J}{\partial \phi^i}) (A + B_1). \]  

(34)

If and only if the condition

\[ A_{i}^{ijk} = B_{i}^{ijk} = C_{ij}^k = 0, \]
\[ \frac{\partial D_{i}^{ijk}}{\partial \phi^i} + (ijkl \text{ cyclic}) = 0, \]  

(35)

is satisfied, the action (33) is invariant under the gauge transformation

\[ \delta \phi^i = J^i_j c^j + P^{ij} t_{1j}, \]

\[ \delta A^i = d c^j - P^{ij} t_{2j} + \left( -P^{kl} H_{jkl} - \frac{\partial J^j_k}{\partial \phi^j} + \frac{\partial J^j_i}{\partial \phi^i} \right) (A^i c^k + \frac{\partial P^{ki}}{\partial \phi^i} (A^i t_{1k} - c^j B_{1k}), \]

\[ \delta B_{1i} = dt_{1i} - J^i_j t_{2j} + \left( J^i_j H_{jkl} + J^i_j H_{kil} + J^i_j H_{ijl} + \frac{\partial Q_{jk}}{\partial \phi^j} + \frac{\partial Q_{ki}}{\partial \phi^i} + \frac{\partial Q_{ij}}{\partial \phi^j} \right) (A^i c^k + \frac{\partial P^{jk}}{\partial \phi^j} B_{1j} t_{1k}, \]

\[ \delta B_{2i} = dt_{2i} - \frac{\partial J^j_k}{\partial \phi^i} (B_{2j} c^k + t_{2j} A^k) - \frac{\partial P^{jk}}{\partial \phi^i} (B_{2j} t_{1k} + t_{2j} B_{1k} ) \]

\[ + \frac{\partial}{\partial \phi^i} \left( J^m_j H_{klm} + J^m_j H_{ljm} + J^m_j H_{jkm} + \frac{\partial Q_{kl}}{\partial \phi^j} + \frac{\partial Q_{lj}}{\partial \phi^i} + \frac{\partial Q_{jk}}{\partial \phi^j} \right) (A^i A^k c^l \]

\[ + \frac{1}{2} \frac{\partial}{\partial \phi^i} \left( -P^{km} H_{jkl} - \frac{\partial J^m_k}{\partial \phi^j} + \frac{\partial J^m_j}{\partial \phi^k} \right) (A^i A^k t_{1m} + 2A^i c^k B_{1m} ) \]

\[ + \frac{1}{2} \frac{\partial^2 P^{kl}}{\partial \phi^i \partial \phi^j} (2A^i B_{1k} t_{1l} + c^j B_{1k} B_{1l}), \]  

(36)

We can obtain (29) as the \( H = 0 \) theory in (33). Moreover the action (33) (and (29)) is invariant under the \( b \)-transformation (18) and (24) by simple calculation if we define the \( b \)-transformation on the fields

\[ \hat{\phi}^i = \phi^i, \]

\[ \hat{A}^i = A^i, \]  

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\[ B_{1i} = B_{1i} + b_{ij} A^j, \]
\[ B_{2i} = B_{2i} - \frac{1}{2} \partial b_{jk} A^j A^k. \] (37)

The condition which the action is gauge invariant is not \( D_{ijk} = 0 \) but in (32), and \( D_{H_{ijk}} = 0 \) but \( \partial D_{H_{ijk}}/\partial \phi^i + (ijkl \text{ cyclic}) = 0 \) in (35). This is because our model is \( b \)-transformation invariant and \( H_{ijk} \) has \( b \)-transformation ambiguity (24), that is, \( H \) is defined as a cohomology class in \( H^3(M) \) in a (twisted) generalized complex structure.

Let us consider mirror transformation in our model. Mirror transform is exchange of the complex structure \( J_{ij} \) and the symplectic structure \( P_{ij} \) and \( Q_{ij} \).

A simplest mirror transformation \( m_0 \) is exchange
\[ J_{ij} A^j \leftrightarrow P_{ij} B_{1j}, \]
\[ -J_{ij} B_{1j} \leftrightarrow Q_{ij} A^j. \] (38)

The action (33) is invariant under this symmetry.

A general mirror transform \( m : TM \oplus T^*M \rightarrow TM \oplus T^*M \) is a \( Z_2 \) transformation with
\[ m^2 = 1 \]
\[ m(J_{ij} A^j + P_{ij} B_{1j}) = J_{ij} A^j + P_{ij} B_{1j}, \]
\[ m(-J_{ij} B_{1j} + Q_{ij} A^j) = -J_{ij} B_{1j} + Q_{ij} A^j. \] (39)

We can easily confirm that the action (33) is invariant under \( m \).

4 Derivation from 3D Nonlinear Gauge Theory

In the paper [22], We have proposed 3D nonlinear BF theory as a general topological field theory of Schwarz type in three dimension. A 3D (twisted) generalized complex sigma model is a special model of a 3D nonlinear BF theory.

Let \( X \) be a three-dimensional manifold and \( M \) be a target manifold of a smooth map \( \phi : X \rightarrow M \) with local coordinate expression \( \{ \phi^i \} \). We also have a vector bundle \( E \) over \( M \) with \( A^a \) a section of \( T^*X \otimes \phi^*(E) \). We further introduce \( B_{1a} \) as a section of \( T^*X \otimes \phi^*(E^*) \) and \( B_{2i} \) as a section of \( \wedge^2 T^*X \otimes \phi^*(T^*M) \). Hereafter, the letters \( a, b, \cdots \) represent indices on the fiber of \( E \) and \( i, j, \cdots \) represent indices on \( M \) and \( T^*M \).
3D nonlinear BF theory with nonlinear gauge symmetry has the following action

\[ S = S_0 + S_1; \]
\[ S_0 = \int_X \left( -B_{2i} d\phi^i + B_{1a} dA^a \right), \]
\[ S_1 = \int_X \left( f_{2i}^1 (\phi) B_{2i} A^a + f_{2b}^1 (\phi) B_{2i} B_{1b} + \frac{1}{6} f_{3abc} (\phi) A^a A^b A^c \right) \]
\[ + \frac{1}{2} f_{4ab}^c (\phi) A^a A^b B_{1c} + \frac{1}{2} f_{5a}^{bc} (\phi) A^a B_{1b} B_{1c} + \frac{1}{6} f_{6}^{abc} (\phi) B_{1a} B_{1b} B_{1c}, \] (40)

where the structure functions \( f_1, \ldots, f_6 \) satisfy the identities

\[ f_{1e}^i f_{2}^{je} + f_{2}^{ie} f_{1}^j = 0, \]
\[ -\frac{\partial f_{1e}^i}{\partial \phi^j} f_{1}^{ib} + \frac{\partial f_{1b}^i}{\partial \phi^j} f_{1}^{ic} + f_{1}^{ie} f_{4bc} + f_{2}^{ie} f_{3eb} = 0, \]
\[ f_{1b}^j \frac{\partial f_{2}^{ic}}{\partial \phi^j} - f_{2}^{jc} \frac{\partial f_{1b}^i}{\partial \phi^j} + f_{1}^{ie} f_{5b}^{cc} - f_{2}^{ie} f_{4eb} = 0, \]
\[ -f_{2}^{jb} \frac{\partial f_{2}^{ic}}{\partial \phi^j} + f_{2}^{jc} \frac{\partial f_{2}^{ib}}{\partial \phi^j} + f_{1}^{ie} f_{6}^{eb} + f_{2}^{ie} f_{5e}^{bc} = 0, \]
\[ -f_{1}^a \frac{\partial f_{4eb}}{\partial \phi^j} + f_{2}^{jd} \frac{\partial f_{3abc}}{\partial \phi^j} + f_{4e}^{[a} d f_{4bc}^{]e} + f_{5e}^{[a} f_{5c}^{d} = 0, \]
\[ -f_{1}^a \frac{\partial f_{5b}^{cd}}{\partial \phi^j} - f_{2}^{jc} \frac{\partial f_{4ab}^{d]}}{\partial \phi^j} + f_{3eab} f_{6}^{cd} + f_{4e}^{[a} d f_{5b}^{]c} + f_{4eb}^{e} f_{5e}^{cd} = 0, \]
\[ -f_{1}^a \frac{\partial f_{6}^{bcd}}{\partial \phi^j} + f_{2}^{jb} \frac{\partial f_{5a}^{c]}}{\partial \phi^j} + f_{4ea}^{[b} f_{6}^{cd}]e + f_{5e}^{[bc} f_{5a}^{d]e} = 0, \]
\[ -f_{2}^{ja} \frac{\partial f_{6}^{bcd]]}}{\partial \phi^j} + f_{6}^{[ab} f_{5e}^{cd]} = 0, \]
\[ -f_{1}^a \frac{\partial f_{3eab}}{\partial \phi^j} + f_{4[a}^{e} f_{3cd]}^{e} = 0. \] (41)

This theory has a Courant algebroid structure. A Courant algebroid [17] is a vector bundle \( E \rightarrow M \) with a nondegenerate symmetric bilinear form \( \langle \cdot , \cdot \rangle \) on the bundle, a bilinear operation (a Dorfman bracket) \( \circ \) on \( \Gamma(E) \) (the space of sections on \( E \)), and a bundle map (called the anchor) \( \rho : E \rightarrow TM \) satisfying the following properties:

1) \( e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3), \)
2) \( \rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)], \)
3) \( e_1 \circ F e_2 = F(e_1 \circ e_2) + (\rho(e_1) F) e_2, \)
4) \( e_1 \circ e_2 = \frac{1}{2} \mathcal{D}\langle e_1 , e_2 \rangle, \)
5) \( \rho(e_1) \langle e_2 , e_3 \rangle = \langle e_1 \circ e_2 , e_3 \rangle + \langle e_2 , e_1 \circ e_3 \rangle, \) (42)
where $e_1, e_2, \text{ and } e_3$ are sections of $E$; $F$ is a function on $M$; $\mathcal{D}$ is a map from functions on $M$ to $\Gamma(E)$ and is defined by $\langle \mathcal{D} F, e \rangle = \rho(e) F$.

If we take a local basis, Eq.(41) is equivalent to the relations of structure functions of a Courant algebroid on a vector bundle $E \oplus E^*$ over $M$: Symmetric bilinear form $\langle \cdot, \cdot \rangle$ is defined from the natural pairing of $E$ and $E^*$. That is, $\langle e_a, e_b \rangle = \langle e^a, e^b \rangle = 0$ and $\langle e_a, e^b \rangle = \delta^a_b$ if $\{ e_a \}$ is a basis of sections of $E^*$ and $\{ e^a \}$ is that of $E$. The bilinear form $\circ$ and the anchor $\rho$ are represented as follows:

$$
e^a \circ e^b = -f_5^{ab}(\phi)e^c - f_6^{abc}(\phi)e_c,
\ne^a \circ e_b = -f_4^{bc}a(\phi)e^c + f_5^{ac}(\phi)e_c,
\ne_a \circ e_b = -f_3^{abc}(\phi)e^c - f_4^{ab}c(\phi)e_c,
\rho(e^a) = -f_2^{ia}(\phi)\frac{\partial}{\partial \phi^i},
\rho(e_a) = -f_1^i(\phi)\frac{\partial}{\partial \phi^i}.
$$

(43)

If we take $E = TM$ and $E^* = T^*M$, and we set

$$
f_1^i = -J^i_j,
f_2^{ij} = -P^{ij},
\begin{align*}
f_{3ijk} &= J^l_iH_{jkl} + J^l_jH_{kil} + J^l_kH_{ijl} + \frac{\partial Q_{jk}}{\partial \phi^i} + \frac{\partial Q_{ki}}{\partial \phi^j} + \frac{\partial Q_{ij}}{\partial \phi^k}, \\
f_{4ij}^k &= -P^{kl}H_{ijl} - \frac{\partial J^k_j}{\partial \phi^i} + \frac{\partial J^k_i}{\partial \phi^j}, \\
f_{5ij}^{jk} &= \frac{\partial P^{jk}}{\partial \phi^i}, \\
f_{6}^{ijk} &= 0,
\end{align*}
$$

(44)

we can find that 3D nonlinear gauge theory (40) reduces to the (twisted) generalized complex sigma model (33), and (41) reduces to (35).

## 5 Zucchini Model as a Boundary Action

Zucchini has proposed a topological sigma model with a generalized complex structure on two dimensional worldsheet [20]. He has called the model the Hitchin sigma model.
First we assume $H = 0$. The Zucchini’s action is

$$S_Z = \int_\Sigma B_{1i}d\phi^i + \frac{1}{2}P^{ij}B_{1i}B_{1j} + \frac{1}{2}Q_{ij}d\phi^id\phi^j + J^i_1B_{1i}d\phi^i,$$  \hspace{1cm} (45)

where $\Sigma$ is a two dimensional world sheet and $\phi^i$ and $B_{1i}$ are restricted on $\Sigma$. We consider this model as a boundary theory. That is, we consider a three dimensional world volume $X$ such that $\Sigma = \partial X$, and we regard the action (45) as

$$S_Z = \int_X \left( B_{1i}d\phi^i + \frac{1}{2}P^{ij}B_{1i}B_{1j} + \frac{1}{2}Q_{ij}d\phi^id\phi^j + J^i_1B_{1i}d\phi^i \right)$$

$$= \int_X dB_{1i}d\phi^i + \frac{1}{2}\frac{\partial P^{ij}}{\partial \phi^k}d\phi^kd\phi^kB_{1i}B_{1j} + P^{ij}dB_{1i}B_{1j} + \frac{1}{2}\frac{\partial Q_{ij}}{\partial \phi^k}d\phi^kd\phi^jd\phi^j$$

$$\quad + \frac{\partial J^i}{\partial \phi^k}d\phi^kB_{1i}d\phi^j + J^i_1dB_{1i}d\phi^i.$$  \hspace{1cm} (46)

We introduce a 1-form $A^i$ and a 2-form $B_{2i}$ such that $A^i = d\phi^i$ and $B_{2i} = -dB_{1i}$. If we introduce two Lagrange multiplier fields, a 2-form $Y_{2i}$ and a 1-form $Z^i$, we obtain an equivalent action

$$S_Z = \int_X -B_{2i}A^i + \frac{1}{2}\frac{\partial P^{ij}}{\partial \phi^k}A^kB_{1i}B_{1j} - P^{ij}B_{2i}B_{2j} + \frac{1}{2}\frac{\partial Q_{ij}}{\partial \phi^k}A^iA^jA^k$$

$$\quad + \frac{\partial J^i}{\partial \phi^k}A^kB_{1i}A^j - J^i_1B_{2i}A^i$$

$$\quad + (A^i - d\phi^i)Y_{2i} + (B_{2i} + dB_{1i})Z^i.$$  \hspace{1cm} (47)

Moreover we redefine $Y_{2i}$ and $Z^i$ as $Y^\prime_{2i} = Y_{2i} - B_{2i}$ and $Z^\prime_i = Z^i - A^i$. Then (47) is

$$S_Z = S_a + S_{Zb};$$

$$S_a = \int_X -Y^\prime_{2i}d\phi^i + dB_{1i}Z^\prime_i + Y^\prime_{2i}A^i + B_{2i}Z^\prime_i + B_{2i}A^i,$$

$$S_{Zb} = S$$

$$= \int_X -B_{2i}d\phi^i + dB_{1i}A^i - J^i_1B_{2i}A^i - P^{ij}B_{2i}B_{1j} + \frac{1}{2}\frac{\partial Q_{jk}}{\partial \phi^i}A^iA^jA^k$$

$$\quad + \frac{1}{2}\left( \frac{\partial J^i_j}{\partial \phi^i} + \frac{\partial J^i_i}{\partial \phi^j} \right)A^iA^jB_{1k} + \frac{1}{2}\frac{\partial P^{jk}}{\partial \phi^i}A^iB_{1j}B_{1k}.$$  \hspace{1cm} (48)

$S_{Zb}$ coincides with (a partial derivative of) our generalized complex sigma model (29) and $S_a$ are terms independent of a generalized complex structure. Therefore $S_{Zb}$ and $S_Z$ are constructed from the same generalized complex structure. We have found that three dimensional topological field theory equivalent to the Zucchini’s model, which is derived from the same generalized complex structure.
We make a similar discussion for a twisted generalized complex structure with $H \neq 0$. From (33), it is natural to consider the action

$$S_T = S_a + S_{Tb};$$

$$S_a = \int_X -Y_2'd\phi^i + dB_1Z_i + Y_2'A^i + B_2Z^i + B_2A^i,$$

$$S_{Tb} = S_H$$

$$= \int_X -B_2d\phi^i + B_1dA^i - J_{ij}B_2A^i - P^{ij}B_2B_{1j} + \frac{1}{2} \left( J_{ij}H_{jkl} + \frac{\partial Q_{jk}}{\partial \phi^i} \right) A^iA^jA^k$$

$$+ \frac{1}{2} \left( -P^{kl}H_{ijkl} - \frac{\partial J_{jkl}}{\partial \phi^i} - \frac{\partial J_{jkl}}{\partial \phi^i} \right) A^iA^jB_{1k} + \frac{1}{2} \frac{\partial P_{jk}}{\partial \phi^i} A^iB_{1j}B_{1k}. \quad (49)$$

We rewrite the action (49) by the procedure with the modification from (48) to (45). If we remove $Y_2i, Z_i, A^i$ and $B_2i$ by means of the equations of motion, we obtain

$$S_T = \int_{\Sigma} B_{1i}d\phi^i + \frac{1}{2} P^{ij}B_{1j}B_{1i} + \frac{1}{2} Q_{ij}d\phi^id\phi^j + J_{ij}B_{1i}d\phi^j$$

$$+ \frac{1}{2} \int_X J_{ij}H_{jkl}d\phi^i d\phi^j d\phi^k - P^{kl}H_{ijkl}B_{1k}d\phi^i d\phi^j,$$

the second line in (50) is 'Wess-Zumino' terms, which is integration on $X$. However these 'Wess-Zumino' terms are different from Zucchini's $H$-term. If we consider the following 3D action as the beginning

$$S_{ZH} = S_a + S_{ZHb};$$

$$S_a = \int_X -Y_2'i d\phi^i + dB_1Z_i + Y_2'A^i + B_2Z^i + B_2A^i,$$

$$S_{ZHb} = \int_X -B_2d\phi^i + B_1dA^i - J_{ij}B_2A^i - P^{ij}B_2B_{1j} + \frac{1}{2} \left( H_{ijkl} + \frac{\partial Q_{jk}}{\partial \phi^i} \right) A^iA^jA^k$$

$$+ \frac{1}{2} \left( -\frac{\partial J_{jkl}}{\partial \phi^i} - \frac{\partial J_{jkl}}{\partial \phi^i} \right) A^iA^jB_{1k} + \frac{1}{2} \frac{\partial P_{jk}}{\partial \phi^i} A^iB_{1j}B_{1k}, \quad (51)$$

we obtain a boundary action

$$S_{ZH} = \int_{\Sigma} B_{1i}d\phi^i + \frac{1}{2} P^{ij}B_{1j}B_{1i} + \frac{1}{2} Q_{ij}d\phi^id\phi^j + J_{ij}B_{1i}d\phi^j$$

$$+ \frac{1}{2} \int_X H_{ijk}d\phi^i d\phi^j d\phi^k,$$

which coincides with Zucchini's proposal.

The point is $b$-transformation property of the action (45). The Zucchini’s action (45) is not invariant under the $b$-transformation

$$S_{ZH} = S_{Z} - \int_{\Sigma} b_{ij}d\phi^id\phi^j,$$

$$\hat{S}_{ZH} = S_{ZH} - \int_{\Sigma} b_{ij}d\phi^id\phi^j,$$

$$S_{ZH} = S_{ZH} - \int_{\Sigma} b_{ij}d\phi^id\phi^j.$$
where the $b$-transformation is defined as
\[
\hat{\phi}^i = \phi^i, \\
\hat{B}_{1i} = B_{1i} + b_{ij}d\phi^j.
\]
(54)

The equivalent 3D action (48) also changes under the $b$-transformation
\[
\hat{S}_Z = S_Z - \frac{1}{2} \int_X \frac{\partial b_{jk}}{\partial \phi^i} A^i A^j A^k - \frac{1}{2} \int_X d(b_{ij} A^i A^j),
\]
(55)
where $b$-transformation is defined by (18), (37) and
\[
\hat{Y}''_{2i} = \frac{1}{2} \frac{\partial b_{jk}}{\partial \phi^i} A^i A^j + b_{ij} dA^j - \frac{\partial b_{jk}}{\partial \phi^i} A^j d\phi^k - J^i_k \frac{\partial b_{jl}}{\partial \phi^j} A^j A^k - P_{lk} \frac{\partial b_{jl}}{\partial \phi^j} A^j B_{1k},
\]
\[
\hat{Z}''_i = Z''_i,
\]
(56)
however in (37), the $b$-transformation of $B_{2i}$ is changed to
\[
\hat{B}_{2i} = B_{2i} - d(b_{ij} A^j),
\]
(57)
because of consistency with equations of motion.

Since $H$ is closed, we can locally write $H$ by a 2-form $q$ on $M$ as
\[
H_{ijk} = \frac{1}{3} \left( \frac{\partial q_{jk}}{\partial \phi^i} + \frac{\partial q_{ki}}{\partial \phi^j} + \frac{\partial q_{ij}}{\partial \phi^k} \right).
\]
(58)
Since there exists the $\frac{\partial b_{jk}}{\partial \phi^i} A^i A^j A^k$ term in (55), the $H$-term can be absorbed to $Q$ by a local $b$-transformation $q_{ij} = b_{ij}$ in the action (51), and we obtain (48). In other words, the $H$ term in (51) is consistent up to $H$-exact terms as a global theory, the theory is meaningful only as a cohomology class in $H^3(M)$. It describes gerbe gauge transformation dependence [20].

If we set $Q_{ij} = J^j_i = 0$ in (52), we obtain the WZ-Poisson sigma model
\[
S_{WZP} = \int_{\Sigma} B_{1i} d\phi^i + \frac{1}{2} P^{ij} B_{1i} B_{1j} + \frac{1}{2} \int_X H_{ijk} d\phi^i d\phi^j d\phi^k,
\]
(59)
and from (51), we obtain an equivalent 3D sigma model
\[
S_{WZP} = S_a + S_{WZPb};
\]
\[
S_a = \int_X -Y''_{2i} d\phi^i + dB_{1i} Z''_i + Y''_{2i} A^i + B_{2i} Z''_i + B_{2i} A^i,
\]
\[
S_{WZPb} = \int_X -B_{2i} d\phi^i + dB_{1i} A^i - P^{ij} B_{2i} B_{1j} + \frac{1}{2} H_{ijk} A^i A^j A^k
\]
\[
+ \frac{1}{2} \frac{\partial P^{jk}}{\partial \phi^i} A^i B_{1j} B_{1k},
\]
(60)
6 Conclusions and Discussion

We have constructed a three-dimensional topological sigma model based on a (twisted) generalized complex structure on a target generalized complex manifold. The theory is invariant under the diffeomorphism on the world volume, the $b$-transformation on the (twisted) generalized complex structure and the mirror symmetries in a generalized complex geometry.

We derive the Zucchini’s two dimensional topological sigma model with a generalized complex structure as a boundary action of our three dimensional action plus terms independent of a generalized complex structure. In case of a twisted generalized complex structure, the $H$ term in the Zucchini action is different from our proposal, however the ‘Wess-Zumino’ terms in the Zucchini action can be generated locally by the $b$-transformation because Zucchini action has a gerbe gauge transformation dependence.

This model will be useful to investigate mirror symmetry. We have analyzed only a classical theory in this paper. It is important to investigate a quantum theory. There exist several topological and non topological string theories with a generalized complex structure other than the Zucchini’s model. Relations to other known theories based on a generalized complex structure will be interesting.

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References


