Anti-de Sitter 3-dimensional gravity with torsion

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Abstract

Using the canonical formalism, we study the asymptotic symmetries of the topological 3-dimensional gravity with torsion. In the anti-de Sitter sector, the symmetries are realized by two independent Virasoro algebras with classical central charges. In the simple case of the teleparallel vacuum geometry, the central charges are equal to each other and have the same value as in general relativity, while in the general Riemann-Cartan geometry, they become different.

keywords: gravity; torsion; asymptotic symmetry.

1 Introduction

Three-dimensional (3d) gravity has been used as a theoretical laboratory to test some of the conceptual problems of both classical and quantum gravity [1, 2]. One can identify several particularly important achievements in the development of these ideas. (a) Brown and Henneaux demonstrated that, under suitable asymptotic conditions, the asymptotic symmetry of 3d gravity has an extremely rich structure, described by two independent canonical Virasoro algebras with classical central charges [3]. (b) Soon after that, Witten found that general relativity with a cosmological constant (GRΛ) can be formulated as a Chern-Simons gauge theory [4]. The equivalence between gravity and an ordinary gauge theory was crucial for a deeper understanding of quantum gravity. (c) Next, the discovery of the black hole solution by Bañados, Teitelboim and Zanelli had a powerful impact on 3d gravity [5]. It turned out that the Virasoro algebra of the asymptotic symmetry plays a central role in our understanding of the quantum nature of black hole [6-11].

Following a widely spread belief that the dynamics of gravity is to be described by general relativity, investigations of 3d gravity have been carried out mostly in the realm of Riemannian geometry. However, since the early 1990s, the possibility of Riemann-Cartan geometry has been also explored [12-18]; it is a geometry in which both the curvature and the torsion are present as independent geometric characteristics of spacetime [19, 20]. In this way, one expects to clarify the influence of geometry on the dynamical content of spacetime.

Dynamics of a theory is determined not only by its action, but also by the asymptotic conditions. The dynamical content of asymptotic conditions is best seen in topological theories, where the non-trivial dynamics is bound to exist only at the boundary. General action for topological 3d gravity in Riemann–Cartan spacetime has been proposed by Mielke and Baekler [12, 13]. This model is our starting point for exploring the structure of 3d gravity with torsion. In particular, we shall investigate

the existence of the black hole with torsion, and
the asymptotic structure of 3d gravity with torsion.

We restrict ourselves to the anti-de Sitter (AdS) sector of the theory, with negative effective cosmological constant. For a particular choice of parameters, the Mielke-Baekler action leads to the teleparallel (Weizenböck) geometry in vacuum \([21, 22, 20]\), defined by the requirement of vanishing curvature, which we choose as the simplest framework for studying the influence of torsion on the spacetime dynamics.

The paper is organized as follows. In Sect. 2 we introduce Riemann–Cartan spacetime as a general geometric arena for 3d gravity with torsion, and discuss the teleparallel description of gravity in vacuum. In Sect. 3 we construct the teleparallel black hole solution. Then, in Sect. 4, we introduce the concept of asymptotically AdS configuration, and show that the related asymptotic symmetry is the same as in general relativity—the conformal symmetry. In the next section, the gauge structure of the theory is incorporated into the canonical formalism by investigating the Poisson bracket (PB) algebra of the asymptotic generators. The asymptotic symmetry is characterized by two independent canonical Virasoro algebras with classical central charges, the values of which are the same as in Riemannian spacetime of general relativity. In Sect. 6 we discuss the general case of Riemann-Cartan geometry, and show that the related classical central charges are different. Finally, Sect. 7 is devoted to concluding remarks.

Our conventions are given by the following rules: the Latin indices refer to the local Lorentz frame, the Greek indices refer to the coordinate frame; the first letters of both alphabets \((a, b, c, ...; \alpha, \beta, \gamma, ...)\) run over 1,2, the middle alphabet letters \((i, j, k, ...; \mu, \nu, \lambda, ...)\) run over 0,1,2; the signature of spacetime is \(\eta = (+, -, -);\) totally antisymmetric tensor \(\varepsilon_{ijk}\) and the related tensor density \(\varepsilon_{\mu\nu\rho}\) are both normalized so that \(\varepsilon_{012} = 1.\)

## 2 Riemann-Cartan gravity in 3d

Theory of gravity with torsion can be formulated as Poincaré gauge theory (PGT), with an underlying spacetime structure described by Riemann-Cartan geometry \([19, 20]\).

**PGT in brief.** The basic gravitational variables in PGT are the triad \(b^i = b^{ij}_{\mu}dx^\mu\) and the Lorentz connection \(A^{ij}_{\mu}dx^\mu\) (1-forms). Metric tensor \(g\) is not an independent variable, it is defined in terms of the triad field: \(g = b^i \otimes b^i \eta_{ij}\), where \(\eta_{ij} = (+, -, -).\) The field strengths corresponding to the gauge potentials \(b^i\) and \(A^{ij}\) are the torsion \(T^i\) and the curvature \(R^{ij}\) (2-forms). In 3d, we can simplify the notation by introducing the duals of \(A^{ij}\) and \(R^{ij}\): \(\omega_i = -\frac{1}{2} \varepsilon_{ijk}A^{jk};\) \(R_i = -\frac{1}{2} \varepsilon_{ijk}R^{jk}\). Gauge symmetries of the theory are local translations and local Lorentz rotations, parametrized by \(\xi^\mu\) and \(\theta^i\), respectively:

\[
\begin{align*}
\delta_0 b^i_{\mu} &= -\varepsilon^{j k}b^j_{\mu}\theta^k - (\partial_{\mu}\xi^\rho)b^i_{\rho} - \xi^\rho \partial_{\rho} b^i_{\mu} \\
\delta_0 \omega^i_{\mu} &= -\nabla_{\mu}\theta^i - (\partial_{\mu}\xi^\rho)\omega^i_{\rho} - \xi^\rho \partial_{\rho} \omega^i_{\mu},
\end{align*}
\]

where \(\nabla_{\mu}\theta^i = \partial_{\mu}\theta^i + \varepsilon^{j k}\omega^j_{\mu}\theta^k\) is the covariant derivative of \(\theta^i\). The related field strengths, the torsion and the curvature, are given by the expressions

\[
\begin{align*}
R^i &= d\omega^i + \frac{1}{2} \varepsilon^{j k} \omega^j \omega^k \equiv \frac{1}{2} R^i_{\mu\nu} dx^\mu dx^\nu, \\
T^i &= db^i + \varepsilon^{j k} \omega^j b^k \equiv \frac{1}{2} T^i_{\mu\nu} dx^\mu dx^\nu,
\end{align*}
\]

where wedge product signs are omitted for simplicity.
In PGT, the triad and the connection are related to each other by the *metricity condition*: $\nabla g = 0$. The geometric interpretation of the connection implies a useful identity, which relates Lorentz connection $A$ and (Riemannian) Levi-Civita connection $\Delta$:

$$A_{ijk} = \Delta_{ijk} + K_{ijk}, \quad (2.3)$$

where $K_{ijk} = -(T_{ijk} + T_{kij} - T_{jki})/2$ is the contortion.

**Topological action.** In general, gravitational dynamics is defined by Lagrangians which are at most quadratic in field strengths. Omitting the quadratic terms, Mielke and Baekler proposed a *topological* model for 3d gravity [12, 13], with the action

$$I = aI_1 + \Lambda I_2 + \alpha_3 I_3 + \alpha_4 I_4 + I_M, \quad (2.4a)$$

where

$$I_1 = 2 \int b^i R_i,$$

$$I_2 = -\frac{1}{3} \int \varepsilon_{ijk} b^i b^j b^k,$$

$$I_3 = \int \left( \omega^i d\omega_i + \frac{1}{3} \varepsilon_{ijk} \omega^i \omega^j \omega^k \right),$$

$$I_4 = \int b^i T_i, \quad (2.4b)$$

and $I_M$ is a matter action. The first term with $a = 1/16\pi G$ is the usual Einstein-Cartan action, the second term is a cosmological term, $I_3$ is a Chern-Simons action for the connection, and $I_4$ is an action of the translational Chern-Simons type. The Mielke-Baekler model can be thought of as a generalization of \(\text{GR}_\Lambda\) ($\alpha_3 = \alpha_4 = 0$) to a topological gravity theory in Riemann-Cartan spacetime.

**Field equations.** Variation of the action with respect to triad and connection yields the field equations:

$$\varepsilon^{\mu\nu\rho} \left[ a R_{i\nu\rho} + \alpha_4 T_{i\nu\rho} - \Lambda \varepsilon_{ijk} b^i b^j b^k \right] = \tau^{\mu i},$$

$$\varepsilon^{\mu\nu\rho} \left[ \alpha_3 R_{i\nu\rho} + a T_{i\nu\rho} + \alpha_4 \varepsilon_{ijk} b^i b^j b^k \right] = \sigma^{\mu i},$$

where $\tau$ and $\sigma$ are the matter energy-momentum and spin tensors, respectively. For our purposes—to explore exact vacuum solutions and the asymptotic structure—it is sufficient to consider the field equations in vacuum, where $\tau = \sigma = 0$. For $\alpha_3 \alpha_4 - a^2 \neq 0$, these equations take the simple form

$$T_{ijk} = A \varepsilon_{ijk}, \quad \quad (2.5a)$$

$$R_{ijk} = B \varepsilon_{ijk}, \quad \quad (2.5b)$$

where

$$A = \frac{\alpha_3 \Lambda + \alpha_4 a}{\alpha_3 \alpha_4 - a^2}, \quad B = -\frac{(\alpha_4)^2 + a\Lambda}{\alpha_3 \alpha_4 - a^2}.$$

Thus, the vacuum configuration is characterized by constant torsion and constant curvature.
Using the PGT identity (2.3), one can express the Riemann-Cartan curvature $R^{ij}_{\mu\nu}(A)$ in terms of its Riemannian piece $\tilde{R}^{ij}_{\mu\nu}(\Delta)$ and the contortion. This geometric identity, combined with the field equations (2.5), leads to

$$\tilde{R}^{ij}_{\mu\nu} = -\Lambda_{\text{eff}}(b^i_\mu b^j_\nu - b^j_\nu b^i_\mu), \quad \Lambda_{\text{eff}} = B - \frac{1}{4} A^2,$$

where $\Lambda_{\text{eff}}$ is the effective cosmological constant. Equation (2.6) can be considered as an equivalent of the second field equation (2.5b). Looking at (2.6) as an equation for the metric, one concludes that our spacetime is maximally symmetric [23]:

$$\Lambda_{\text{eff}} < 0 \Rightarrow \text{anti-de Sitter sector},$$

$$\Lambda_{\text{eff}} > 0 \Rightarrow \text{de Sitter sector}.$$

**Teleparallelism in vacuum.** There are two interesting special cases of the general Mielke-Baekler model.

- For $\alpha_3 = \alpha_4 = 0$, the torsion vanishes and the vacuum geometry becomes Riemannian, with the field equations $T_{ijk} = 0$, $R_{ijk} = B \varepsilon_{ijk}$. This case defines GR$_{\Lambda}$, and corresponds to Witten’s choice [4].
- For $(\alpha_4)^2 + a\Lambda = 0$, the curvature vanishes, and the vacuum geometry is teleparallel: $T_{ijk} = A \varepsilon_{ijk}$, $R_{ijk} = 0$ [16]. The vacuum field equations are “geometrically dual” to those of GR$_{\Lambda}$.

Having in mind our intention to study the influence of torsion on the gravitational dynamics, we restrict our attention to the teleparallel geometry as the simplest framework involving torsion. In this case, the field equations take the form

$$T_{ijk} = A \varepsilon_{ijk},$$

$$\tilde{R}^{ij}_{\mu\nu} = -\Lambda_{\text{eff}}(b^i_\mu b^j_\nu - b^j_\nu b^i_\mu),$$

where the effective cosmological constant becomes negative:

$$\Lambda_{\text{eff}} = -\frac{1}{4} A^2 \equiv -\frac{1}{\ell^2} < 0.$$

Although the general structure of spacetime, in the presence of matter, is described by Riemann-Cartan geometry, the field equations (2.7) imply that the effective dynamics in vacuum is teleparallel.

Since the field equations do not involve $\alpha_3$ ($A = -\alpha_4/a$), we can take, without loss of generality, that $\alpha_3 = 0$. In summary, we adopt the following restrictions on parameters: $\alpha_3 = 0$, $\alpha_4 = -2a/\ell$, $\Lambda = -4a/\ell^2$ (and $a = 1/16\pi G$), whereupon the action (2.4) reduces to

$$I = a \int d^3 x \varepsilon^{\rho\mu\nu} \left[ b^i_\rho \left( R_{i\mu\nu} - \frac{1}{\ell} T_{i\mu\nu} \right) + \frac{4}{3\ell^2} \varepsilon_{ijk} b^i_\rho b^j_\mu b^k_\nu \right].$$

### 3 Exact vacuum solutions

We start our study of 3d gravity with torsion by an analysis of the exact classical solutions in vacuum. Our search for the exact solutions is based on the following procedure:

- For a given $\Lambda_{\text{eff}}$, find a solution of Eq. (2.7b) for the metric. This step is very simple, since the form of the metric for maximally symmetric 3d spaces is well known [23].
- Given the metric, find a solution for the triad, such that $g = b^i \otimes b^j \eta_{ij}$.
- Finally, use Eq. (2.7a) to determine the connection $\omega^i$. 
Teleparallel black hole. Equation (2.7b) represents the Riemannian condition for maximal symmetry, and it has a well known solution for the metric—the BTZ black hole [5]. In the static coordinates \( x^\mu = (t, r, \varphi) \) \((0 \leq \varphi < 2\pi)\), the BTZ metric has the form (in units \( G = 1 \)):

\[
\begin{align*}
    ds^2 &= N^2 dt^2 - N^{-2} dr^2 - r^2 (d\varphi + N_\varphi dt)^2, \\
    N^2 &= \left( -2m + \frac{r^2}{\ell^2} + \frac{J^2}{r^2} \right), \\
    N_\varphi &= \frac{J}{r^2}.
\end{align*}
\]

(3.1)

As we shall see later, the parameters \( m \) and \( J \) define the conserved charges—energy and angular momentum. Given the metric (3.1), the triad field can be chosen in the simple, “diagonal” form,

\[
\begin{align*}
    b^0 &= N dt, \\
    b^1 &= N^{-1} dr, \\
    b^2 &= r (d\varphi + N_\varphi dt),
\end{align*}
\]

(3.2a)

and the related connection is obtained by solving (2.7a):

\[
\begin{align*}
    \omega^0 &= N dx^- , \\
    \omega^1 &= N^{-1} \left( \frac{1}{\ell} + \frac{J}{r^2} \right) dr, \\
    \omega^2 &= - \left( \frac{r}{\ell} - \frac{J}{r} \right) dx^- ,
\end{align*}
\]

(3.2b)

where \( x^\pm = t/\ell \pm \varphi \). Equations (3.2) define the teleparallel black hole [16]. For Riemann-Cartan black hole, see Refs. 15 and 18.

Teleparallel AdS solution. In Riemannian geometry with negative \( \Lambda \), there is a general solution with maximal number of Killing vectors (the solutions of the Killing equation \( \delta_0 g_{\mu\nu} = 0 \)), which is called the anti-de Sitter solution, AdS\(_3\). Locally, it can be obtained from (3.1) by the replacement \( J = 0, 2m = -1 \). Although AdS\(_3\) and the black hole are locally isometric, they are globally distinct [5, 23].

Similarly, in the teleparallel geometry, the general, maximally symmetric solution defines the teleparallel AdS\(_3\). It can be obtained from the black hole (3.2) by the same replacement \( (J = 0, 2m = -1)\):

\[
\begin{align*}
    b^0 &= f dt, \\
    b^1 &= f^{-1} dr, \\
    b^2 &= r d\varphi, \\
    \omega^0 &= f dx^- , \\
    \omega^1 &= \frac{1}{\ell f} dr, \\
    \omega^2 &= - \frac{r}{\ell} dx^- ,
\end{align*}
\]

(3.3)

with \( f^2 = 1 + r^2/\ell^2 \).

One should stress that in Riemann-Cartan geometry, one can define the generalized isometries by the requirements \( \delta_0 b^\mu_\mu = 0, \delta_0 \omega^\mu_\mu = 0 \), which differ from the Killing equation in Riemannian geometry. When applied to the teleparallel AdS solution (3.3), these requirements define six independent solutions \( \xi^{(k)} \) and \( \theta^{(k)} \) \((k = 1, \ldots , 6)\) for the allowed \( \xi \) and \( \theta \), displayed in Appendix A of Ref. 16. The related symmetry group is the six-parameter AdS group \( SO(2,2) \).

4 Asymptotic conditions

Spacetime outside localized matter sources is described by the vacuum solutions of the field equations. Thus, matter has no influence on the local properties of spacetime in the source-free regions, but it can change its global properties. The global properties can be expressed geometrically by the symmetry properties of the asymptotic configurations, which are, on the other hand, closely related to the gravitational conservation laws.

Returning to 3d gravity with \( \Lambda_{\text{eff}} < 0 \), let us note that maximally symmetric AdS solution (3.3) has the role analogous to the role of Minkowski space in the \( \Lambda_{\text{eff}} = 0 \) case. Following this analogy,
we could choose that all dynamical variables approach to the configuration (3.3) in such a way, that the resulting asymptotic symmetry is $SO(2,2)$, the maximal symmetry of (3.3). However, the important black hole geometries would be thereby excluded, as they are not $SO(2,2)$ invariant. Such a situation motivates us to introduce the concept of the AdS asymptotic behaviour, defined by the following requirements [3, 24]:

(a) the asymptotic conditions should include the black hole geometries;
(b) they should be invariant under the action of the AdS group $SO(2,2)$;
(c) the asymptotic symmetries should have well defined canonical generators.

**AdS asymptotics.** According to (a), the asymptotic form of the black hole configuration, defined by Eqs. (3.2a) and (3.2b), should be included in the set of asymptotic states we are searching for. The second requirement (b) is realized by acting on the black hole solution with all possible $SO(2,2)$ transformations, and demanding the resulting configurations to belong to our set of asymptotic states.

The procedure just described leads to the following asymptotic form of the triad field:

$$
\tilde{b}^i_\mu = \begin{pmatrix}
\frac{r}{\ell} + O_1 & O_4 & O_1 \\
O_2 & \frac{\ell}{r} + O_3 & O_2 \\
O_1 & O_4 & r + O_1
\end{pmatrix} \equiv \begin{pmatrix}
\frac{r}{\ell} & 0 & 0 \\
0 & \frac{\ell}{r} & 0 \\
0 & 0 & r
\end{pmatrix} + \tilde{B}^i_\mu .
$$

(4.1a)

The real meaning of the result is clarified by the following remark. The set of all $SO(2,2)$ transformations is defined by six pairs $(\xi(k), \theta(k))$, hence, strictly speaking, the family of black hole triads obtained by the action of these transformations is parametrized by six real parameters, say $\sigma_i$. The meaning of (4.1a) is slightly different: any $c/r^n$ term is supposed to be of the form $c(t, \varphi)/r^n$, i.e. constants $c = c(\sigma_i)$ of the six parameter family are promoted to functions $c(t, \varphi)$. This is the simplest way to characterize the asymptotic behaviour of $\tilde{b}^i_\mu$. The triad family (4.1a) generates the Brown-Henneaux asymptotic form of the metric [3], but clearly, it is not uniquely determined by it.

In a similar manner, we can use the torsion equation of motion (2.7a) to obtain the asymptotic form of the connection:

$$
\omega^i_\mu = \begin{pmatrix}
\frac{r}{\ell^2} + O_1 & O_2 & -\frac{r}{\ell} + O_1 \\
O_2 & \frac{1}{r} + O_3 & O_2 \\
-\frac{r}{\ell^2} + O_1 & O_2 & \frac{r}{\ell} + O_1
\end{pmatrix} \equiv \begin{pmatrix}
\frac{r}{\ell^2} & 0 & -\frac{r}{\ell} \\
0 & \frac{1}{r} & 0 \\
-\frac{r}{\ell^2} & 0 & \frac{r}{\ell}
\end{pmatrix} + \Omega^i_\mu .
$$

(4.1b)

All the higher order terms $B^i_\mu$ and $\Omega^i_\mu$ are considered to be arbitrary and independent of each other. One can verify that the asymptotic conditions (4.1) are indeed invariant under the action of the AdS group $SO(2,2)$.

**Asymptotic symmetries.** We are now going to examine the symmetries of the above asymptotic configurations, i.e. to find out the subset of the gauge transformations which leaves the set of asymptotic states (4.1) invariant. The parameters of these transformations are determined by the relations

$$
\epsilon^{ijk}\theta_j b_{k\mu} - (\partial_\mu \xi^\rho)b^j_\rho - \xi^\rho \partial_\rho b^j_\mu = \delta_0 \tilde{B}^i_\mu ,
$$

$$
-\partial_\mu \theta^i + \epsilon^{ijk}\theta_j \omega_{k\mu} - (\partial_\mu \xi^\rho)\omega^j_\rho - \xi^\rho \partial_\rho \omega^j_\mu = \delta_0 \Omega^i_\mu .
$$

(4.2a)

(4.2b)
Acting on a specific field satisfying (4.1), these transformations change the form of the non-leading terms $B_{i\mu}$, $\Omega_{i\mu}$. One should stress that the symmetry transformations defined in this way differ from the usual symmetries, which act according to the rule $\delta_0 b_{i\mu} = 0$, $\delta_0 \omega_{i\mu} = 0$.

We find the restricted gauge parameters as follows [16]. The symmetric part of (4.2a) multiplied by $b_{i\nu}$ (six relations) yields

$$\xi^0 = \ell \left[ T + \frac{1}{2} \left( \frac{\partial^2 T}{\partial t^2} \right) \frac{\ell^4}{r^2} \right] + O_4,$$

$$\xi^2 = S - \frac{1}{2} \left( \frac{\partial^2 S}{\partial \varphi^2} \right) \frac{\ell^2}{r^2} + O_4,$$

$$\xi^1 = -\ell \left( \frac{\partial T}{\partial t} \right) r + O_1,$$

where the functions $T(t, \varphi)$ and $S(t, \varphi)$ satisfy the conditions

$$\frac{\partial T}{\partial \varphi} = \ell \frac{\partial S}{\partial t}, \quad \frac{\partial S}{\partial \varphi} = \ell \frac{\partial T}{\partial t}.$$

These equations define the two-dimensional conformal group at large distances, in accordance with the Brown-Henneaux result for GRΛ [3].

The remaining three components of (4.2a) are used to determine $\theta^i$:

$$\theta^0 = -\frac{\ell^2}{r} T_{,02} + O_3,$$

$$\theta^2 = \frac{\ell^3}{r} T_{,00} + O_3,$$

$$\theta^1 = T_{,2} + O_2,$$

while the conditions (4.2b) produce no new limitations on the parameters.

Rewriting Eqs. (4.4) in the form $\partial_\pm (T \mp S) = 0$, with $2\partial_\pm = \ell \partial_0 \pm \partial_2$, we find that the general solution is given by

$$T + S = f(x^+), \quad T - S = g(x^-),$$

where $f$ and $g$ are two arbitrary, periodic functions.

Parameters $(T, S)$ define the conformal symmetry in the asymptotic region of our spacetime. In addition to the conformal transformations, the complete gauge group defined by (4.3) contains also the residual (or pure) gauge transformations, characterized by the higher order terms that remain after imposing $T = S = 0$. As we shall see, the residual gauge transformations do not contribute to the values of the conserved charges (their generators vanish weakly), and consequently, they can be ignored in our discussion of the conserved charges. This is effectively done by introducing the improved definition of the asymptotic symmetry:

- the asymptotic symmetry group is defined as the factor group of the gauge group determined by (4.3), with respect to the residual gauge group.

In conclusion, the asymptotic behaviour (4.1) defines the most general configuration space of the theory that respects our requirements (a) and (b) formulated at the beginning of this section. In order to verify the status of the last requirement (c), it is necessary to explore the canonical structure of the theory.
5 Canonical structure of the asymptotic symmetry

After having introduced the notion of the asymptotic symmetry group, we now continue with the related canonical analysis. We construct the improved form of the general gauge generator, assuming the asymptotic conditions (4.1) and (4.3), and prove the conservation of the corresponding charges; then, we investigate the canonical algebra of asymptotic generators [16].

Hamiltonian and constraints. Starting from the definition of the canonical momenta \( (\pi^i_\mu, \Pi^i_\mu) \), corresponding to the Lagrangian variables \((b^i_\mu, \omega^i_\mu)\), we use the Dirac procedure for the constrained dynamical systems to explore the dynamical structure of the theory.

The primary constraints are of the form:

\[
\phi_i^0 \equiv \pi_i^0 \approx 0, \quad \Phi_i^0 \equiv \Pi_i^0 \approx 0, \\
\phi_i^\alpha \equiv \pi_i^\alpha + \frac{2a}{\ell} \varepsilon^{\alpha\beta\gamma} b_i^\beta \approx 0, \quad \Phi_i^\alpha \equiv \Pi_i^\alpha - 2a \varepsilon^{\alpha\beta\gamma} b_i^\beta \approx 0.
\]

Up to an irrelevant divergence, the total Hamiltonian reads

\[
\mathcal{H}_T = b_i^0 \mathcal{H}_i + \omega_i^0 \mathcal{K}_i + u_i^0 \pi_i^0 + v_i^0 \Pi_i^0, \tag{5.1}
\]

where

\[
\mathcal{H}_i = -a \varepsilon^{\alpha\beta} \left( R^\alpha_{\beta\gamma} - \frac{2}{\ell} T^\alpha_{\beta\gamma} + \frac{4}{\ell^2} \varepsilon_{ijk} b^i_\alpha b^j_\beta \right) - \nabla^\beta \phi_i^\beta + \frac{2}{\ell} \varepsilon_{imn} b^m_\beta \phi^n_\beta,
\]

\[
\mathcal{K}_i = -a \varepsilon^{\alpha\beta} \left( T^\alpha_{\beta\gamma} - \frac{2}{\ell} \varepsilon_{ijk} b^i_\alpha b^j_\beta \right) - \nabla^\beta \Phi_i^\beta - \varepsilon_{imn} b^m_\beta \phi^n_\beta.
\]

The constraints are classified as follows: \((\pi^i_0, \Pi^i_0, \mathcal{H}_i, \mathcal{K}_i)\) are first class, while \((\phi_i^\alpha, \Phi_i^\alpha)\) are second class.

Canonical generators. Gauge symmetries of a dynamical system are best described by the canonical generator, which is constructed using the general method of Castellani [25]. Expressed in terms of the conveniently chosen parameters \( \xi^\mu \) and \( \theta^i \), the gauge generator is given by

\[
G = -G_1 - G_2, \\
G_1 \equiv \dot{\xi}^\rho \left( b^i_\rho \pi_i^0 + \omega^i_\rho \Pi_i^0 \right) + \xi^\rho \left[ b^i_\rho \mathcal{H}_i + \omega^i_\rho \mathcal{K}_i + (\partial_\rho b^0_\alpha) \pi_i^0 + (\partial_\rho \omega^0_\alpha) \Pi_i^0 \right], \\
G_2 \equiv \dot{\theta}^i \Pi_i^0 + \theta^i \left[ \mathcal{K}_i - \varepsilon_{ijk} \left( b^0_\alpha \pi^k_0 + \omega^0_\alpha \Pi^k_0 \right) \right]. \tag{5.2}
\]

Here, the time derivatives \( \dot{b}^i_\mu \) and \( \dot{\omega}^i_\mu \) are shorts for \( u^i_\mu \) and \( v^i_\mu \), respectively, and the integration symbol \( \int d^2x \) is omitted in order to simplify the notation.

The transformation law of the fields, defined by the Poisson bracket \( \delta_0 \phi \equiv \{ \phi, G \} \), is in complete agreement with the gauge transformations (2.1) on shell.

Asymptotics of the phase space. To complete the analysis of the asymptotic structure of phase space, we need to define the behaviour of momentum variables at large distances. Our procedure is based on the following general principle:

- the expressions that vanish on-shell should have an arbitrarily fast asymptotic decrease, as no solution of the field equations is thereby lost.
Applied to the primary constraints of the theory, this principle gives the asymptotic behaviour of
the momenta $\pi_i^\mu$ and $\Pi_i^\mu$. The same principle can be also applied to the secondary constraints
and the true equations of motion, producing a refined form of the original asymptotic conditions.

We are now ready to discuss the impact of the adopted boundary conditions on the form of
the canonical generator.

**The improved generator.** The canonical symmetry generators act on dynamical variables via
the PB operation, which is defined in terms of functional derivatives. Hence, the phase-space
functionals representing the gauge generators must have *well defined functional derivatives*. Our
general gauge generator $G$ does not meet this requirement, but the problem is corrected by adding
suitable boundary terms [26].

The improved gauge generator $\tilde{G}$ is found to have the following form:

\[
\tilde{G} = G + \Gamma ,
\]

\[
\Gamma = -\int_{0}^{2\pi} d\phi \left( \ell T E^1 + S M^1 \right),
\]

where the integration goes over the circle at infinity (the boundary of the spatial section of space-
time), and

\[
E^\alpha = 2\alpha \varepsilon^{\alpha \beta 0} \left( \omega^0_\beta + \frac{1}{\ell} b_\beta^2 - \frac{1}{\ell} b_0^\beta b^0_\beta \right),
\]

\[
M^\alpha = -2\alpha \varepsilon^{\alpha \beta 0} \left( \omega^2_\beta + \frac{1}{\ell} b^0_\beta - \frac{1}{\ell} b^2_\beta \right) b^0_\beta .
\]

The adopted asymptotic behaviour guarantees finiteness of $\Gamma$, hence, $\tilde{G}$ is a well defined generator
(finite and differentiable functional). Thus, the requirement (c), defined at the beginning of section
4, is automatically satisfied.

As we can see, the surface term $\Gamma$ depends only on the parameters $(T, S)$, and not on the higher
order terms in (4.3). Thus, it is only the asymptotic generators that have non-trivial surface terms,
or charges. On the other hand, the residual gauge generators are characterized by vanishing $\Gamma$, and
can only have zero charges [3, 24].

Two special cases of the improved generator (5.5) are of particular importance: the time
translation generator $\tilde{G}[\xi^0]$, and the spatial rotation generator $\tilde{G}[\xi^2]$. For $\xi^0 = 1$ and $\xi^2 = 1$,
the corresponding surface terms have the meaning of energy and angular momentum, respectively:

\[
E_0 = \int_{0}^{2\pi} E^1 d\phi , \quad M_0 = \int_{0}^{2\pi} M^1 d\phi .
\]

**Canonical algebra.** We now wish to find the PB algebra of the improved generators, which
contains important information regarding the symmetry structure of the asymptotic dynamics.

Introducing the notation $G' \equiv G[T', S']$, $G'' \equiv G[T'', S'']$, and so on, the PB algebra is found
to have the form

\[
\{ \tilde{G}^{m}, \tilde{G}' \} = \tilde{G}^{m'} + C^{m'} ,
\]

where the parameters $T''$, $S''$ are determined by the relations

\[
T'' = T' S''_2 - T'' S'_2 + S' T''_2 - S'' T'_2 , \quad S'' = S' S''_2 - S'' S'_2 + T' T''_2 - T'' T'_2 ,
\]

and $C^{m'} \equiv C[T', S'; T'', S'']$ is the central term of the canonical algebra:

\[
C^{m'} = 2\alpha \ell \int d\phi \left( S''_2 T'_{22} - S'_2 T''_2 \right).
\]
Conservation laws. Direct calculation based on the PB algebra (5.5) shows that the asymptotic generator $\tilde{G}[T, S]$ is conserved:

$$\frac{d}{dt} \tilde{G}[T, S] = \frac{\partial \tilde{G}}{\partial t} + \{\tilde{G}, \tilde{H}_T\} \approx 0. \tag{5.6}$$

This also implies the conservation of the boundary term $\Gamma$.

To test the obtained result, we calculate the values of all the conserved charges for the BTZ black hole solution (3.2). Recalling that $a = 1/(16\pi G) = 1/(4\pi) \ (\text{in units } 4G = 1)$, we obtain

$$E_0(\text{black hole}) = m, \quad M_0(\text{black hole}) = J.$$  

Thus, the black hole parameters $m$ and $J$ have the meaning of energy and angular momentum, respectively. One can also show that there are no other independent conserved charges [16].

Central charge. The PB algebra (5.7) can be brought to a more familiar form by using the representation in terms of Fourier modes. Indeed, after introducing

$$2L_n = -\tilde{G}[T = S = e^{inx}], \quad 2\bar{L}_n = -\tilde{G}[T = -S = e^{inx}],$$

the canonical algebra takes the form of two independent Virasoro algebras with classical central charges:

$$\{L_n, L_m\} = -i(n - m)L_{n+m} - 2\pi i a \ell n^3 \delta_{n,-m},$$

$$\{\bar{L}_n, \bar{L}_m\} = -i(n - m)\bar{L}_{n+m} - 2\pi i a \ell n^3 \delta_{n,-m},$$

$$\{L_n, \bar{L}_m\} = 0. \tag{5.7}$$

Upon the redefinition of the zero modes, $L_0 \rightarrow L_0 + \pi a \ell$, $\bar{L}_0 \rightarrow \bar{L}_0 + \pi a \ell$, we obtain the standard form of the Virasoro algebras. Using the string theory normalization of the central charge, we have

$$c_1 = c_2 = 12 \cdot 2\pi a \ell = \frac{3\ell}{2G}. \tag{5.8}$$

Thus, two central charges in the teleparallel theory coincide with each other, and with the Brown–Henneaux central charge, defined in Riemannian GR$_A$.

The form (5.7) of the asymptotic algebra shows that central term for the AdS subgroup, generated by $(L_{-1}, L_0, L_1), (\bar{L}_{-1}, \bar{L}_0, \bar{L}_1)$, vanishes. This is a consequence of the fact that the AdS subgroup is an exact symmetry of the vacuum (3.3) [3].

6 Central charges in Riemann-Cartan gravity

We now return to the general Riemann-Cartan action (2.4) to discuss the form of the corresponding asymptotic symmetry [18].

Starting from the Chern-Simons Lagrangian, $L_{CS}(A) = A^i dA_i + \frac{1}{3} \varepsilon_{ijk} A^i A^j A^k$, and introducing the new variables $A^i = \omega^i + qb^i$ and $\bar{A}^i = \omega^i + \bar{q}b^i$, with $q \neq \bar{q}$, one can derive the important identity

$$\kappa_1 L_{CS}(A) - \kappa_2 L_{CS}(\bar{A}) = 2ab^i R_i - \frac{1}{3} \Lambda \varepsilon_{ijk} b^i b^j b^k$$

$$+ \alpha_3 L_{CS}(\omega) + \alpha_4 b^i T_i + \alpha d(b^i \omega_i), \tag{6.1a}$$
where
\[
a = \kappa_1 q - \kappa_2 \bar{q}, \quad \Lambda = -(\kappa_1 q^3 - \kappa_2 \bar{q}^3),
\]
\[
\alpha_3 = \kappa_1 - \kappa_2, \quad \alpha_4 = \kappa_1 q^2 - \kappa_2 \bar{q}^2.
\]
(6.1b)

Comparing with (2.4), we can rewrite this identity in the form:
\[
\kappa_1 I_{CS}[A] - \kappa_2 I_{CS}[\bar{A}] = I_G + a \int d(b^i \omega_i),
\]
(6.2)

where \(I_G\) denotes the gravitational action in (2.4). The additional surface integral on the right hand side is just a correction which improves the differentiability of \(I_G\) under the boundary conditions (4.1). Thus, Riemann-Cartan gravity in 3d can be formulated as a Chern-Simons gauge theory. The role of the coefficients \(\kappa_1\) and \(\kappa_2\) is to define central charges of the asymptotic symmetry.

Since \(q \neq \bar{q}\), one can easily find the inverse of (6.1b):
\[
q = -\frac{A}{2} + \frac{1}{\ell}, \quad \bar{q} = -\frac{A}{2} - \frac{1}{\ell},
\]
\[
\kappa_1 - \kappa_2 = \alpha_3, \quad \kappa_1 + \kappa_2 = \ell \left( a + \frac{A}{2} \alpha_3 \right),
\]
(6.3)

where \(\ell = \sqrt{-\Lambda_{\text{eff}}}\) is real. These relations clarify the role of four parameters appearing in the action (2.4). In particular, combining (6.2) and (6.3) one concludes that the gravitational theory with \(\alpha_3 \neq 0\) has conformal symmetry with two different central charges:
\[
c_{1,2} = 12 \cdot 4\pi \kappa_{1,2} = 24\pi \left[ a\ell + \alpha_3 \left( \frac{A\ell}{2} + 1 \right) \right].
\]
(6.4)

In the complementary sector with \(\alpha_3 = 0\) [GR\(\Lambda\) and our teleparallel theory (2.8)], the central charges \(c_1\) and \(c_2\) are equal to each other.

### 7 Concluding remarks

- 3d gravity with torsion, defined by Eq. (2.8), possesses the teleparallel black hole solution, a generalization of the Riemannian BTZ black hole.
- Assuming the AdS asymptotic conditions, the canonical asymptotic symmetry is realized by two commuting Virasoro algebras with central extensions:
  - in GR\(\Lambda\) and in the teleparallel theory: \(c_1 = c_2 = 3\ell/2G\),
  - in Riemann-Cartan theory: \(c_1 \neq c_2\).

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References


