Number of photons and brilliance of the radiation from a crystalline undulator

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ABSTRACT

The scheme for accurate quantitative treatment of the radiation from a crystalline undulator in presence of the dechanneling and the photon attenuation is presented. The number of emitted photons and the brilliance of electromagnetic radiation generated by ultra-relativistic positrons channeling in a crystalline undulator are calculated for various crystals, positron energies and different bending parameters. It is demonstrated that with the use of high-energy positron beams available at present in modern colliders it is possible to generate the crystalline undulator radiation with energies from hundreds of keV up to tens of MeV region. The brilliance of the undulator radiation within this energy range is comparable to that of conventional light sources of the third generation but for much lower photon energies.

Keywords: crystalline undulator, dechanneling, photon attenuation, brilliance

1. INTRODUCTION

In this paper new results from the theory of electromagnetic radiation emitted by a bunch of ultra-relativistic positrons channeling through a periodically deformed crystal (a crystalline undulator) are reported. We formulate the approximation for effective analytical and numerical analysis of the characteristics of the undulator radiation with account for the influence of two main parasitic effects, the positron dechanneling and the photon attenuation. The developed formalism is applied to calculate the number of the emitted photons and the brilliance of the radiation formed in crystalline undulators.

In a crystalline undulator there appears, in addition to a well-known channeling radiation, the radiation of an undulator type which is due to the periodic motion of channeling particles which follow the bending of the crystallographic planes. The parameters of the undulator radiation can be easily varied by changing the energy of beam particles and the parameters of crystal bending. The feasibility of this scheme was explicitly demonstrated for in Refs. 1, 2. In these papers as well as in the subsequent publications\textsuperscript{3–8} the idea of this new type of radiation, the essential conditions and limitations which must be fulfilled to make possible the observation of the effect were formulated in an adequate form for the first time. A number of corresponding numerical results were presented to illustrate the developed theory. The importance of the ideas suggested and discussed in the cited papers has also been realized by other authors resulting in a significant increase of the number of publications in the field during last years\textsuperscript{9–17} but, unfortunately, often without proper citation.\textsuperscript{11–17} A detailed review of the results obtained in this newly arisen field as well as a historical survey of the development of all principal ideas and related phenomena can be found in Ref. 18.

The mechanism of the photon emission by means of a crystalline undulator is illustrated in Fig. 1. The \((yz)\)-plane in the figure is a cross section of an initially linear crystal, and the \(z\)-axis represents the cross section of a midplane of two neighbouring non-deformed crystallographic planes (not drawn in the figure) spaced by the interplanar distance \(d\).

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Under certain conditions the ultra-relativistic positrons will channel in the periodically bent channel. The trajectory of a particle contains two elements. Firstly, there are channeling oscillations due to the action of the interplanar potential. Their typical frequency $\Omega_{\text{ch}}$ depends on the positron energy $\varepsilon$ and the parameters of the interplanar potential. Secondly, there are oscillations related to the periodicity of the distorted midplane, - the undulator oscillations, whose frequency is $\omega_0 = 2\pi c/\lambda_u$.

The spontaneous emission of photons is associated with both of these oscillations. The typical frequency of the channeling radiation is $\omega_{\text{ch}} \approx 2\gamma^2\Omega_{\text{ch}}$ where $\gamma = \varepsilon/mc^2$ is the relativistic Lorenz factor. The undulator oscillations give rise to the photons with frequency $\omega_u \approx 4\gamma^2\omega_0/(2 + p^2)$ where $p$ is the undulator parameter, $p = 2\pi\gamma(a/\lambda_u)$. If strong inequality $\omega_0 \ll \Omega_{\text{ch}}$ is met than the frequencies of the channeling radiation and the undulator radiation are also well separated, $\omega \ll \omega_{\text{ch}}$. In this case the characteristics of the undulator radiation are practically independent on the channeling oscillations but depend on the shape of the periodically bent midplane.

There are essential features which distinguish a crystalline undulator from a conventional one based on the action of the periodic magnetic (or electric) field on the projectile. In the latter the beam of particles and the photon flux move in vacuum whereas in the proposed scheme they propagate through a crystalline media. Therefore, to prove that the crystalline undulator is feasible, it is necessary to analyze the influence of the interaction of both beams with the crystal constituents. On the basis of such analysis one can formulate the conditions which must be met and define the ranges of parameters (which include $\varepsilon$, $a$, $\lambda_u$ and also the crystal length $L_u$ and the photon energy $\hbar\omega$) within which all the criteria are fulfilled. In full this analysis was carried out very recently and the feasibility of the crystalline undulator was demonstrated in an adequate form for the first time in Refs. 1, 2 and in Refs. 3–8.

For further referencing let us briefly mention the conditions which must be met in a crystalline undulator.

A stable planar channeling of an ultra-relativistic positron in a periodically bent crystal occurs if the maximum centrifugal force, $F_{\text{ct}}$, is less than the maximal force due to the interplanar field, $F_{\text{int}}$. Notating the ratio $F_{\text{ct}}/F_{\text{int}}$ as $C$ one formulates this condition as follows$^{1,2,19}$:

$$C = (2\pi)^2 \frac{\varepsilon}{U_{\text{max}}^r} \frac{a}{\lambda_u^2} \ll 1. \tag{1}$$

There are two essentially different regimes of the radiation formation in a periodically bent crystals. They are defined by the magnitude of the ratio $a/d$. In the case of low amplitudes, $a/d \ll 1$, the characteristic frequencies of the channeling radiation and the undulator radiation become compatible $\omega_u \sim \omega_{\text{ch}}$. This results in the loss of
the monochromaticity of the radiation, since the channeling radiation is essentially non-monochromatic due to
noticeable deviations of the interplanar potential from a harmonic form. Additionally, in this case the intensity of
undulator radiation is small compared with that of the channeling radiation.3, 4

On the contrary, in the limit \( a \gg d \) not only the characteristic frequencies are well separated, \( \omega_{\text{u}}/\omega_{\text{ch}} \approx C d/a \ll 1 \), but also the undulator radiation intensity is higher than the intensity of the channeling radiation.1, 2, 4

As a result, if one is only interested in the spectral distribution of the undulator radiation, one may disregard the
channeling oscillations and assume that the projectile moves along the centerline of the bent channel. Therefore,
the criterion which is imposed on the relative magnitudes of \( d, a \) and \( \lambda_u \) is as follows

\[ d \ll a \ll \lambda_u. \tag{2} \]

The second inequality ensures that the crystal is deformed elastically, and its structure and symmetry are not
affected by the deformation.

The term 'undulator' implies that the number of undulator periods, \( N_u \), is large. Only in this limit does
the radiation formed during the passage of a bunch of relativistic particles through a periodic system bear the
features of undulator radiation (narrow, well-separated peaks in spectral-angular distribution) rather than those
of synchrotron radiation. Hence, the following strong inequality, which entangles the period \( \lambda_u \) and the length
of a crystal \( L_u \) must be met in the crystalline undulator1, 2:

\[ N_u = \frac{L_u}{\lambda_u} \gg 1. \tag{3} \]

The coherence of the radiation, emitted in the crystalline undulator, takes place if the energy of the channeling
particle does not change noticeably during with the penetration distance. For ultra-relativistic projectiles the
main source of energy losses are the radiative losses. Therefore, it is important to establish the range of energies
for which the parameters of undulator radiation formed in a perfect periodic crystalline structure are stable. In
Ref. 3 a comprehensive quantitative analysis of the radiative loss of energy, \( \Delta \varepsilon \), due to the channeling and the
undulator radiation was carried out. It was established that the relative radiative losses \( \Delta \varepsilon/\varepsilon \) become large if
the initial energy of the positron bunch is \( \varepsilon > 10 \text{ GeV} \). For lower energies of positrons

\[ \varepsilon < 10 \text{ GeV}, \tag{4} \]

the radiative losses are small, \( \Delta \varepsilon < 0.01 \varepsilon \).

As was pointed out Refs. 1, 2, 5–7 two phenomena, the dechanneling effect and the photon attenuation, lead to
severe limitation on the length of a crystalline undulator.

If the dechanneling effect is neglected, one may unrestrictedly increase the intensity of the undulator radiation
by considering larger \( N_u \)-values. In reality, random scattering of the channeling particle by the electrons and
nuclei of the crystal leads to a gradual increase of the particle energy associated with the transverse oscillations
in the channel. As a result, the transverse energy at some distance from the entrance point exceeds the depth
of the interplanar potential well, and the particle leaves the channel. The mean penetration distance covered
by a channeling particle is called the dechanneling length. For given crystal and projectile the dechanneling
length \( L_d = L_d(\varepsilon, C) \) depends on the energy \( \varepsilon \) and on the parameter \( C \) (see (1)). To calculate the dechanneling
length one can either apply the diffusion theory to describe the multiple scattering20, 21 or carry out a computer
simulation of the scattering process of the projectile from the crystal constituents.5, 22 Alternatively, to estimate
\( L_d(\varepsilon, C) \) one can use the approximate formulae.2, 22 For an ultra-relativistic positron the dechanneling length in
straight channels (i.e. \( C = 0 \)) in various crystals lies within the interval \( L_d(\varepsilon, 0) \approx (0.05...0.08) \varepsilon \text{ (GeV)} \).18
i.e. does not exceed several millimeters at GeV energies of a positron. For a periodically bent channel the
dechanneling length decreases as \( C \) grows following, approximately, the law \( L_d(\varepsilon, C) \approx (1 - C)^2 L_d(\varepsilon, 0) \).5, 22

The propagation of photons emitted in a crystalline undulator is strongly influenced by the atomic and the
nuclear photoeffects, the coherent and incoherent scattering on electrons and nuclei, the electron-positron pair
production. All these processes lead to the decrease in the intensity of the photon flux as it propagates through
the crystal. A quantitative parameter, which accounts for all these effects and defines the scale within which the
The spectral-angular distribution of the energy \( E \) indicates that the propagation of positrons and photons occurs in vacuum. The results from the general theory of undulator radiation (see, e.g., Refs. 25–27). We use the term ‘ideal undulator’ can be written in the following form

\[
\int_0^\infty \frac{d^3E}{h\omega d\Omega} = S(\omega, \theta, \phi) D_N(\tilde{\eta}) .
\]  

Here \( \theta \ll 1 \) and \( \phi \) are the emission angles with respect to the undulator axis, \( d\Omega = \theta d\theta d\phi \) is the solid angle of the emission. The function \( S(\omega, \theta, \phi) \), which does not depend on the undulator length, is given by

\[
S(\omega, \theta, \phi) = \frac{\alpha}{4\pi^2} \frac{\omega^2}{\gamma^2 \omega_0^2} \left\{ p^2 |I_1|^2 + \frac{\gamma^2 \theta^2}{2} |I_0|^2 - 2p\gamma \theta \cos \varphi \Re (I_0^* I_1) \right\} ,
\]

\[
I_m = \int_0^{2\pi} d\psi \cos^m \psi \exp \left( i \left[ \eta \psi + \frac{p^2 \omega}{8\gamma^2 \omega_0} \sin(2\psi) - \frac{p\omega}{\gamma \omega_0} \theta \cos \varphi \sin \psi \right] \right) , \quad m = 0, 1 .
\]

Here \( \alpha \approx 1/137, \omega_0 = 2\pi c/\lambda_0, p \) is the undulator parameter and the parameter \( \eta \) is given by

\[
\eta = \frac{\omega}{2\gamma^2 \omega_0} \left( 1 + \gamma^2 \theta^2 + \frac{p^2}{2} \right) .
\]
The factor $D_{N_u}(\tilde{\eta})$ on the right-hand side of (5) is defined as follows

$$D_{N_u}(\tilde{\eta}) = \left( \frac{\sin N_u \pi \tilde{\eta}}{\sin \pi \tilde{\eta}} \right)^2,$$

where $\tilde{\eta} = \eta - n$ and $n$ is a positive integer such that $n - 1/2 < \eta < n + 1/2$.

For $N_u \gg 1$ the function $D_{N_u}(\tilde{\eta})$ has a sharp and powerful maximum in the point $\tilde{\eta} = 0$, where $D_{N_u}(0) = N_u^2$. The width of the peak $\Delta \tilde{\eta}_n$ is equal to $1/N_u$. This behaviour of $D_{N_u}(\tilde{\eta})$ results in a peculiar form of the spectral-angular distribution of undulator radiation which clearly distinguishes it from other types of electromagnetic radiation formed by a charge moving in external fields. Namely, for each value of the emission angle $\theta$ the spectral distribution consists of a set of narrow and equally spaced peaks (harmonics). The peak intensity is proportional to $N_u^2$. This factor reflects the constructive interference of radiation emitted from each of the undulator periods and is typical for any system which contains $N_u$ coherent emitters.

The values $\omega_n$ of the harmonics frequencies follow from the condition that parameter $\eta$ becomes an integer (this corresponds to $\tilde{\eta} = 0$). In particular, in the case of the forward emission, $\theta = 0$, the harmonics frequencies are defined from the relation

$$n = \frac{1}{2}\gamma^2 \frac{\omega_n}{\omega_0} \left( 1 + \frac{p^2}{2} \right),$$

where $\gamma$ and $p$ are relativistic particles. These quantities are the $n$th harmonic. Using $\Delta \tilde{\eta}_n = 1/N_u$ and accounting for (10) one derives

$$\Delta \Omega_n = \frac{\pi}{\gamma^2} \frac{1 + p^2/2}{nN_u}, \quad \frac{\Delta \omega_n}{\omega_n} = \frac{1}{nN_u}.$$  (12)

Formulae (11)-(12) allow one to calculate the number of photons $\Delta N_{\omega_n}$ of energy $\omega = \omega_n - \Delta \omega_n/2, \omega_n + \Delta \omega_n/2$ emitted by a beam particle within the cone $\Delta \Omega_n$:

$$\Delta N_{\omega_n} = \frac{\Delta \Omega_n}{\omega_n} \frac{\Delta \omega_n}{\omega_n} = \pi \alpha N_u Q_n(p) \frac{\Delta \omega_n}{\omega_n}.$$  (13)

where $Q_n(p) = 4z \left[ J_{(n-1)/2}(z) - J_{(n+1)/2}(z) \right]^2$.

Let us introduce two other quantities which characterize the radiation formed in an undulator and are closely related to the number of the emitted photons, but also take into account the properties of the beam of ultra-relativistic particles. These quantities are the flux and the brilliance (see, e.g. Ref. 27).

The flux $F_n$ describes the number of photons per second of the $n$th harmonic emitted in the cone $\Delta \Omega_n$ and in a given bandwidth. A quantitative definition of this quantity, measured in (photons/s/0.1%BW) (the abbreviation 'BW' stands for the bandwidth $\Delta \omega_n/\omega_n$), is given by the following formula:

$$F_n = \frac{\Delta N_{\omega_n}}{10^3(\Delta \omega_n/\omega_n)} \frac{I}{e} = 10^{-3} \pi \alpha N_u Q_n(p) \frac{I}{e} = 1.431 \times 10^{14} N_u Q_n(p) I [A],$$  (14)

where $I$ is the electric current of the beam. In the latter expression $I$ is measured in Amperes.
The general definition of brilliance of the photon source of a finite size is given in terms of the number of photons of energy $\hbar \omega$ emitted in the cone $\Delta \Omega$ per unit time interval, unit source area, unit solid angle and per bandwidth. To calculate this quantity is it necessary to know the beam sizes $\sigma_x$, $\sigma_y$ and angular divergencies $\phi_x$, $\phi_y$ in two perpendicular directions, as well as the divergence angle of the radiation and the ‘size’ of the photon beam. The brilliance of undulator radiation can be related to the flux $F_n$ as follows:\n
$$B_n = \frac{F_n}{(2\pi)^2 \epsilon_x \epsilon_y}.$$\n
(15)\n
Here $\epsilon_{x,y} = \sqrt{\sigma_{x,y}^2 + \sigma_{x,y}^2}$ are the total emittance of the photon source in the $x$ and $y$ directions, with $\phi_n = \sqrt{\Delta \Omega_n / 2\pi}$ being the angular width of the $n$th harmonic and $\sigma_n = \lambda_n / 4\pi \phi_n$ is the ‘apparent’ source size taken in the diffraction limit.\n
To obtain brilliance in the units $\left( \text{photons/s/mrad}^2/\text{mm}^2/0.1\%\text{BW} \right)$ the quantities $\sigma_{x,y}$ and $\sigma_n$ must be measured in millimeters whereas the angular variables $\phi_{x,y}$ and $\phi_n$ - in milliradians.

3. CHARACTERISTICS OF RADIATION FORMED IN A CRYSTALLINE UNDULATOR

In an ideal undulator the beam of particles and the emitted photons propagate in vacuum. In a crystalline undulator, due to the interactions with crystal atoms, the particles can dechannel, and thus be lost for further motion through the undulator. Additionally, the photons emitted inside the crystal can be absorbed or scattered while making their way out from the crystal. Therefore, it is necessary to account for the processes of dechanneling and photon attenuation. In what follows we carry out the qualitative analysis of the influence of these two processes on the characteristics of the radiation formed in a crystalline undulator.

3.1. Spectral-angular distribution in presence of the dechanneling and attenuation

Let the crystal length, the amplitude and period of bending and the energy $\varepsilon$ satisfy the conditions (1)-(4).

A positron, which enters the crystal at small incident angle with respect to the curved crystallographic plane, penetrates through the crystal following the bending of its channel. However, due to random scattering by the electrons and nuclei of the crystal the energy of the transverse oscillations of the positron in the channel increases, and finally the particle leaves the channel, becoming lost for the crystalline undulator. Although the rigorous treatment of the dechanneling process cannot be implemented by analytical means only, it is possible to develop a model approach based on the assumption that the probability $w(z)$ for a particle to penetrate at a distance $z$ along the undulator axis ($z \in [0,L_u]$) can be described by the exponential decay law

$$w(z) = \exp (-z/L_d).$$\n
(16)\n
In intermediate formulae when referring to the dechanneling length we omit its arguments $\varepsilon$ and $C$.\n
With the effect of dechanneling taken into account the spectral-angular distribution of the radiated energy per one particle can be written as follows:

$$\frac{d^3E}{\hbar d\omega d\Omega} = \frac{d^3E^{(-)}}{\hbar d\omega d\Omega} + \frac{d^3E^{(+)}}{\hbar d\omega d\Omega}.$$\n
(17)\n
The first term is the contribution to from all the processes in which the particle dechannels somewhere inside the crystal. To calculate this term one notices that the quantity $L_d^{-1}dz \exp (-z/L_d)$ defines the probability of a particle to channel through the distance $z$ and then dechannel within the interval $dz$. Such a particle emits the radiation which corresponds to the undulator of the length $z$ and the number of periods $z/\lambda_u$. Therefore

$$\frac{d^3E^{(-)}}{\hbar d\omega d\Omega} = \int_0^{L_d} \frac{dz}{L_d} e^{-z/L_d} \frac{d^3E^{(\text{att})}(z)}{\hbar d\omega d\Omega}.$$\n
(18)
and in this undulator. In the final formula this limitation will be omitted. Throughout the text the notations formulae we assume that the ratio $N_z$ described by (5) where one substitutes $j \lambda_u$ on its way to a distant detection point $R_0$ ($R_0 \gg L_u$).

The integral in (18) is also evaluated with the help of (5) where one substitutes $D_{N_z}(\eta_j)$ defined in (9). Therefore one can write

$$
\frac{d^3 E^{(+)}(z)}{\hbar d\omega d\Omega} = e^{-L_u/L_d} \frac{d^3 E^{(att)}(L_u)}{\hbar d\omega d\Omega}.
$$

The second term on the right-hand side of (17) is due to the process when the projectile channels through the whole length $L_u$. Its probability is given by the factor $\exp (-L_u/L_d)$. Therefore one can write

$$
\frac{d^3 E^{(+)}(z)}{\hbar d\omega d\Omega} = \frac{d^3 E^{(att)}(L_u)}{\hbar d\omega d\Omega}.
$$

If the photon attenuation is neglected, then to calculate (19) one uses (5) instead of $d^3 E^{(att)}(L_u)/\hbar d\omega d\Omega$. The integral in (18) is also evaluated with the help of (5) where one substitutes $N_u$ with $z/\lambda_u$. Such approach was applied Ref. 5 with the only difference that in the cited paper to calculate (18) and (19) we used the discrete probabilities instead of the continuous distribution function (16). The use of the latter implies that the dechanneling effect is small over the scale of one undulator period and, therefore, $L_d \gg \lambda_u$.

Now let us turn to the derivation of the quantity $d^3 E^{(att)}(z)/\hbar d\omega d\Omega$ which is the spectral-angular distribution of radiation formed in the undulator of the length $z \leq L_u$ in presence of the attenuation. In the intermediate formulae we assume that the ratio $N_z = z/\lambda_u$ is an integer number which corresponds to the number of periods in this undulator. In the final formula this limitation will be omitted. Throughout the text the notations $L_u$ and $N_u$ are reserved for the length of the crystal and the number of undulator periods within $L_u$.

As mentioned above, if one neglects the photon attenuation effect, the distribution $d^3 E^{(att)}(z)/\hbar d\omega d\Omega$ is described by (5) where one substitutes $N_u$ with $N_z$. The only quantity in (5) which depends on the number of undulator periods is the factor $D_{N_z}(\eta_j)$ defined in (9). This factor appears in the formula for spectral-angular distribution as a result of squaring the modulus of a coherent sum of the amplitudes of electromagnetic waves emitted from spatially different but similar parts of the undulator. In more detail, $D_{N_z}(\eta_j)$ is given by

$$
D_{N_z}(\eta_j) = \sum_{j=1}^{N_z} \exp \left( i k R_0 - 2\pi \eta_j \right)^2.
$$

The argument $(k R_0 - 2\pi \eta_j) (k = \omega/c$ is the wavenumber) stands for the phase of the electromagnetic wave emitted within the $j$th period of the undulator and detected at some distant point $R_0$ from the undulator. It is assumed that the quantities $L_u$, $z$ and $R_0$ satisfy the relations: $z \leq L_u \ll R_0$.

In a crystalline undulator a photon emitted within the $j$th period in the direction of the point $R_0$ can be absorbed within the distance $L_u - j \lambda_u$ while propagating through the crystal, see Fig. 2. To account for this possibility one can assume that the wavenumber becomes complex, $k \rightarrow \omega/c + i\mu/2$. The quantity $\mu = \mu(\omega)$ defines the attenuation length $L_u(\omega) = \mu^{-1}(\omega)$ within which the photon flux is reduced by a factor of $e$. For a complex $k$ the phase factor $e^{ik R_0}$, which in an ideal undulator is the same for all periods $j = 1 \ldots N_z$, is replaced.

![Figure 2](image-url) Illustration of the photon attenuation in a crystalline undulator. A photon (the long-dashed line), emitted within the $j$th period of the undulator of the length $L_u$, can be absorbed (or scattered) in the part of crystal of thickness $L_u - j \lambda_u$ on its way to a distant detection point $R_0$ ($R_0 \gg L_u$).
with \( e^{ik_R_0 e^{-\mu(L_u - j\lambda_u)}} \), and a proper expression for \( D_{N_u} \) is

\[
D_{N_z}(\bar{\eta}) \to D_{N_z}^{(\text{att})}(\bar{\eta}) = \left| \sum_{j=1}^{N_z} e^{2i\pi \bar{\eta} j} e^{-\Phi(L_u - j\lambda_u)} \right|^2 = \frac{e^{-\mu L_u} + e^{\mu z} - 2e^{\mu z/2} \cos(2\pi \bar{\eta} N_z)}{1 + e^{\mu \lambda_u} - 2e^{\mu \lambda_u/2} \cos(2\pi \bar{\eta})},
\]

(20)

The spectral-angular distribution of radiation in presence of the photon attenuation acquires the form

\[
\frac{d^3E^{(\text{att})}(z)}{h\omega \, d\Omega} = S(\omega, \theta, \varphi) D_{N_z}^{(\text{att})}(\bar{\eta}).
\]

(21)

In the limit \( \mu \to 0 \) (i.e., when there is no attenuation) the factor \( D_{N_z}^{(\text{att})}(\bar{\eta}) \) becomes equal to \( D_{N_z}(\bar{\eta}) \) from (9), and the right-hand side of (21) reduces to that of Eq. (5).

To derive the explicit expression for the spectral-angular distribution of the radiated energy from a crystalline undulator one uses (21) in (17)–(19). Let us note here, that although the expression (20) was obtained for the spectral-angular distribution (17) of radiation formed in the crystalline undulator in the form similar to (5) being of the order of magnitude \( \lambda_u \), and the right-hand side of (23) its main features can be easily understood. Firstly, we notice that if the dechanneling is neglected, \( L_d \to \infty \) (or \( \kappa_d \to 0 \)), the function \( D_{N_u}(\bar{\eta}) \) reproduces \( D_{N_u}^{(\text{att})}(\bar{\eta}) \) from (20). In another limit \( \kappa_d = \kappa_a = 0 \) (i.e., no attenuation and dechanneling) eq. (23) reduces to the definition of the factor \( D_{N_u}(\bar{\eta}) \) which characterizes the ideal undulator. In the case when only the attenuation effect is neglected the limit of \( D_{N_u}(\bar{\eta}) \) can also be easily evaluated. In either of these cases the main maximum of \( D_{N_u}(\bar{\eta}) \) is located in the point \( \bar{\eta} = 0 \), i.e. when the parameter \( \eta \) reduces to an integer, and, therefore, the harmonics frequencies are still defined by (10). The maximum value \( D_{N_u}(0) \) can be presented as follows:

\[
D_{N_u}(0) = 4N_d^2 \left[ \frac{e^{-x\kappa_a}}{(1-x)(2-x)} - \frac{e^{-x\kappa_d}}{x(1-x)} + \frac{2e^{-(2+x)\kappa_d/2}}{x(2-x)} \right],
\]

(25)

where the quantity \( N_d = L_d/\lambda_u \) stands for the number of undulator periods within \( L_d \), and the ratio

\[
x = \frac{\kappa_a}{\kappa_d} = \frac{L_d}{L_a}
\]

(26)

does not depend on the crystal length \( L_u \).
The width of the central peak $\Delta \tilde{\eta}$, which in the case of an ideal undulator equals to $1/N$, is increased due to the photon attenuation and the dechanneling. Formally, the additional widths are due to the factors $1/(\kappa_d^2 + 16 N^2 \sin^2 \pi \tilde{\eta})$ and $1/((2\kappa_d - \kappa_a)^2 + 4\phi^2)$ which enter (23). The widths associated with these factors are, respectively, $\Delta \tilde{\eta}_1 = \kappa_a/(2N_\pi)$ and $\Delta \tilde{\eta}_2 = (2\kappa_d - \kappa_a)/(2N_\pi)$. Thus, the total width of the peak is:

$$\Delta \tilde{\eta} = \sqrt{N_a^2 + (\Delta \tilde{\eta}_1)^2 + (\Delta \tilde{\eta}_2)^2} = \frac{1}{N_a} \sqrt{1 + \frac{(\kappa_a - \kappa_d)^2 + \kappa_d^2}{4\pi^2}}$$  \hspace{0.5cm} (27)

The additional widths lead to the enlargement of the solid angle $\Delta \Omega_n$ of the emission cone in the forward direction. In accordance with (27) one derives

$$\Delta \Omega_n = \frac{\pi}{\gamma^2} \frac{1 + p^2/2}{nN_a} \sqrt{1 + \kappa_d^2 \left( \frac{x - 1}{2} \right)^2 + 1 \frac{1}{4\pi^2}}.$$  \hspace{0.5cm} (28)

The formulae for the number of photons $\Delta N_{\omega_n}$ emitted in the cone $\Delta \Omega_n$ as well the corresponding flux of radiation $F_n$ one derives similarly to how it was done in Sect. 2 for an ideal undulator. The result is:

$$\Delta N_{\omega_n} = \pi \alpha N_{\text{eff}}(x, \kappa_d) Q_n(p) \frac{\Delta \omega_n}{\omega_n}$$  \hspace{0.5cm} (29)

$$F_n = 1.431 \times 10^{14} N_{\text{eff}}(x, \kappa_d) Q_n(p) I [A].$$  \hspace{0.5cm} (30)

The difference between these equations and formulae (13) and (14) is that the number of undulator periods $N_a$, met in the latter, is substituted with the effective number of periods, $N_{\text{eff}}(x, \kappa_d)$, which is defined as follows:

$$N_{\text{eff}}(x, \kappa_d) = \frac{D_{N_a}(0)}{N_a} \sqrt{1 + \kappa_d^2 \left( \frac{x - 1}{2} \right)^2 + 1 \frac{1}{4\pi^2}} \equiv N_d f(x, \kappa_d)$$  \hspace{0.5cm} (31)

$$f(x, \kappa_d) = \frac{4}{\kappa_d} \left[ \frac{e^{-x\kappa_d}}{(1 - x)(2 - x)} - \frac{e^{-\kappa_d}}{x(1 - x)} + \frac{2e^{-(2x+1)\kappa_d/2}}{x(2 - x)} \right] \sqrt{1 + \kappa_d^2 \left( \frac{x - 1}{2} \right)^2 + 1 \frac{1}{4\pi^2}}.$$  \hspace{0.5cm} (32)

We use these equations in the subsequent section to define the optimal length of a crystalline undulator.

### 3.2. Optimal length of a crystalline undulator

In the case of an ideal undulator one can, in principle, increase infinitely the length of the undulator. This will result in the increase of the number of photons, the photon flux, and the brilliance since they are proportional to the number of periods. The limitations on the values of $L_a$ and $N_a$ are mainly of a technological nature.

The situation is different for a crystalline undulator, where the number of channeling particles and the number of photons which can emerge from the crystal decrease with the growth of $L_a$. It is seen from (32) that if $L_a \to \infty$ then the parameters $\kappa_d = L_a/L_d$ and $x\kappa_d = L_a/L_d$ also become infinitely large, and the effective number of periods goes to zero leading to $\Delta N_{\omega_n}, F_n \to 0$. This is quite natural result, since in the limit $L_a \gg L_d$ all particles leave the channeling mode and, thus, do not undulate in the most part of the crystal, whereas all emitted photons are absorbed inside the crystal if $L_a \gg L_d$. Another formal (and physically trivial) fact, which follows from (31) and (32), is that $N_{\text{eff}}(x, \kappa_d) = 0$ also for a zero-length undulator, when $L_a = 0$. Vanishing of a positively defined quantity $N_{\text{eff}}(x, \kappa_d)$ at two extreme boundaries suggests that there exists the length $\bar{L}(x)$ for which the effective number of periods (taken for fixed values of $L_a$, $L_d$ and $\lambda_n$) attains the maximum.

To define the value of $\bar{L}(x)$ or, what is equivalent, the quantity $\bar{\kappa}_d(x) = \bar{L}(x)/\kappa_d$, one carries out the derivative of $f(x, \kappa_d)$ with respect to $\kappa_d$ and equalizes it to zero. The analysis of the resulting equation shows that for each value of $x = L_d/L_a \geq 0$ there is only one root $\bar{\kappa}_d$. Hence, the equation defines, in an inexplicit form, a single-valued function $\bar{\kappa}_d(x) = \bar{L}(x)/L_d$ which ensures the maximum of $N_{\text{eff}}(x, \kappa_d)$ for given $L_a$, $L_d$, and $\lambda_n$.

It is important to note that the crystal length enters Eqs. (29)-(30) only via the ratio $\kappa_d$. All other quantities, met in these formulae as well as in (31) and (32), are independent on the length of the crystal. Therefore, the quantity $\bar{L}(x)$ ensures the highest values of $\Delta N_{\omega_n}$ and $F_n$ for the radiation formed in the crystalline undulator. In this sense $\bar{L}(x)$ can be called the optimal length of the undulator which corresponds to a given value of $x$. 
The dependences of $\bar{\kappa}_d(x) = \bar{L}(x)/L_d$ and of the ratio $f(x, \bar{\kappa}_d(x)) = N_{\text{eff}}(x, \bar{\kappa}_d(x))/N_d$ on $x = L_d/L_a$ are presented in Fig. 3. For a given crystalline structure, the dechanneling length $L_d$ is uniquely defined by the energy $\varepsilon$ and the parameters of bending $a$ and $\lambda_u$. On the other hand, the attenuation length $L_a$ is the function of $\omega$. Therefore, fixing $\varepsilon$, $a$, $\lambda_u$ and $\omega$ one calculates $x = L_d/L_a$ and, then, using the dashed curve in the figure finds the optimal length of the crystalline undulator $\bar{L}(x)$ which accounts for the dechanneling effect and the photon attenuation. Simultaneously, from the solid curve one finds the effective number of the undulator periods $N_{\text{eff}}(x, \bar{\kappa}_d(x))$ which defines the number of emitted photons, the flux and the brilliance of radiation.

4. NUMERICAL RESULTS

From (29) follows, that to find the number of photons $\Delta N_{\omega_n}$ emitted in a crystalline one has to calculate two factors. The factor $Q_n(p)$ (see (13)) depends on the harmonic number $n$ and on the undulator parameter $p$ which, in turn, is defined by the values of $\varepsilon$, $a$ and $\lambda_u$ through the relation $p = 2\pi\gamma a/\lambda_u$. The second factor, $N_{\text{eff}}(x, \kappa_d)$ depends on $L_u$, $\lambda_u$, $L_d = L_d(\varepsilon, C)$ and $L_a = L_a(\omega)$. It was explained in Sect. 3.2 that once the quantities $\lambda_u$, $L_d(\varepsilon, C)$ and $L_a(\omega)$ are known the length of the crystal can be fixed by the condition $L_u = \bar{L}$ which results in the maximum values of $N_{\text{eff}}(x, \kappa_d)$ and $F_n$ with respect to $L_u$. The numerical data presented below in this section was obtained for the optimal length of undulator.

Therefore, to calculate $\Delta N_{\omega_n}$ one fixes, in addition to the crystallographic plane, the values of $n$, $\varepsilon$, $a$ and $\lambda_u$ (the three latter are subject to the conditions (1)–(4)) which uniquely define the quantities $p$, $C$, $\omega_n$, $L_a(\omega)$. However, there is some uncertainty with respect to the magnitude of the dechanneling length. This uncertainty is not intrinsic to the case of a periodically bent crystal but rather reflects the stochastic nature of the interaction of a channeling particle with crystal constituents. As mentioned above to calculate the dechanneling length one can apply the diffusion theory to describe the multiple scattering or carry out numerical simulations of the scattering process. Alternatively, one can use model-dependent analytic expressions for $L_d(\varepsilon, C)$. In the present paper we utilize the approach, presented in Ref. 2, and approximate $L_d(\varepsilon, C)$ with

$$L_d(\varepsilon, C) = (1 - C)^2 L_d(\varepsilon, 0), \quad L_d(\varepsilon, 0) = \frac{256}{9\pi^2} \frac{a \gamma}{m c^2 r_0} \frac{\varepsilon}{\Lambda}. \quad (33)$$
Here where \( r_0 = 2.8 \times 10^{-13} \) cm is the electron classical radius, \( mc^2 = 0.511 \) MeV is the electron rest energy, \( a_{TF} \) is the Thomas-Fermi radius of the crystal atom. The parameter \( C \) is defined by Eq. (1). The quantity \( L_d(\varepsilon, 0) \) stands for the dechanneling length of a positron in a straight crystal.\(^5\) The quantity \( \Lambda = \ln \sqrt{2\pi mc^2/I - 23/24} \), with \( I \) denoting the (average) ionization potential of the crystal atom, is the Coulomb logarithm characterizing the ionization losses of an ultra-relativistic particle in amorphous media. For a quick estimation of \( L_d(\varepsilon, 0) \) (in cm) one can re-write the right-hand side of the second equation from (33) as \( 2a_{TF} d \varepsilon/\Lambda \), with \( a_{TF} \) and \( d \) measured in \( \text{Å} \) and \( \varepsilon \) in GeV. The values of \( a_{TF} \) and \( d \), are presented in Table 1.

**Table 1.** Parameters \( d \), \( a_{TF} \) and \( U'_{\max} \) for different crystals and channels.

<table>
<thead>
<tr>
<th></th>
<th>C (111)</th>
<th>Si (111)</th>
<th>Ge (111)</th>
<th>W (110)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d ) (Å)</td>
<td>1.54</td>
<td>2.35</td>
<td>2.45</td>
<td>2.24</td>
</tr>
<tr>
<td>( a_{TF} ) (Å)</td>
<td>0.258</td>
<td>0.194</td>
<td>0.148</td>
<td>0.112</td>
</tr>
<tr>
<td>( U'_{\max} ) (GeV/cm)</td>
<td>9.23</td>
<td>8.58</td>
<td>17.5</td>
<td>57.4</td>
</tr>
</tbody>
</table>

To calculate the brilliance of a crystalline undulator (which one obtains by using (30) in (15)) it is necessary to specify the parameters of a positron bunch, which are the current \( I \), the beam sizes \( \sigma_{x,y} \) and angular divergencies \( \phi_{x,y} \). We used the parameters of the positron beams from several modern high-energy \( e^-e^+ \) colliders. These parameters are summarized in Table 2. The data on \( \varepsilon \), \( \sigma_{x,y} \), \( l \), \( N' \) and \( I \) (which is an average beam current) are taken from Ref. 24. The beam divergencies \( \phi_{x,y} \) were calculated using the data on the transverse emittance (not presented in the table) and the beam size \( \sigma_{x,y} \). The peak current \( I \), which is defined as the electric current of a single bunch, was calculated as \( I (\Lambda) \approx 48N'/l \) with \( l \) in cm.

**Table 2.** Positron energy \( \varepsilon \), bunch length \( l \), number of particles per bunch \( N' \), beam sizes \( \sigma_{x,y} \), beam divergencies \( \phi_{x,y} \), and a positron peak current \( I \) for several modern high-energy \( e^-e^+ \) colliders.\(^{24}\)

<table>
<thead>
<tr>
<th></th>
<th>DAΦNE (Frascati)</th>
<th>VEPP-2000 (Russia)</th>
<th>BEPC-II (China)</th>
<th>PEP-II (SLAC)</th>
<th>KEKB (KEK)</th>
<th>CERS-C (Cornell)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varepsilon ) (GeV)</td>
<td>0.700</td>
<td>1.0</td>
<td>1.9-2.1</td>
<td>2.5-4</td>
<td>3.5</td>
<td>6</td>
</tr>
<tr>
<td>( l ) (cm)</td>
<td>1-2</td>
<td>4</td>
<td>1.3</td>
<td>1</td>
<td>0.65</td>
<td>1.2</td>
</tr>
<tr>
<td>( N' ) (units 10^{13})</td>
<td>3-9</td>
<td>16</td>
<td>4.8</td>
<td>6.7</td>
<td>7.3</td>
<td>1.15</td>
</tr>
<tr>
<td>( \sigma_x ) (mm)</td>
<td>0.800</td>
<td>0.125</td>
<td>0.380</td>
<td>0.157</td>
<td>0.110</td>
<td>0.300</td>
</tr>
<tr>
<td>( \sigma_y ) (mm)</td>
<td>0.0048</td>
<td>0.125</td>
<td>0.0057</td>
<td>0.0047</td>
<td>0.0024</td>
<td>0.0057</td>
</tr>
<tr>
<td>( \phi_x ) (mrad)</td>
<td>0.375</td>
<td>2</td>
<td>0.379</td>
<td>0.153</td>
<td>0.164</td>
<td>0.500</td>
</tr>
<tr>
<td>( \phi_y ) (mrad)</td>
<td>0.208</td>
<td>2</td>
<td>0.544</td>
<td>0.319</td>
<td>0.417</td>
<td>0.439</td>
</tr>
<tr>
<td>( I ) (Å)</td>
<td>144-216</td>
<td>192</td>
<td>177</td>
<td>322</td>
<td>539</td>
<td>46</td>
</tr>
</tbody>
</table>

The results of our calculations are presented in Figs. 4 and 5. The choice of the crystals was motivated by the fact that C, Si, Ge and W crystals are frequently used in channeling experiments (see, e.g., Ref. 22). An additional reason is that for a given photon frequency the magnitude of \( L_d(\omega) \) rapidly decreases with the growth of atomic number of the constituent atoms. Therefore, by comparing the results obtained for different crystals one can investigate the influence of the photon attenuation on the formation of the radiation in a crystalline undulator.

Graphs (a) in Fig. 4 present the dependence of the maximal number of emitted photons of the first harmonic \( (n = 1) \) per bandwidth \( \Delta\omega_1/\omega_1 \) and per a positron versus the ratio \( a/d \). The curves were calculated for the positron energies indicated in Table 2 (for BEPC-II and PEP-II colliders we used the values \( \varepsilon = 2 \) GeV and \( \varepsilon = 3 \) GeV, respectively). For each crystal and for each \( \varepsilon \) value the dependences \( (\Delta N_{em}/BW)_{\max} \) were obtained as follows. There are two independent variables, \( \lambda_0 \) and \( a \), which, (for fixed crystal, energy and harmonic number \( n \)) define all other quantities on the right-hand side of (29). For practical purposes it is more convenient to chose the
ratio $a/d > 1$ and the parameter $C < 1$ (see Eq. (1)) as the independent variables. Then, for each pair $(a/d, C)$ one finds $\lambda_u, p = 2\pi a/\lambda_u, Q_1(p)$, the dechanneling length $L_d(\varepsilon, C)$ and the number of periods $N_d = L_d(\varepsilon, C)/\lambda_u$, the fundamental harmonic frequency $\omega_1$ (see Eq. (10)) and the attenuation length $L_a(\omega_1)$, and the value of $N_{\text{eff}}(x, \kappa_d(x))$ which corresponds to the optimal undulator length calculated for $x = L_d(\varepsilon, C)/L_a(\omega_1)$ (see (31)-(32) and Fig. 3 and Sect. 3.2). As a result, one finds the magnitude of $\Delta N_{\omega_1}/\text{BW}$. Finally, scanning through all $(a/d, C)$ values one determines the highest possible value of the number of photons per BW, $(\Delta N_{\omega_1}/\text{BW})_{\text{max}}$, as a function of $a/d$. Having done this one also finds the dependence $\omega_1 = \omega_1(a/d)$ (graphs ‘(b)’ in Fig. 4) as well all other characteristics of the undulator as functions of $a/d$.

Let us briefly discuss the behaviour of obtained dependences. Firstly, as it is seen from the graphs (a), for a fixed amplitude $a$ the quantity $(\Delta N_{\omega_1}/\text{BW})_{\text{max}}$ is an increasing function of a positron energy $\varepsilon$. This feature becomes clear if one analyzes the $\varepsilon$ dependence of the product $Q_1(p) N_d f(x, \kappa_d(x))$ which defines the number of emitted photons (see Eqs. (29) and (31)). All three factors are increasing functions of energy (although it is not too obvious for $f(x, \kappa_d(x))$).

Another feature of the curves $(\Delta N_{\omega_1}/\text{BW})_{\text{max}}$ is that they are decreasing function of $a/d$ in the region $a/d > 1$. To a great extent this is a consequence of the photon attenuation in the crystal. Indeed, as the ratio $a/d$ increases the undulator period $\lambda_u$ increases too, in order to maintain the inequality $C \ll 1$ (see Eq. (1)). Larger values of $\lambda_u$ results in lowering of the emitted photon energy (see Eq. (10) and the graphs (b) in Fig. 4) and, consequently, to the decrease of the attenuation length, $L_a(\omega_1)$. This, in turn, leads to the increase of the ratio $x = L_d/L_a$ which defines the magnitude of $f(x, \kappa_d(x))$. This factor, as it is seen from Fig. 3, rapidly falls off for $x > 0.1$, and this feature manifests itself in the dependence of $(\Delta N_{\omega_1}/\text{BW})_{\text{max}}$ on $a/d$. In the case of crystals consisting of heavy atoms the dependence acquires additional features, which are due to the fact that the ionization potentials, $I_0$, of the inner atomic subshells of such atoms lie within the energy range $1 \ldots 100$ keV. The photons with the energy just above the threshold are absorbed much more efficiently than those with the lower energies. As a result, the dependence of $L_a(\omega)$ in the vicinity of the threshold becomes a saw-like. For $\omega < I_0$ the attenuation length noticeably (up to the order of magnitude) exceeds $L_a(\omega)$ for $\omega \geq I_0$. This effect results in the irregularities of the dependence $(\Delta N_{\omega_1}/\text{BW})_{\text{max}}$ on $a/d$, in which Figs. 3 are mostly pronounced for diamond and tungsten crystals.

In the opposite limit, when $a/d \ll 1$ the number of the emitted photons goes to zero. This tendency, which is seen explicitly for all the curves (but the CERS-C one) in the case of C and Si crystals, is also clear and is due to the fact that the case $a = 0$ corresponds to the linear crystal, i.e. the absence of the crystalline undulator.

In Fig. 5 we present the peak brilliance of the crystalline undulators based on different crystals (as indicated) and calculated using the parameters of the positron beams from Table 2. The data refer to the emission in the first and the third harmonics in the forward direction. It is seen that in contrast to the number of the emitted photons which is the same, by the order of magnitude, for all colliders, the magnitudes of the peak brilliance for different beams differ by orders of magnitude. To the largest extent this is due to the quality of the beam, which includes, apart from the beam current $I$, its size and angular divergency, see, Eqs. (14) and (15). For all crystals and over the whole range of photon energies the product $\varepsilon_p\varepsilon_q$ of the photon source emittances is the smallest for the KEKB collider (labeled as ‘5’ in the graphs in Fig. 5). As a result, this beam, which does not lead to the highest values of $(\Delta N_{\omega_1}/\text{BW})_{\text{max}}$, ensures the largest peak brilliance of the crystalline undulator radiation. The peak brilliance for the KEKB positron beam is on the level of $(4 \ldots 20) \times 10^{22}$ (photons/s/mrad²/mm²/0.1%/BW) for the photon energies within $1 \ldots 10$ MeV range. These values can be compared with the peak brilliance of the light sources of the third generation. The peak brilliance on the level $10^{21} \ldots 10^{23}$ in the $100$ keV range of photon energies by means of the undulators based on the action of magnetic field is planned to be achieved within several projects. The data from Fig. 5 demonstrate that it is feasible to produce the radiation of the same level of brilliance but for much higher energies by means of crystalline undulators.

5. CONCLUSION

Theoretical investigations show that it is entirely realistic to use a crystalline undulator for generating spontaneous radiation in a wide range of photon energies. The parameters of such an undulator, being subject to the restrictions mentioned in Sect. 1, can be easily tuned by varying the parameters of the bending, the positron...
energy and by choosing different channels. The large range of energies available in modern colliders together with the wide range preparation of periodically bent crystalline structures allow one to generate the crystalline undulator radiation with energies from hundreds of keV up to tens of MeV region. The brilliance of the undulator radiation within this energy range is comparable to that of conventional light sources of the third generation but for much lower photon energies.

The experimental efforts are needed for the verification of numerous theoretical predictions. Such efforts will certainly make this field of endeavor even more fascinating than as it is already and will possibly lead to the practical development of a new type of tunable and monochromatic radiation sources.

ACKNOWLEDGMENTS
AVK acknowledges the support from the Alexander von Humboldt Foundation.

REFERENCES
Figure 4. Graphs (a): the maximal number of photons of the first harmonic \((n = 1)\) per a bandwidth \(\Delta\omega_1/\omega_1\) and per a positron as a function of the ratio \(a/d\) calculated for the positron energies in various colliders (see Table 2) as indicated. Graphs (b): the corresponding values of the fundamental harmonic energy (see Eq. (10) with \(n = 1\)). See also explanations in the text. Each vertical pair of the graphs (a) and (b) correspond to the positron channeling in the particular periodically bent channel as indicated in the graphs (a). The legend refers to all graphs in the figure.
Figure 5. Peak brilliance of the undulator radiation in the forward direction calculated for four channels as indicated in each graph. The solid curves correspond to the radiation in the fundamental harmonic $n = 1$, the dashed curves refer to $n = 3$. In each graph the enumerated sets of the solid and the dashed curves correspond to the parameters of the positron beams in different colliders (see Table 2). 1: DAΦNE, 2: VEPP-2000, 3: BEPC-II, 4: PEP-II, 5: KEKB, 6: CERS-C.