Non-relativistic classical mechanics for spinning particles

Giovanni Salesi
Università Statale di Bergamo, Facoltà di Ingegneria, Italy; and
Istituto Nazionale di Fisica Nucleare–Sezione di Milano, Italy

We study the classical dynamics of non-relativistic particles endowed with spin. Non-vanishing Zitterbewegung terms appear in the equation of motion also in the small momentum limit. We derive a generalized work-energy theorem which suggests classical interpretations for tunnel effect and quantum potential.

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I. SPIN AND ZITTERBEWEGUNG IN NON-NEWTONIAN MECHANICS

The basic kinematical feature of the motion of spinning particles is the so-called Zitterbewegung [1–7], i.e., the high frequency jitter-motion first described by Schrödinger [1] in the Thirties. Because of Zitterbewegung velocity and momentum of a spinning particle are non-proportional but independent quantities

\[ \mathbf{v} \not\parallel \mathbf{p}. \]

Actually in Dirac theory the velocity and momentum operators are not proportional

\[ \hat{\mathbf{v}} = \alpha \mathbf{c} \parallel \hat{\mathbf{p}} = -i \hbar \nabla. \]

Furthermore \( \hat{\mathbf{v}} \), differently from \( \hat{\mathbf{p}} \), does not commute with the Dirac Hamiltonian \( \hat{H} = \alpha \mathbf{c} \cdot \hat{\mathbf{p}} + \beta mc^2 \) so that, while \( \hat{\mathbf{p}} \) is a constant quantity, \( \hat{\mathbf{v}} \) is not. Therefore the dynamical structure of Quantum Mechanics is intrinsically non-Newtonian

\[ \mathbf{v} \neq \frac{\mathbf{p}}{m}, \quad \mathbf{a} \neq \frac{1}{m} \frac{d\mathbf{p}}{dt} = \frac{\mathbf{F}}{m}. \]

As a consequence Newton’s Law and (in the absence of external forces) Galileo’s Principle of Inertia do not hold anymore, and the free motion is not in general uniform rectilinear. Zitterbewegung is well depicted through the celebrate Gordon decomposition [8] of the Dirac probability current:

\[ j^\mu = \bar{\psi} \gamma^\mu \psi = \frac{1}{2m} \left[ \nabla \left( \bar{\psi} \mathbf{p} \psi \right) - \left( \bar{\psi} \mathbf{p} \psi \right) \nabla \right] + \frac{1}{m} \partial_\nu \left( \bar{\psi} \hat{S}^{\mu\nu} \psi \right) \]

\( \bar{\psi} \equiv \psi^\dagger \gamma^0, \quad \hat{\mathbf{p}}^\mu \equiv i \hbar \partial^\mu, \) and \( \hat{S}^{\mu\nu} \equiv i \hbar (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)/4 \) represents the spin tensor operator. The first term in the r.h.s. is associated with the translational motion of the CM; whilst, the non-Newtonian term in the r.h.s. is related to the existence of the spin, and describes the Zitterbewegung rotational motion.

Analogous Gordon-like decompositions of the conserved currents can be written for spin-1 bosons and for spin-\( \frac{3}{2} \) fermions (in the Proca and Rarita-Schwinger theories, respectively), as well as for NR particles, in Pauli’s and Schrödinger’s theories[3,9,10].

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†e-mail: salesi@unibg.it
1In fact, following Landau [9], we can write a NR Gordon-like decomposition of the conserved Pauli current

\[ j = \frac{i \hbar}{2m} \left[ \left( \nabla \psi^\dagger \right) \psi - \psi^\dagger \nabla \psi \right] + \frac{\hbar}{m} \nabla \times \left( \psi^\dagger \sigma \psi \right), \]
In a recent paper of ours [11] it was proposed a classical particle theory in which Zitterbewegung arises quite naturally. For the above considerations we called that theory Non-Newtonian Mechanics (NNM). The classical motion of spinning particles was therein described without recourse to particular models or special formalisms, and without employing Clifford algebras, or classical spinors (appearing in all supersymmetric-like classical models, as the Barut-Zanghì one [5]), but simply by generalizing the usual spinless theory. Newtonian Mechanics is re-obtained as a particular case of that theory: namely for spinless systems with no Zitterbewegung.

Let us remark that:

a) NNM does not fix either the dimension or the metric of the spacetime (we could also have $D \neq 4$ representations of the quantized theory);

b) NNM (see below) is the most natural, straight and economic extension of the Newtonian Mechanics which allows a classical theory of spin;

c) the Lagrangian and the equations of motion do not contain explicitly the Lorentz factor: so that the ordinary

After contracting the above equation with $p_{\mu}$ and exploiting the on-shell constraint $p^2 = m^2$, we obtain the Zitterbewegung equation:

$$v^\mu = \frac{p^\mu + \dot{S}^{\mu \nu} p_\nu}{m^2}.$$  

The proper time $\tau$ is defined as the time measured in the Center-of-Mass Frame (CMF) where, by definition, the 3-momentum vanishes, $p = 0$. The reference system, where by definition the speed vanishes, $v \equiv dx/d\tau = 0$, is instead the Rest Frame (RF) different from the CMF for spinning particles endowed with Zitterbewegung. We assume also the on-shell constraint $p^2 = m^2$, which implies, as shown in Ref. [11], the “Dirac constraint” $p_\mu v^\mu = m$. In the absence of external fields, by only requiring spacetime isotropy and homogeneity ($J^{\mu \nu} = \dot{p}^\mu = 0$), we get

$$\dot{S}^{\mu \nu} = -\dot{L}^{\mu \nu} = p^\mu v^\nu - p^\nu v^\mu.$$  

where $\psi$ is a Pauli 2-components spinor and $\sigma$ is the usual Pauli vector $(2 \times 2)$ matrix. Also the above current appears as a sum of a Newtonian part which, in the spinless (Newtonian) limit ($h \sigma \rightarrow 0$), is parallel to the classical momentum; and of a non-Newtonian spin part due to the spin, which vanishes only for spinless bodies, but not in the NR limit, i.e. for small angular momentum $p$. The Schrödinger theory, a particular case of Pauli’s, corresponds to constant spin, $s = \psi \sigma \bar{\psi}/\psi \bar{\psi}$ = const. Nevertheless the Zitterbewegung term of current (1) does not vanish since the curl of quantity $\psi^\dagger \sigma \bar{\psi} = \psi^\dagger \psi s = \rho$ is not zero (the probability density is in general not constant). As the divergence of a curl vanishes identically, also the Zitterbewegung term is conserved. Therefore it is incorrect to assume, as usual, the Newtonian term as equal to the whole Schrödinger current. This assumption leads to severe problems in the interpretation of the nonvanishing mean kinetic energy for stationary states described by real wavefunctions.
which in the CMF reduces to the spin 3-vector $s$. Since the momentum is constant, we can re-write the Zitterbewegung equation in a symmetric form

$$v^\mu = \frac{p^\mu}{m} - \frac{\ddot{W}^\mu}{m}.$$  \hspace{1cm} (6)

From Eq. (3) it follows that in a generic frame the trajectory is a helix around the constant direction of $p$. We found in Ref. [11] various consequences of the above Zitterbewegung equation, among which:

i) luminal or superluminal “global” motions are allowed, without violating Special Relativity, provided that the energy-momentum, and any related signal or information, travel with a subluminal (average) speed;

ii) except for the case of polarized particles ($s \parallel p$), the Zitterbewegung motion has a nonvanishing component along the momentum;

iii) the ratio between the time durations measured in a generic inertial frame and in the CMF is not constant and differs from the instantaneous Lorentz factor which instead constitutes the mean value of $dt/d\tau$ on a Zitterbewegung period. In general we can say that a non-linear relation occurs between infinitesimal time durations measured in different inertial frames.

We can build up, in a relativistically covariant way, also in the presence of fields, a Lagrangian formulation of NNM through a straightforward generalization of the Newtonian Lagrangian $L^{(0)} = \frac{1}{2} m v^2$ to Lagrangians containing time-derivatives of the velocity up to the $n$-th order:

$$L^{(n)} = \frac{1}{2} m v^2 + \frac{1}{2} k_1 \dot{v}^2 + \frac{1}{2} k_2 \ddot{v}^2 + \cdots - U \equiv \sum_{i=0}^{n} \frac{1}{2} k_i v^{(2i)} - U,$$ \hspace{1cm} (7)

where $U$ is a scalar potential due to external forces, the $k_i$ are constant scalar coefficients endowed with alternate signs [11], $k_0 = m$, and $v^{(i)} \equiv d^i v/ d\tau^i$. The Euler-Lagrange equation of motion

$$\frac{\partial L}{\partial x^\mu} = \frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial \ddot{x}^\mu} - \cdots$$

gives a constant-coefficients $n$-th order differential equation, which appears as a generalization of Newton’s Law $F = ma$, in which non-Newtonian Zitterbewegung terms appear:

$$- \frac{\partial U}{\partial x_\mu} = m a^\mu - k_1 \ddot{a}^\mu + k_2 \dddot{a}^\mu - \cdots \equiv \sum_{i=0}^{n} (-1)^i k_i a^{(2i)}.$$  \hspace{1cm} (8)

The canonical momentum

$$\frac{\partial L}{\partial \dot{x}_\mu} - \frac{\partial L}{\partial \ddot{x}_\mu} - \cdots$$

conjugate to $x^\mu$ writes

$$p^\mu = m v^\mu - k_1 \dot{v}^\mu + k_2 \ddot{v}^\mu - \cdots \equiv \sum_{i=0}^{n} (-1)^i k_i v^{(2i)}.$$ \hspace{1cm} (9)

from which we get the Zitterbewegung equation for $L^{(n)}$:

$$v^\mu = \frac{p^\mu}{m} + \frac{k_1}{m} \dot{v}^\mu - \frac{k_2}{m} \ddot{v}^\mu - \cdots = \frac{p^\mu}{m} - \sum_{i=1}^{n} (-1)^i \frac{k_i}{m} v^{(2i)}.$$  \hspace{1cm} (10)

Through the Noether Theorem, by satisfying the symmetry under rotations, the classical spin can be unequivocally defined employing only classical kinematical quantities. We get the spin tensor and the spin vector, respectively (hereafter we refer to the first order theory; in Ref. [11] it was made also for $n > 1$):

$$S^{\mu \nu} = k_1 (v^\mu a^\nu - v^\nu a^\mu),$$  \hspace{1cm} (11)

$$s = k_1 (v \times a).$$ \hspace{1cm} (12)

The Hamiltonian representation of the theory is performed in NNM by introducing, besides the zero order momentum $p^\mu$ given in (9), a first order momentum $\pi^\mu$ canonically conjugate to $q^\mu \equiv v^\mu$:
\[ \pi^\mu \equiv \frac{\partial L}{\partial \dot{q}_\mu} = \frac{\partial L}{\partial \dot{\pi}_\mu} = k_1 a^\mu . \] (13)

Thus the spin tensor can be expressed also in a canonical way:

\[ S^{\mu\nu} = q^\mu \pi^\nu - q^\nu \pi^\mu . \] (14)

The Poisson brackets are here defined as follows

\[ \{ f, g \} \equiv \left( \frac{\partial f}{\partial x_\mu} \frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu} \frac{\partial g}{\partial x_\mu} \right) + \left( \frac{\partial f}{\partial q_\mu} \frac{\partial g}{\partial \pi_\mu} - \frac{\partial f}{\partial \pi_\mu} \frac{\partial g}{\partial q_\mu} \right) . \]

The scalar Hamiltonian, which conserves because of the \( \tau \)-reparametrization invariance, is

\[ H(\tau; x, p; q, \pi) = p_\mu \dot{x}^\mu + \pi_\mu \dot{q}^\mu - L = \frac{1}{2} m q^2 + \frac{\pi^2}{2k_1} + U . \] (15)

The action \( S = \int L d\tau \) can be put in the characteristic form

\[ S = \int p_\mu dx^\mu + \pi_\mu dq^\mu - H d\tau , \] (16)

from which

\[ p^\mu = \frac{\partial S}{\partial x_\mu}, \quad \pi^\mu = \frac{\partial S}{\partial q_\mu}, \quad H = - \frac{\partial S}{\partial \tau} . \] (17)

Besides the standard couple of Hamilton equations, we have a non-Newtonian couple of Hamilton equations, applying to the second order pair of canonical variables \((q^\mu, \pi^\mu)\)

\[ \begin{array}{c}
\frac{\partial H}{\partial p_\mu} = \dot{x}^\mu \\
\frac{\partial H}{\partial x_\mu} = -\dot{p}^\mu \\
\frac{\partial H}{\partial q_\mu} = \dot{q}^\mu \\
\frac{\partial H}{\partial \pi_\mu} = -\dot{\pi}^\mu .
\end{array} \] (18)

(for \( n > 1 \) we have to introduce other higher order momenta up to the \( n \)-th order entering \( n + 1 \) systems of pairs of Hamilton equations). As shown in Ref. [11], the above Hamilton equations are fully equivalent to the Euler-Lagrange equation, that is to generalized Newton’s Law (8).

**II. THREE-DIMENSIONAL AND NON-RELATIVISTIC THEORY**

Let us fix the only (apart the mass \( m \)) NNM free parameter \( k_1 \) to \(-\hbar^2/4mc^4\), so that (considering for simplicity free particles, \( U = 0 \)):

\[ \pi^\mu = - \frac{\hbar^2}{4mc^4} a^\mu \] (19)

\[ L = \frac{1}{2} m v^2 - \frac{\hbar^2}{8mc^4} a^2 \] (20)

\[ H = pq - \frac{1}{2} m q^2 - \frac{2mc^4}{\hbar^2} \pi^2 \] (21)

\[ S^{\mu\nu} = \frac{\hbar^2}{4mc^4} (a^\mu v^\nu - a^\nu v^\mu) . \] (22)
Extending NNM to macroscopic bodies we find, as expected, a Newtonian behavior with vanishing spin and Zitterbewegung because of the extreme smallness of \( k_1 = -\hbar^2/4mc^4 \) when \( m \to \infty \).

The Zitterbewegung equation of motion (10) now reduces to

\[
v^\mu = \frac{p^\mu}{m} - \frac{\hbar^2}{4m^2c^4} \ddot{\alpha}^\mu,
\]

whose general solution oscillates with the so-called “Compton frequency” \( \omega_c = 2mc^2/\hbar \) (\( E^\mu, H^\mu \) are constant spacelike 4-vectors fixing the “internal” initial conditions whilst \( \dot{p}^\mu \) fixes the “external” one)

\[
v^\mu = \frac{p^\mu}{m} + E^\mu \cos(\omega_c \tau) + H^\mu \sin(\omega_c \tau).
\]

The Hamilton generalized equations in NNM, globally equivalent to Eq. (23), become

\[
\begin{align*}
\frac{\partial \mathcal{H}}{\partial p_\mu} &= q^\mu = \dot{x}^\mu \\
\frac{\partial \mathcal{H}}{\partial x_\mu} &= 0 = -\dot{p}^\mu \\
\frac{\partial \mathcal{H}}{\partial \pi_\mu} &= -\frac{4mc^4}{\hbar^2} \pi^\mu = \ddot{q}^\mu \\
\frac{\partial \mathcal{H}}{\partial \pi_\mu} &= p^\mu - mq^\mu = -\ddot{\pi}^\mu.
\end{align*}
\]

As a consequence, also the second-order canonical variables \( q \) and \( \pi \) are harmonic oscillator coordinates with the Compton frequency

\[
\ddot{q}^\mu + \omega_c^2 q^\mu = 4mp^\mu \\
\ddot{\pi}^\mu + \omega_c^2 \pi^\mu = 0.
\]

In Appendix it is shown that NNM is strictly related to the Proper Time Dirac theory [2,11,12], a representation of SO(3,2) Lie algebra. Actually, some quantum Heisenberg equations \( \hat{G} = i [\hat{H}, \hat{G}] \) have classical counterparts in the Poisson brackets equations \( \dot{G} = \{ \mathcal{H}, G \} \) if we take the operator \( \hat{H} \) equal to the scalar Hamiltonian \( \hat{p}_\mu \gamma^\mu - m \) in Proper-Time Dirac theory\(^2\) and \( \mathcal{H} \) equal to NNM Hamiltonian, Eq. (21).

Just as it occurs in the NR limit of the Dirac theory (i.e. in quantum Pauli’s and Schrödinger’s theories), we have a classical Zitterbewegung motion also in the NR limit of NNM. Actually we have a motion even in the CMF. In fact, for \( p \to 0 \) we have \( \dot{S}^{ik} \to 0 \) (since the spin 3-vector conserves in NR Mechanics) but \( \ddot{S}^{0i} \not\to 0 \) \((S^{0i} \not\text{is not required to conserve}) \), so that from Eq. (3) we have \( v^i \to -\dot{S}^{0i}/m \neq 0 \). Therefore it is physically meaningful to study the NR NNM.

To this end it is sufficient to recall that in the NR limit the ordinary time coordinate \( t \) reduces to the proper time \( \tau \) so that all the time derivatives in Zitterbewegung equation (23) can now be meant as taken with respect to \( \tau \).

From the spatial part of (23) we get the 3-momentum of a free particle

\[
p = mv + \frac{\hbar^2}{4mc^4} \ddot{\alpha}.
\]

In the presence of an external force \( F \) and of a potential \( U(x) \), we can write

\[
\frac{dp}{dt} = -\nabla U = F,
\]

an then, by time-differentiating Eq. (27),

\[
F = ma + \frac{\hbar^2}{4mc^4} \dddot{\alpha}
\]

where it appears a non-Newtonian term which becomes important only when \( \dddot{\alpha} \) is very large.

\(^2\)The very Dirac equation \( \hat{p}_\mu \gamma^\mu - m = 0 \) arises by taking \( \hat{H} \), according to Dirac’s definition, “weakly zero”.
Let us derive the interesting non-Newtonian extension of the Work-Kinetic Energy Theorem holding in classical Newtonian mechanics \( T = \int \mathbf{F} \cdot d\mathbf{x} = \frac{1}{2} m \mathbf{v}^2 \). We now have

\[
T = \int \mathbf{F} \cdot d\mathbf{x} = \int \left( m\mathbf{a} + \frac{\hbar^2}{4mc^4}\mathbf{a} \right) \cdot d\mathbf{x} = \int \left( m\mathbf{a} + \frac{\hbar^2}{4mc^4}\mathbf{a} \right) \cdot \mathbf{v}dt.
\]

Taking in account that identically \( \mathbf{a} \cdot \mathbf{v} = \frac{d}{dt} \left( \frac{\mathbf{v}^2}{2} \right) \) and \( \mathbf{a} \cdot \mathbf{v} = -\frac{d}{dt} \left( \frac{\mathbf{a}^2}{2} - \mathbf{a} \cdot \mathbf{v} \right) \) we can write a non-Newtonian Work-Kinetic Energy Theorem where, besides the usual Newtonian term, a term appears which depends on the first two derivatives of the velocity

\[
T = \frac{1}{2} m \mathbf{v}^2 - \frac{\hbar^2}{4mc^4} \left( \frac{\mathbf{a}^2}{2} - \mathbf{a} \cdot \mathbf{v} \right).
\]

(31)

It is easy to show that the total mechanical energy \( T + U \) is a conserved quantity because turns out to be equal, up to a constant, to the 3D NR limit of Hamiltonian (15), that is \( \mathbf{p} \cdot \mathbf{q} - \frac{1}{2} m \mathbf{q}^2 - 2m\pi^2 + U \).

There is a noticeable consequence of this theorem: the non-existence of the classical “potential barrier”, the classical analogue of the quantum tunnel effect. In fact from the conservation of the energy

\[
E = T + U = \frac{1}{2} m \mathbf{v}^2 - \frac{\hbar^2}{4mc^4} \left( \frac{\mathbf{a}^2}{2} - \mathbf{a} \cdot \mathbf{v} \right) + U
\]

(32)

we see that, even in the space regions where \( E < U \), quantity \( \mathbf{v}^2 \) can be positive, for the counterbalancing presence in \( T \) of the non-Newtonian term.

Let us also remark the analogy between the conservation equation

\[
E = \frac{1}{2} m \mathbf{v}^2 - \frac{\hbar^2}{4mc^4} \left( \frac{\mathbf{a}^2}{2} - \mathbf{a} \cdot \mathbf{v} \right) + U
\]

(33)

and the “quantum Hamilton-Jacobi” equation for the Madelung fluid of the Schrödinger wave-Mechanics (\( \psi \equiv \sqrt{\rho} e^{i\phi}/\hbar \))

\[
-\partial_t \phi = \frac{1}{2m} (\nabla \phi)^2 + \frac{\hbar^2}{4m} \left[ \frac{1}{2} \left( \nabla \rho / \rho \right)^2 - \frac{\Delta \rho}{\rho} \right] + U.
\]

(34)

Actually, we are induced to relate the quantum phase and the classical action

\[
-\partial_t \phi \iff E = -\partial_t S \quad -\frac{\nabla \phi}{m} \iff \mathbf{v} = \frac{\nabla S}{m}
\]

(35)

as well as the so-called “quantum potential” and the non-Newtonian kinetic term

\[
\frac{\hbar^2}{4m} \left[ \frac{1}{2} \left( \nabla \rho / \rho \right)^2 - \frac{\Delta \rho}{\rho} \right] \iff \frac{\hbar^2}{4mc^4} \left( \frac{1}{2} \mathbf{a}^2 - \mathbf{a} \cdot \mathbf{v} \right).
\]

(36)

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Appendix — Correspondences between NNM and Proper-Time Dirac Theory

For $k_1 = -1/4m$ (hereafter we adopt, as usual, $\hbar = c = 1$) NNM is strictly related to the Proper-Time Dirac theory. As a matter of fact we can find exact correspondences between quantum and classical time-evolution equations. That is, between the Heisenberg equations $\hat{\dot{G}} = i \left[ \hat{H}, \hat{G} \right]$ and the Poisson brackets equations $\dot{G} = \{H, G\}$ (taking $\hat{H} = \hat{p}_\mu \gamma^\mu - m$ and $\mathcal{H} = pq - \frac{1}{2} mq^2 - 2m\pi^2$). For example we have

\[
\hat{\dot{p}}^\mu = i \left[ \hat{H}, \hat{p}^\mu \right] = 0
\]

\[
\hat{\dot{S}}^{\mu\nu} = i \left[ \hat{H}, \hat{S}^{\mu\nu} \right] = \hat{\hat{\pi}}^{\mu\gamma} - \hat{\hat{\pi}}^{\gamma\mu} ;
\]

which, taking in account that $\gamma^\mu = i \left[ \hat{H}, x^\mu \right] = \hat{\nu}^\mu$, correspond to the classical equations of conservation of the linear and angular momenta

\[
\hat{\dot{p}}^\mu = \{H, p^\mu\} = 0
\]

\[
\hat{\dot{S}}^{\mu\nu} = \{H, S^{\mu\nu}\} = p^{\mu\nu} - p^{\nu\mu}.
\]

Furthermore we have for the 4-acceleration operator

\[
\hat{\dot{a}}^\mu = i \left[ \hat{H}, \hat{a}^\mu \right] = 4 \hat{\hat{S}}^{\mu\nu} \hat{\hat{p}}_\nu ;
\]

correspondingly, in NNM, by contracting both sides of (22) with $p_\nu$ and exploiting the on-shell constraint $p^2 = m^2$, we have

\[
S^{\mu\nu} p_\nu = \frac{a^\mu}{4} .
\]

Notice that, differently from equations (39), (40), the last equation holds only in NNM(1) and not for larger $n$ anymore.

Besides the above correspondences, let us now recover in Proper Time Dirac theory just the NNM Zitterbewegung equation, Eq.(23). Let us apply the Heisenberg equation to the 4-acceleration operator $\hat{\dot{a}}^\mu$

\[
\hat{\dot{a}}^\mu = i \left[ \hat{H}, \hat{\dot{a}}^\mu \right] = i \left[ \hat{\hat{\pi}}_{\lambda} \gamma^\lambda - m, 4 \hat{\hat{S}}^{\mu\nu} \hat{\hat{p}}_\nu \right] = -4 \hat{\hat{\pi}}^{\gamma} - 4 \hat{\hat{\pi}}^{\mu} \hat{\hat{\pi}}_{\lambda} \gamma^\lambda ,
\]

and consider only the physical on-shell states which satisfy both Dirac and Klein–Gordon equation, $\hat{\hat{\pi}}_{\lambda} \gamma^\lambda \psi = m \psi$ and $\hat{\hat{\pi}}^2 \psi = m^2 \psi$. When applying to the vectors of that Hilbert subspace we can write

\[
\hat{\dot{a}}^\mu = -4m^2 \hat{\check{v}}^\mu + 4m \hat{\check{p}}^\mu ,
\]

which can be put just in the form of an operator version of the Zitterbewegung equation, Eq.(23):

\[
\hat{\check{v}}^\mu = \frac{\hat{\check{p}}^\mu}{m} - \frac{\hat{\dot{a}}^\mu}{4m^2} .
\]