Algorithmic complexity of quantum states

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(Dated: 22nd December 2004)

Abstract

In this paper we give a definition for the Kolmogorov complexity of a pure quantum state. In classical information theory the algorithmic complexity of a string is a measure of the information needed by a universal machine to reproduce the string itself. We define the complexity of a quantum state by means of the classical description complexity of an (abstract) experimental procedure that allows us to prepare the state with a given fidelity. We argue that our definition satisfies the intuitive idea of complexity as a measure of “how difficult” it is to prepare a state. We apply this definition to give an upper bound on the algorithmic complexity of a number of states.

PACS numbers:
I. INTRODUCTION AND NOTATION

Algorithmic information has provided a concise notion of randomness for individual objects. It has also revealed deep connections between thermodynamics and the theory of classical computation [17]. The algorithmic complexity (or randomness) of an object—usually a binary string—is thereby defined as the length in bits of the shortest program for a universal computer that reproduces the string under question [11].

Quantum theory, on the other hand, has provided a new conceptual basis for the theory of computation. Attempts have been made to also generalize the notion of algorithmic complexity to quantum mechanical objects, described by states in a Hilbert space. These attempts are motivated primarily by the desire to formulate a comprehensive theory of quantum information. We may also expect further insights into the theory of entanglement.

In this paper, we want to characterize the algorithmic complexity of a given quantum state. Proposals have already been made by Vitanyi [16] and Berthiaume et al. [3], who have introduced two possible definitions of quantum algorithmic complexity based on the reproducibility of the state via Turing machines. Gács [4] has instead adopted an approach based on universal probability. Our definition will be closer to the one proposed by Vitanyi, however deviating from his definition in two crucial aspects.

To introduce our definition of the algorithmic complexity of a quantum state, we shall consider the following scenario. Imagine that Alice has created a certain quantum state in her laboratory and wants to describe this state to Bob, who wants to reproduce it in his laboratory. How difficult is it for Alice to describe to Bob the state of her system? We may distinguish the two situations in which Alice and Bob communicate via a classical or a quantum channel, respectively. Depending on the choice of the communication channel, we may arrive at different notions of complexity of a quantum state.

In the first situation, Alice has to use classical information to describe her state to Bob. This appears to be a restriction, on first sight. However, we may always regard the quantum state of a system as the result of some experimental preparation procedure. The complexity of a quantum state is then associated, in a very natural way, with the (classical) description complexity of an experimental preparation procedure. The resulting notion of complexity might therefore also be called preparation complexity [18].

In the second situation, Alice may use quantum information to describe her quantum
state to Bob. In doing this, she has several options. She may send either the quantum state altogether to Bob, or a copy (if available), or the state in some Schumacher compressed form, or some other quantum state which Bob can transform by into the desired state. If we adopt this scenario, we arrive -with Berthiaume et al. 3- at a quite different notion of quantum complexity, which might also be called *encoding complexity*.

In this paper, we shall follow the first approach and identify the algorithmic complexity of a quantum state with its preparation complexity i.e. the classical description complexity of the preparation procedure. Although the second approach looks “more quantum”, it lacks an important feature that we usually associate with the *description* of an object. Even if Bob has the state sent by Alice in his hands, he might not know what state he has received. A proper description of the state, on the other hand, will allow him to reproduce the state himself.

To be able to communicate, Alice and Bob must first have agreed on a common language -a non-trivial problem in linguistics which they are using to describe their preparation procedure. Ideally, they will use the same “toolbox” to compose their experiments and the same words when referring to elements of this toolbox. In quantum information theory, we abstract from a particular physical system in which a quantum state is realized. The experimental toolbox is thereby replaced by a set of elementary operations on a Hilbert space with a given tensor-product structure and dimension [19]. The toolbox in quantum information theory is thus a gross abstraction from an experimental scenario. Here it the toolbox will include the possibility to prepare some standard reference state, and a finite set of elementary unitary transformations. A complete preparation procedure is then described as a sequence of unitary transformations (a quantum circuit) applied to the reference state.

Considering that a quantum state can be characterized by a circuit with which the state can be prepared, we want to define the complexity of a state referring to that of the circuit itself. It is known that a finite state of gates (constituting a complete basis) is suitable to prepare any state (up to an arbitrary precision); through a sufficient coding, thus, the circuit itself can be reduced to a (classical) string whose Kolmogorov complexity is well defined and which can be associated to the original state. In this way the algorithmic complexity of a state satisfies the intuitive idea of complexity as a measure of “how difficult” it is to prepare a state.
From now on we will represent with $Q_N$ the space generated by $N$ qubits and with $Q_N \ni \ket{0} = \ket{0}_N = \ket{0}_{(1)} \ket{0}_{(2)} \cdots \ket{0}_{(N)}$ the null vector (where $\ket{0}_{(i)} \in Q_i^{(i)}$ is an element of the computational basis $\{\ket{0}_i, \ket{1}_i\}$ of $Q_i^{(i)}$).

We will represent with $C|0\rangle$ the result of the application of a circuit $C$ on the null vector; if $\langle \varphi |C|\text{null}\rangle^2 \geq 1 - \varepsilon$ we will say that $C$ prepares $\ket{\varphi}$ with precision $\varepsilon$ (where $0 \leq \varepsilon \leq 1$).

When saying that $C$ prepares $\ket{\varphi}$ we mean that $C|0\rangle = \ket{\varphi}$.

In particular we will be interested in building quantum circuits from a fixed set of gates: as we want to be able to reproduce any state (at least up to a given precision) this set must constitute a complete gate basis.

**Example 1 (Standard basis).** An example of a complete gate basis is the so called standard basis $B = \{H, S, T, C\}$, where $H$ is the Hadamard gate, $S$ the phase gate, $T$ a $\frac{\pi}{8}$ gate and $C$ the controlled not:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \; \; ; \; \; S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \; \; ; \; \; T = \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix} \; \; ; \; \; C \ket{0} = \ket{0} \; \; ; \; \; C \ket{1} = \ket{1} \; \; .$$

$T$ represents a $\pi/4$ rotation about the $z$ axis, while $HTH$ a $\pi/4$ rotation around the $x$ axis.

Given a fixed (finite) number of gates only a countable set of states can be prepared exactly. If we consider a complete (finite) gate basis it is though possible to reproduce any unitary transformation $U$ (and thus any state $\ket{\varphi}$) up to an arbitrary precision. Considering that the definition of the complexity of a quantum state will be based on its preparation by means of a quantum circuit, it is therefore necessary to introduce a precision parameter in such definition. We will nevertheless start defining the algorithmic complexity on the set of states that can be exactly prepared with circuits built from a fixed basis. In this case it obviously is not necessary to introduce this parameter; it will appear only when generalizing this definition to arbitrary states.

**II. CLASSICAL KOLMOGOROV COMPLEXITY**

The definition of algorithmic complexity proposed by Kolmogorov [4, 11] is meant to give an answer to the question: “Is a (classical) sequence random?”
Algorithmic complexity gives a definition of randomness very close to the intuitive idea of “structurless” and is based on the concept of algorithmic *reproducibility* of a sequence. In practice, the (classical) Kolmogorov complexity \( K_{cl} \) of a (binary) string \( \omega \) is defined as the length of the shortest program that, running on a universal Turing machine, gives \( \omega \) as output.

It follows quite easily from the definition that the algorithmic complexity of a string \( \omega = \omega_1\omega_2 \cdots \) can grow at most linearly with the length of \( \omega \): it is in fact always possible to reproduce the string by means of a program of the form: “*write \( \omega_1, \omega_2 \cdots \)”.

This actually means that the length of a string constitutes (up to a constant) an upper bound for the complexity of the string itself:

\[
K_{cl}(\omega_n) \leq l(\omega_n) + O(1) = n + O(1) .
\] (1)

Naturally there are sequences for which this upper bound is far too large: it is easily shown, for example, that the complexity of a periodic string grow only *logarithmically* with the length of the sequence. A string is said to be complex (or structureless, or random) if its algorithmic complexity grows linearly with its length; these are the strings typically generated by random sources (such as, for example, a coin toss).

We will want to find a “natural” upper bound also for the complexity of a quantum state. A difficulty arises from the fact there seems to be no natural quantum counterpart to the classical “number of bits in the string”. We thus find it necessary to look for another quantity, classically related to the number of bits, for which such a counterpart exists.

In order to do this, let us consider the set of all infinite binary strings: it is easily shown that this set is isomorphic to the unit interval \([0, 1]\). Through this isomorphism it is possible to construct a (normalized) measure on the set of infinite strings. Any \( n \)-bit binary string \( \omega_n \) identifies the set of infinite strings whose first \( n \) bits coincide with \( \omega_n \): the volume of this set (a ball, \( B_{\omega_n} \)) is \( V(B_{\omega_n}) \sim 2^{-n} \).

The unit interval is thus divided in \([V(B_{\omega_n})]^{-1} \sim 2^n \) subintervals, each identified by a \( n \)-bit sequence \( \omega_n^{(i)} \) with \( i = 1, 2, \cdots, 2^n \). Once we have numbered all the \( n \)-bit sequences it follows immediately that each of them can be reproduced by a program that specifies its index \( i \), that is, by a program that requires at most (up to some constant) \( n = -\log 2^{-n} = -\log V(B_{\omega_n}) \) bits. This simple “counting” argument gives an upper bound for the complexity
of an $n$-bit string which coincides with the one given by equation (1):

$$K_{cl}(\omega_n) \leq -\log V(B_{\omega_n}) .$$

The advantage of this reasoning is that it can be easily adopted to find an upper bound
to the complexity of quantum states.

In the quantum case, in fact, we will be looking at a circuit (or quantum Turing machine
or any other appropriate model) that reproduces a normalized quantum state $|\varphi\rangle$ up to a
fixed (given) precision $\varepsilon$. This means that the circuit must prepare some quantum state $|\psi\rangle$
such that $|\langle \psi | \varphi \rangle|^2 \geq 1 - \varepsilon$: the set of all these states (any of which is acceptable as output
for the circuit) defines a ball in the $2^N$-dimensional space $Q_N$, with volume $V$ such that
$V \sim 2^{-N \varepsilon^2 2^N}$. This means that if $K^\varepsilon(|\varphi\rangle)$ is the complexity of the state $|\varphi\rangle \in Q_N$ (when
reproduced with precision $\varepsilon$) we must have:

$$K(|\varphi\rangle) \leq -\log V \Leftrightarrow K^\varepsilon(|\varphi\rangle) \leq -2^N \log \varepsilon + N .$$

In general the linear term can be omitted; we will thus usually consider simply the condition:

$$K^\varepsilon(|\varphi\rangle) \leq -2^N \log \varepsilon .$$

**Remark 1.** We underline that this is a preliminary condition, that should hold true independ-
ently of the way one choses to define quantum algorithmic complexity. It has in fact
no relation to the model chosen to reproduce the state, but depends instead only on a priori
properties such as the dimension of the space where the state is defined and the precision
with which the state must be reproduced.

### III. ALGORITHMIC COMPLEXITY ON A FIXED SET OF STATES

In the following section we will assume to have fixed a complete gate basis $B =$
$\{G_1, G_2, \ldots, G_k\}$ and we will consider only states $|\varphi\rangle$ that can be prepared exactly by
circuits built from $B$.

Once we fix a code (that is, an alphabet $\Omega = \{\omega_1, \omega_2, \ldots, \omega_l\}$), the procedure to compute
the algorithmic complexity of a state $|\varphi\rangle$ the procedure is very simple.

1. With the gates contained in the basis $B$, build a circuit $C^B(|\varphi\rangle)$ such that $C^B(|\varphi\rangle)|0\rangle =$
$|\varphi\rangle$.
2. Code the circuit, obtaining a classical sequence \( \omega^0(C^B) = \omega_{i_1}^{\Omega} \omega_{i_2}^{\Omega} \cdots \omega_{i_m}^{\Omega} \) of symbols \( \omega_{i_j}^{\Omega} = \omega_{i_j}(C^B) \in \Omega \).

Remark 2. Most of the code (that is, excluding some parts, e.g. a “new line” instruction or a way to identify the different quibits, that will be more or less common to all codes) is strictly related to the gate basis. In fact the code can be seen as a function that associates each gate of the basis to a symbol (letter) or group of symbols (word).

4. We have now all the elements to define the algorithmic complexity of a state \(|\varphi\rangle\).

**Definition 1.** The algorithmic complexity of a state, relative to the basis \(B\), the code \(\Omega\) and the circuit \(C^B(|\varphi\rangle)\) is:

\[
K_{\text{Net}}^{\Omega,B,C^B}(|\varphi\rangle) = K_{\text{CL}}(\omega^0(C^B))
\]

5. In general there are more circuits that prepare the same state \(|\varphi\rangle\), and in principle the correspondent complexities can be different. In order to define a property of the state itself (and not related to the circuit used to reproduce it) we consider the following definition.

**Definition 2.** The algorithmic complexity of the state \(|\varphi\rangle\), relative to the code \(\Omega\) and the gate basis \(B\) is:

\[
K_{\text{Net}}^{\Omega,B}(|\varphi\rangle) = \min_{C^B \in \bar{C}_B} K_{\text{Net}}^{\Omega,B,C^B}(|\varphi\rangle)
\]  \hspace{1cm} (5)  

where \(\bar{C}_B\) is the set of all the circuits built with gates from \(B\) that prepare \(|\varphi\rangle\).

Naturally, considering that the choices of code and basis are arbitrary, it is necessary to study how they influence the complexity of the state.

**Proposition 1** (“Asymptotic” invariance of the complexity of a state for code choice). If \(\Omega\) and \(\Omega^\prime\) are two different codes, then, for any state \(|\varphi\rangle\):

\[
K_{\text{Net}}^{\Omega,B}(|\varphi\rangle) = K_{\text{Net}}^{\Omega^\prime,B}(|\varphi\rangle) + k_{\Omega,\Omega^\prime},
\]  \hspace{1cm} (6)  

where \(k_{\Omega,\Omega^\prime}\) is a constant that depends only on \(\Omega\) and \(\Omega^\prime\).
This can be seen as follows. For every $C^B \in \tilde{C}^B$, let $\omega^\Omega(C^B) = \omega_{i_1}^\Omega \omega_{i_2}^\Omega \cdots \omega_{i_m}^\Omega$ and $\tilde{\omega}^\Omega(C^B) = \omega_{i_1}^\Omega \omega_{i_2}^\Omega \cdots \omega_{i_n}^\Omega$ be the strings that code $C^B$ using respectively codes $\Omega$ and $\tilde{\Omega}$.

$k_{\Omega, \tilde{\Omega}}$ represents the length of a “dictionary” with which it is possible to translate the description made using one code to that made using the other. Since both codes are finite, such dictionary is finite too. The invariance is asymptotical since, in general, $k'_{\Omega, \tilde{\Omega}}$ can be very big and its relevance is lost only for $K_{Net}^B(|\varphi|) \gg 1$.

**Remark 3.** We underline that in general there is no corresponding invariance property related to the basis choice (see Sec. VII). Nevertheless there are cases in which such an invariance does hold true. This happens, for example, when we consider a coarsening of the gate basis, that is if we consider two gate bases $B$ and $\tilde{B}$, one of which $B$ constituted of non-elementary gates that can be built with gates from $B$ (e.g. $B$ contains a Toffoli gate, while $B$ contains Hadamard and C-not). In this case, in fact, any circuit made by gates from $B$ can be reproduced by one made by gates from $B$. The string that codes this circuit will in general be longer than the one that of the original circuit, but their complexities will change only for a (small) constant $k_{B, \tilde{B}}$ (that represent the length of a “dictionary” between the two gate bases).

Considering the code-invariance property we can from now on omit explicating the dependence on the code (we can imagine to fix it once and for all) and write simply: $K_{Net}^B(|\varphi|)$.

**IV. ALGORITHMIC COMPLEXITY FOR ARBITRARY STATES**

We want to generalize to arbitrary states what we have seen before. In this case it is necessary to introduce the precision parameter $\varepsilon$: we can expect, in fact, that the greater the precision with which the state must be reproduced by the circuit, the longer will be the description of the circuit itself.

**Remark 4.** The fact that the description of the circuit becomes longer does not necessarily mean that the complexity of the string that codes it (and thus that of the state prepared by the circuit) increases. In fact, we can imagine some states that can be prepared with better and better precisions by simply iterating the application of a particular gate (or set of gates). In this case, the length of the string that codes the circuit would naturally grow with the
precision, but not so the complexity of the circuit. However this will not hold true in general, so it is necessary to keep the explicit dependence on the precision parameter.

The precision parameter enters in the definition of the algorithmic complexity of the state $|\varphi\rangle$ at the very first step, that is in building the circuit that prepares it. When considering an arbitrary state in the Hilbert space we will in fact need to specify the precision up to which the circuit must prepare the state. We will represent with $C^B_\varepsilon(|\varphi\rangle)$ a circuit (built with gates from $B$) that prepares $|\varphi\rangle$ with precision $\varepsilon$ and with $\omega(\omega^B(|\varphi\rangle))$ the (classical) sequence that codes $C^B_\varepsilon(|\varphi\rangle)$.

**Definition 3.** The algorithmic complexity of state $|\varphi\rangle$, relative to code $\Omega$ and gate basis $B$ with precision parameter $\varepsilon$ is:

$$K^{\Omega,B,\varepsilon}_{Net}(|\varphi\rangle) = \min_{C^B_\varepsilon \in \Omega^B(|\varphi\rangle)} K_{Cl}(\omega(\omega^B(|\varphi\rangle))),$$

where $\omega^B(|\varphi\rangle)$ is the set of all the circuits built with gates from $B$ that prepare $|\varphi\rangle$ with precision $\varepsilon$.

**Remark 5.** The proof of the code-invariance of the complexity of a state, seen in the previous section, did not require that the state was reproduced exactly by the circuits; thus the code-invariance property still holds true. Therefore we can again we omit explicitating the dependence on the code and write simply:

$$K^{B,\varepsilon}_{Net}(|\varphi\rangle).$$

**V. COMPLEXITY AND PRECISION**

In this paragraph we want to verify that our definition of a complexity satisfies the preliminary condition given in Section III. In order to do this it is necessary to estimate the upper bound of the algorithmic complexity of an arbitrary state $|\varphi\rangle$.

It is known that using only the (continuous) set of all 1-qubit gates, plus the controlled not (C), it is possible to reproduce any unitary operation $U$ over $\mathcal{Q}_N$ using $O(N^2 4^N)$ gates. However, if one is interested to reproduce the action of a unitary operation on one particular (given) state only $O(N^2 2^N)$ such gates are sufficient; this number of gates is thus sufficient to prepare any state $|\varphi\rangle$ from the given initial state $|0\rangle$. 

9
We consider now the Solvay-Kitaev theorem\[10\] which implies that any circuit acting on \(Q_N\) built with \(m\) 1-qubit gates and \(C\)'s can be reproduced up to precision \(\varepsilon\) using \(\mathcal{O}\left(m \log^c \left( \frac{N}{\varepsilon} \right) \right)\) gates from a finite gate basis (\(c \in [1,2]\) is a constant whose exact value is yet not known).

It follows immediately that the action of any unitary transformation on \(|0\rangle\) can be implemented (and thus any \(|\varphi\rangle \in Q_N\) can be prepared) up to precision \(\varepsilon\) via a circuit built only with gates from any finite and complete basis; futhermore, if \(M\) is the number of gates in the circuit, we have:

\[
M = \mathcal{O}\left(N^2 2^N \log^c \left( \frac{N^2 2^N}{\varepsilon} \right) \right) \Rightarrow M \sim -N^2 2^N \log \varepsilon ,
\]

where the last expression is given considering only the leading order in the two variables.

Naturally the length of the string that codes the circuit grows linearly with the number of gates of the circuit itself: in order to code a circuit that prepares a general state \(|\varphi\rangle \in Q_N\) we need thus a word whose length is (proportional to) \(M\). Referring to the definition given in the previous paragraph, in order to say that a state \(|\varphi\rangle\) is complex it is necessary that its complexity (or, equivalently, the complexity of the string \(\omega^B_\varepsilon(|\varphi\rangle)\)) grows linearly with the length \((M)\) of \(\omega^B_\varepsilon(|\varphi\rangle)\). From Equation (9) we obtain immediatly the logarithmic dependence on precision and the exponential dependence on \(N\) that were presented in Section \[11\]as expected upper bounds for the complexity: we have thus

\[
K^{B,\varepsilon}_{\text{Net}}(|\varphi\rangle \in Q_N) \leq -N^2 2^N \log \varepsilon .
\]

**Complex states**

We are now in the position of defining what we mean by a complex state. Algorithmic complexity is an uncomputable quantity: it is in fact based by definition on the classical algorithmic complexity of the string \(\omega^B_\varepsilon(C^B)\) that codes the circuit. As algorithmic complexity is an uncomputable quantity \[1,11,13\] this property passes to algorithmic complexity too.

As defined above, the complexity of a state \(|\varphi\rangle\) is the minimum complexity of a word that codes a circuit in  \(\tilde{C}^B_\varepsilon(|\varphi\rangle)\); this means that there is a circuit \(\tilde{C}^B_\varepsilon(|\varphi\rangle) \in \tilde{C}^B_\varepsilon(|\varphi\rangle)\), coded (using some alphabet) by \(\tilde{\omega}^B_\varepsilon(|\varphi\rangle)\), such that \(K^{B,\varepsilon}_{\text{Net}}(|\varphi\rangle) = K_{\text{Cl}}(\tilde{\omega}^B_\varepsilon(|\varphi\rangle))\).

**Remark 6.** It is in general possible that there are more circuits that satisfy the same condition \(K^{B,\varepsilon}_{\text{Net}}(|\varphi\rangle) = K_{\text{Net}}(\tilde{C}^B_\varepsilon(|\varphi\rangle))\). In this case it is sufficient to chose one of them: the choice is arbitrary and not relevant for the following.
Once we have associated a classical string (the characterizing string $\hat{\omega}_\varepsilon^B(|\varphi\rangle)$) to the state $|\varphi\rangle$ we can introduce the following definition:

**Definition 4.** A quantum state $|\varphi\rangle \in Q_N$ is said to be complex if the classical string $\hat{\omega}_\varepsilon^B(|\varphi\rangle)$ is complex.

As always, a classical string $\omega$, of length $N$, is said to be complex if $K_{C1}(\omega) \sim N$.

**Remark 7.** This definition satisfies the intuitive idea of the complexity of a state. Let us in fact consider the following situation: Alice has obtained a state ($|\varphi\rangle$) and wants Bob to reproduce it (at least with some precision $\varepsilon$). Expecting that a similar situation would arise, they had previously agreed on a common code. All that Alice then has to pass to Bob is the information on how to compose a circuit that prepares $|\varphi\rangle$ with the given precision, and this means sending Bob the string $\omega(C^B_\varepsilon(|\varphi\rangle))$ that codes the circuit. In this case the complexity of the state $|\varphi\rangle$ measures exactly the minimum amount of information that Alice needs to pass to Bob. We underline that this information is not given by the length of the coding string, but by its complexity: this simply reflects the fact that a state could be prepared using a very big circuit (that will be coded by a consequently long string), but a very simple one (again the example of a circuit obtained repeating many times the same set of gates): in this case the amount of information Alice needs will be much smaller than the length of the coding word.

**VI. THE “BASIS PROBLEM”**

The definition we have given for the algorithmic complexity of a state has a very strong dependence on the choice of the basis. Fixing a particular state it is in fact possible to build a particular basis so that the description of $|\varphi\rangle$ is trivial. One could thus argue that the definition has no relevant meaning.

Let us consider again the situation in which Alice prepares a state and wants to describe it to Bob. If they have previously agreed on using a certain gate basis, then Alice has only to describe to Bob the circuit (that means passing to Bob the sequence $\omega_C$). If they have not agreed on a particular gate basis, then Alice could indeed build a circuit using the “best” basis, but in this case she would have to describe the basis itself to Bob, and this would be in general a similarly difficult task.
One might nevertheless wonder whether there is some particular basis that allows to describe all (or almost all) states with simple circuits. If such a basis existed, it would obviously be convenient for Alice and Bob to agree on using that: in this case we would obtain that (almost) all quantum states are non-complex. In the following we will show that such a basis cannot exist as, once any gate basis is fixed, the number of non-complex states is always small in relation to the total number of states.

In classical information theory it is well known that the number of compressible (bit) strings is “small”; more precisely one has:

$$\frac{\#\{\omega_n = \omega_{i_1} \cdots \omega_{i_n} | K_{\text{Cl}}(\omega_n) < c\}}{\#\{\omega_n = \omega_{i_1} \cdots \omega_{i_n}\}} \leq \frac{2^c - 1}{2^n}.$$  

(11)

Now let us consider the quantum case: as we have seen in the previous paragraphs, once we fix a basis $B$ and a precision parameter $\varepsilon$, we can associate to every quantum state $|\varphi\rangle$ a $(-N^2 2^N \log \varepsilon)$-bit string $\hat{\omega}_B^\varepsilon(|\varphi\rangle)$ whose classical algorithmic complexity coincides with the complexity of $|\varphi\rangle$. Applying Eq. (11) to the set of strings $\hat{\omega}_B^\varepsilon$ we obtain:

$$\frac{\#\{|\varphi\rangle \in Q_N | K_B^\varepsilon(|\varphi\rangle) < c\}}{\#\{|\varphi\rangle\}} \leq \frac{2^c - 1}{2^{-N^2 2^N \log \varepsilon}} \sim 2^{N^2 2^N \log \varepsilon + c},$$  

(12)

where with $\#\{|\varphi\rangle\}$ we represent the number of different normalized states that can be prepared with precision $\varepsilon$. Such a relation holds true also in the case when $c = c(N, \varepsilon)$ is a function. State $|\varphi\rangle$ will be non-complex only if its complexity is $o(-N^2 2^N \log \varepsilon)$: this means that the right member of the inequality becomes $2^{N^2 2^N \log \varepsilon + o(-N^2 2^N \log \varepsilon)} \sim 2^{N^2 2^N \log \varepsilon} \ll 1$. Thus, applying equation (12) to these states, one obtains that for any fixed basis $B$, the number of non complex states is exponentially small.

VII. ENTROPY AND COMPLEXITY

In classical information theory one can consider a random source that emits (with a certain probability distribution) letters drawn from some finite alphabet. In this case there is a strong relationship between the Shannon entropy of the source and the algorithmic complexity of the emitted sequences. In particular, if $\omega_n$ is a $n$-letter sequence and $p(\omega_n)$ is its probability, one obtains [5]:

$$|\langle K_{\text{Cl}}(\omega_n) - H \rangle| \leq c,$$  

(13)
where the average is taken over all \( n \)-letter sequences, \( H = -\sum_\omega p(\omega) \log p(\omega) \) is the Shannon entropy of the source and \( c \) is a constant that depends on the probability distribution. When such a distribution does not depend on the length of the sequences (as in the case of Bernoulli sources), this implies that, in the limit of \( n \gg 1 \), the average complexity production of the source coincides with its entropy production:

\[
\frac{\langle K_{\mathrm{Cl}}(\omega_n) \rangle}{n} \xrightarrow{n \to \infty} \frac{H}{n}.
\]  

(14)

It comes quite natural to seek a similar relation in the context of quantum information theory. In order to do this, we will first of all find the corresponding relation in the case of a classical source that emits words (and not letters). After that we will define what we intend by quantum source and find, in this case, the wanted relationship between complexity and entropy.

A. Classical case

Let us consider now a variation of the classical letter-source: in this case we will have a source that emits, with some given probability, words drawn from a finite dictionary \( D \). Each of the words will be a sequence of letters of alphabet \( A \); without loss of generality we can assume all words to have the same length \( l \). The output of such a source will thus be a sequence of words (or sentence). As the Shannon entropy depends only on the probability distribution, once we fix such a distribution it is the same for letter and for word-sources. If we consider the complexity of the emitted message, though, it is evident that there must be some differences. A word-source that emits \( m \) objects will in fact have transmitted a sequence of \( ml \) letters: it is now easy to believe that the complexity of such a sequence can be higher than that of an \( m \)-letter sequence (even though letters and words are emitted following the same probability distribution). This is an immediate consequence of the fact that words are composite objects that have non-zero complexity themselves.

Naturally, when we consider very long output sentences, such a difference becomes negligible. Any sentence can in fact be reduced to a word by a program that associates to each word a symbol: the length of this program will be determined by the complexity of the dictionary (that is, by the complexity of the single words in the dictionary) and will thus be bound by the dictionary length \( l \# D \). This contribution, though, can be extremely relevant
while we consider “short” sentences, that is sentences whose length is comparable to that of the dictionary.

Thus, if $\Omega_m = \omega_{i_1} \cdots \omega_{i_m}$ is an $m$–word sentence, its complexity is given by:

$$K_{C1}(\Omega_m) \simeq K_{C1}(i_m = i_1 \cdots i_m) + \sum_{j=1}^{\#D} K_{C1}(\omega_j) \leq K_{C1}(i_m) + l\#D,$$

(15)

where $i_k$ is the symbol that codes $\omega_{i_k}$, and $l$ is the common length of the words $\omega_{i_k} \in D$.

The sequences $i_m$ can easily be seen as the outputs of a letter-source whose entropy $H$ coincides with that of the considered word-source: for such sequences, thus, the relation expressed by Eq. [13] still holds true. When generalizing it to word-sources, though, it is necessary to consider the contribute of the dictionary, obtaining:

$$|\langle K_{C1}(\Omega_m) - H \rangle - \sum_{j=1}^{\#D} K_{C1}(\omega_j)| \leq c_1.$$

Naturally, being the complexity of the dictionary constant, Eq. [14] remains of the same form for word-sources too:

$$\frac{\langle K_{C1}(\Omega_m) \rangle}{m} \to_{m \to \infty} \frac{H}{m}.$$

(17)

B. Quantum case

Before being able to say anything for the quantum case, it is necessary to specify what we will consider as a quantum source. In principle, any mixed state can be viewed as a quantum source. In the following, though, we will consider a quantum source as a “black box” that emits (with a given probability) pure states drawn from a given set $\{ \varphi_j \}_{j \in \mathbb{D}}$. This corresponds to considering not a general mixed state, but rather a well defined mixture $\{(p_j, |\varphi_j\rangle)\}_{j \in \mathbb{D}}$ of pure states ($|\varphi_j\rangle \in \mathcal{Q}_N$). In this case the source has a kind of “semi-classical” nature: it is in fact quantum only in the sense that it emits quantum states and not (as in the cases considered in the previous paragraphs) classical objects, while it cannot, for example, emit states that are other than the tensor product of the ones in the ensemble. For these sources the Shannon entropy coincides with the Shannon entropy of a classical source that emits objects (letters or words) with the same probability distribution $\{p_j\}_{j \in \mathbb{D}}$.

As we have seen when defining the complexity of a quantum state, once we fix a finite precision $\varepsilon$, it is possible to describe any quantum state by means of a finite word (whose
length depend both on the dimension of the Hilbert space in which the state lives and on the value of $\varepsilon$). This fact allows us to identify a quantum source of this kind with a corresponding word-source. The expression for the complexity of the emitted messages (tensor-product states) follows thus immediately from what seen above:

$$K^{B,\varepsilon}_{\text{Net}}(|\Phi_m\rangle) = |\varphi_{i_1}\rangle|\varphi_{i_2}\rangle\cdots|\varphi_{i_m}\rangle) \simeq K_{\text{Cl}}(i_m = i_1\cdots i_m) + \sum_{j \in D} K^{B,\varepsilon}_{\text{Net}}(|\varphi_j\rangle)$$

$$\leq K_{\text{Cl}}(i_m) - \#D N 2^N \log \varepsilon ,$$

where $i_k$ is the symbol that codes state $|\varphi_{i_k}\rangle$. We underline that, in this case, the contribution due to the complexity of the dictionary can be relevant indeed, being bound by the complexity of the states (that can in principle be very large). The relation between the entropy $H$ of the source (that is, the Shannon entropy of the probability distribution $\{p_j\}_j$) and the complexity of the message is analogously obtained:

$$|\langle K^{B,\varepsilon}_{\text{Net}}(|\Phi_m\rangle) - H \rangle - \sum_{j \in D} K^{B,\varepsilon}_{\text{Net}}(|\varphi_j\rangle)| \leq c .$$

(19)

As in the classical case, though, if we consider the limit of infinitely long state sequences the complexity rate and the entropy rate tend to coincides: in this limit, thus, the source “reveals” its semiclassical nature.

**Remark 8.** Quite naturally one could ask what happens if we have the possibility to apply Schumacher’s noiseless coding theorem [13] and thus compress the emitted states. If $S(\rho)$ is the von Neumann entropy of the source $\rho = \sum_j p_j |\varphi_j\rangle\langle \varphi_j|$, each state can be compressed into a new state $|\varphi'_j\rangle$ in a $(2^{NS(\rho)})$-dimensional Hilbert space. In this case we can rewrite Eq. (18) as:

$$K^{B,\varepsilon}_{\text{Net}}(|\Phi_m\rangle) = |\varphi_{i_1}\rangle|\varphi_{i_2}\rangle\cdots|\varphi_{i_m}\rangle) \simeq K^{B,\varepsilon}_{\text{Net}}(|\Phi'_m\rangle) = |\varphi'_{i_1}\rangle|\varphi'_{i_2}\rangle\cdots|\varphi'_{i_m}\rangle)$$

$$\simeq K_{\text{Cl}}(i_m = i_1\cdots i_m) + \sum_{j \in D} K^{B,\varepsilon}_{\text{Net}}(|\varphi'_j\rangle)$$

$$\leq K_{\text{Cl}}(i_m) - \#D (NS(\rho)) 2^{NS(\rho)} \log \varepsilon .$$

(20)

**VIII. APPLICATIONS AND EXAMPLES**

A. Complexity of copies

One of the properties of classical algorithmic complexity is that obtaining $m$ copies of a given string is (almost) free, being the dependence of the complexity on the number of
copies only logarithmic:

\[ K_{\text{Cl}}(\omega^{(m)} = \underbrace{\omega \odot \cdots \odot \omega}_{m \text{ times}}) \leq K_{\text{Cl}}(\omega) + \mathcal{O}(\log m) . \]

This bound is easily obtained by considering that, once one has a program that reproduces string \( \omega \) it is sufficient to run it \( m \) times to reproduce \( \omega^{(m)} \).

In the case of a quantum system, the situation is slightly more complex. It is first of all necessary to decide exactly what we mean by “preparing \( m \) copies” of a quantum state: in fact either we will require the circuit to prepare \( m \) times a quantum state \( |\varphi\rangle \in Q_N \), with some fixed precision parameter \( \varepsilon \), or it must prepare the global state \( |\varphi\rangle^\otimes m \in Q_N^m \) with the given precision. The two situations are extremely different.

In the first case, we obtain the same relation we have seen in the classical case:

\[ K_{\text{Net}}^{B^\varepsilon}(|\varphi\rangle^{(m)}_{\varepsilon}) = K_{\text{Net}}^{B^\varepsilon}(|\varphi\rangle \cdots |\varphi\rangle_{\varepsilon}) \leq K_{\text{Net}}^{B^\varepsilon,\varepsilon}(|\varphi\rangle) + \mathcal{O}(\log m) , \]

where the expression \( |\varphi\rangle^{(m)}_{\varepsilon} \) wants to remind us that each copy of \( |\varphi\rangle \) is reproduced with precision \( \varepsilon \). This is an immediate consequence of the definition of complexity: a circuit that prepares \( |\varphi\rangle^{(m)}_{\varepsilon} \) can be in fact built by repeating \( m \) times the one that prepares \( |\varphi\rangle \) with precision \( \varepsilon \).

If, instead, we require the state \( |\varphi\rangle^\otimes m \) to be prepared with precision \( \varepsilon \), the situation is different and the classical relation does not (necessarily) hold true any more. In this case the relation between the complexity of \( |\varphi\rangle \) and that of \( |\varphi\rangle^\otimes m \) has the following form:

\[ K_{\text{Net}}^{B^\varepsilon}(|\varphi\rangle^\otimes m) \leq K_{\text{Net}}^{B^\varepsilon,\varepsilon}(|\varphi\rangle^\otimes m) + \mathcal{O}(\log m) \leq -N^2 2^N \log \frac{\varepsilon}{m} . \] (21)

This expression follows immediately by the fact that the state \( |\varphi\rangle^\otimes m \) can be prepared with precision \( \varepsilon \) by a circuit \( \mathcal{C}_{\varepsilon}^{B^\varepsilon}(|\varphi\rangle^\otimes m) \) built with \( m \) identical copies of a smaller circuit \( \mathcal{C}_{\varepsilon/4^m}^{B^\varepsilon}(|\varphi\rangle) \).

The last inequality follows immediately from Eq. (10). We underline that in most cases (that is when \( 4m\varepsilon^{-m-1} \leq 1 \)) this bound is much stricter than the one we could obtain by directly applying (10) with which one has: \( K_{\text{Net}}^{B^\varepsilon}(|\varphi\rangle^\otimes m) \leq -mN^2 2^N \log \varepsilon \).
B. Entanglement and Complexity

Let us consider a state $|\varphi\rangle \in \mathcal{Q}_N$; suppose it can be written as

$$|\varphi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle \otimes \cdots |\varphi_J\rangle,$$

with $|\varphi_j\rangle \in \mathcal{Q}_{N_j}$ such that $\dim(\mathcal{Q}_{N_j}) = 2^{N_j}$, and $\sum_{j=1}^{J} N_j = N$.

This means that the state $|\varphi\rangle$ is not totally entangled, and it can thus be considered as the tensor product of other (possibly entangled) states $|\varphi_j\rangle$.

As a consequence of this fact we have:

$$K_{\text{Net}}^{B,\varepsilon}(|\varphi\rangle) \leq \sum_{j=1}^{J} K_{\text{Net}}^{B,\varepsilon/2J}(|\varphi_j\rangle) \leq - \sum_{j=1}^{J} N_j^2 2^{N_j} \log \frac{\varepsilon}{J}. \quad (22)$$

We want to show that this upper bound is actually stricter than the one obtained for general states in $\mathcal{Q}_N$: let us rewrite the bound given by Eq. (10) as $K_{\text{Net}}^{B,\varepsilon}(|\varphi\rangle) \leq - \sum_{j=1}^{J} \frac{N_j^2 2^N}{J} \log \varepsilon$.

The wanted inequality is proved by considering:

$$N_j^2 2^{N_j} \log \frac{J}{\varepsilon} < N^2 2^N \log \frac{J}{\varepsilon} \leq \frac{N^2 2^N}{2^{J-1}} \log \frac{J}{\varepsilon} < \frac{N^2 2^N}{J} \log \frac{1}{\varepsilon}.$$

Thus, the maximal complexity can be obtained only by a truly $N$-party entangled state (in the sense that it cannot be written as tensor product of states contained in subspaces of $\mathcal{Q}_N$). We stress that this consideration does not imply that all totally entangled states have maximal complexity (as a counterexample it is enough to consider the GHZ states).

It is nevertheless interesting to consider how a property that is characteristic of quantum systems (that is entanglement) has direct effect on the complexity of a state.

**Example 2 (Complexity of a completely separable state).** As an example let us consider a state $|\varphi\rangle \in \mathcal{Q}_N$ of the form $|\varphi\rangle = \bigotimes_{j=1}^{N} |\varphi_i\rangle$, $|\varphi_i\rangle \in \mathcal{Q}_1$. In this case we have:

$$K_{\text{Net}}^{B,\varepsilon}(|\varphi\rangle) \leq \sum_{j=1}^{N} K_{\text{Net}}^{B,\varepsilon/2}(|\varphi_j\rangle) \leq -2N \log \frac{\varepsilon}{N},$$

thus the complexity of a separable state grows only at most linearly with the number of qubits.

**Remark 9.** As we have seen, in general the growth of the complexity of a state with the number of qubits is exponential, and this is substantially different to what happens in the
classic case where the upper bound is linear with the number of bits. In this last example, though, we see that the absence of entanglement re-establishes the classical limit; entanglement thus proves again being a fundamental feature that distinguishes quantum objects from classical ones.

C. Complexity of graph states

Graph states are multi-particle entangled states that can uniquely be described by mathematical graphs, where the vertices of the graph take the role of qubits and edges represent unitary operations between the relative qubits.

Given a graph $G = (V, E)$ one can easily prepare the corresponding graph state using the following procedure:

1. prepare all qubits in state $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$;

2. when there is an edge between two vertices $k$ and $l$ apply a controlled-phase gate between the two qubits. This actually means applying to the two qubits a transformation of the form: $U_{kl} = e^{-i\frac{\pi}{4}(1^{(k)} - \sigma_z^{(k)})/(1^{(l)} - \sigma_z^{(l)})}$.

The resulting state $|\psi\rangle_G$ will be an entangled state uniquely described by the graph $G$.

Once fixed the number $N$ of vertices, there are at most $2^{N(N-1)/2}$ different graphs $G_1, G_2, \cdots, G_{2^{N(N-1)/2}}$ (each vertex can in fact be connected or not to each other vertex by an edge). Correspondingly, given $N$ qubits, we can build at most $2^{N(N-1)/2}$ different graph states $|\psi\rangle_{G_1}, |\psi\rangle_{G_2}, \cdots, |\psi\rangle_{G_{2^{N(N-1)/2}}}$.

Once we define some sort of lexicographic order in the set of all graphs (or equivalently in that of the graph states) only $O(N^2)$ bits are sufficient to specify a determinate state. This value constitutes an upper bound for the complexity of a graph state.

We will now show that this same value can be obtained for the algorithmic complexity of a graph state. As seen above in the more general case, the upper bound for the complexity of a state is obtained finding a bound for the length (and thus for the complexity) of the characterizing string. In the case of graph states this bound is obtained in a simple way: if $N$ is the number of vertices of the graph, the maximum number of edges is $N(N - 1)/2$. This implies that a corresponding graph state can be obtained applying to the $N$ qubits at most $N + N(N - 1)/2$ gates ($N$ Hadamard gates, needed to initially prepare all the qubits
in state $|+\rangle$, and $N(N-1)/2$ controlled-phase gates). If $G$ has $N$ vertices, we have thus:

$$K_{\text{Net}}(|\psi\rangle_G) = K_{\text{CI}}(\hat{\omega}^B(|\psi\rangle_G)) \leq I(\hat{\omega}^B(|\psi\rangle_G)) \lesssim N + N(N-1)/2 \lesssim N^2. \quad (23)$$

As there is no dependence on the precision parameter, Eq. (23) holds true only if we have the possibility to reproduce exactly the controlled-phase and the Hadamard gates.

It is nevertheless possible to obtain a (more general) upper bound for the complexity of this family of states, valid also in the case in which our basis is not of the above type. In order to do this we will use some considerations regarding the complexity of sentences, seen in section VIIA.

In order to prepare a graph state only two different types of gates, controlled-phase and Hadamard, are sufficient. If these gates cannot be reproduced exactly by those in our basis then, naturally, the wanted state can be prepared only with finite precision. As we have seen above, we need at most $\mathcal{O}(N^2)$ of these gates to prepare any arbitrary graph state. To guarantee that the desired state is prepared with precision $\varepsilon$ it is thus enough to reproduce each gate with precision $\varepsilon/N^2$. From the Solvay-Kitaev theorem we know that implies we need to use $\mathcal{O}(-\log \frac{\varepsilon}{N^2})$ gates to simulate each Hadamard (or controlled-phase) gate. Once we code the circuit, to each Hadamard (or controlled-phase) will correspond the same $\mathcal{O}(-\log \frac{\varepsilon}{N^2})$-letter word. Using Eq. (15) (where $\#D = 2$) [22] we obtain:

$$K_{\text{Net}}(|\psi\rangle_G) \leq N^2 - \log \frac{\varepsilon}{N^2}. \quad (24)$$

Different is the case of weighted graph states [3]. These states are generalizations of graph states, in which every edge is specified by a (different) phase. The procedure to prepare these states is analogous to that illustrated for graph states; the only difference is that, in this case, whenever two vertices $k$ and $l$ are connected, one must now apply a transformation of the form $U_{kl} = e^{-i\frac{\pi}{4}(\sigma_z^{(k)}(1^{(k)}) - \sigma_z^{(l)}(1^{(l)})\sigma_z^{(l)})} \cdot$

While the total number of gates in the circuit is still at most $\mathcal{O}(N^2)$, in this case it is not sufficient to consider only Hadamard and controlled-phase gates as, in principle, each phase-gate could be different. When preparing a weighted graph state it is thus necessary to reproduce $\mathcal{O}(N^2)$ different gates with precision $\mathcal{O}(-\log \frac{\varepsilon}{N^2})$. Again we can obtain an upper bound for the complexity of these states by using Eq. (15), only that this time the size of the dictionary does depend on $N$: $\#D \sim N^2$. We have thus:

$$K_{\text{Net}}(|\psi\rangle_{\text{w.\ G.}}) \leq N^2 - N^2 \log \frac{\varepsilon}{N^2} \sim -N^2 \log \frac{\varepsilon}{N^2}. \quad (25)$$
IX. CONCLUSIONS AND OUTLOOK

In this paper we have introduced a new definition for the algorithmic complexity of quantum states. We have defined the complexity of a quantum state as the description complexity of its experimental preparation via a quantum circuit. We have investigated the relation between the Shannon entropy of a source and the algorithmic complexity of the emitted message. We could also straightforwardly apply this definition to find upper bounds for a number of interesting cases. We have seen a relation between entanglement and algorithmic complexity: in particular we have seen that the absence of entanglement reduces the upper bound for the algorithmic complexity to the classical one.

While we have studied the algorithmic complexity of some classes of states, one could pursue this investigation analysing other states, for example states that appear in the context of quantum phase transition and quantum adiabatic computation [12]. Recent results [1] suggest that it could be possible to construct stricter upper bounds for preparation complexity of these states. From a broader perspective, it would be interesting to investigate further connections between algorithmic complexity and entanglement.

We want to thank Fabio Benatti for useful discussions and comments. This work has been supported in part by the Deutsche Forschungsgemeinschaft and the European Union (IST-2001-38877-29227).

Appendix A: Connection with Vitanyi’s Complexity

In [10] the author proposes a definition of quantum algorithmic complexity based on quantum Turing machines. In particular, the algorithmic complexity of a quantum state is given by the following expression:

$$K_{Vit}(|\varphi\rangle) = \min \{ l(p) + \lceil - \log(\langle |\psi| |\varphi\rangle^2) \rceil \} ,$$  \hspace{1cm} (26)

where the minimum is taken over all programs \( p \) (in classical bits) running on a universal Turing machine \( \mathcal{U} \) and such that \( \mathcal{U}(p, |\varphi_0\rangle) = |\psi\rangle \) (where \( |\varphi_0\rangle \) is the initial state of the computer), and \( l(p) \) is the (classical) length of the program. The complexity of \( |\varphi\rangle \) is therefore constituted of two separate terms: the length of a program describing an approximation \( |\psi\rangle \) to \( |\varphi\rangle \) and a term penalizing for a bad approximation.
The author shows that, if $|\varphi\rangle \in \mathcal{Q}_N$, then $2N$ is an upper bound for $K_{\text{Vit}}(|\varphi\rangle)$. This is easily obtained considering the projections of $|\varphi\rangle$ over the vectors of a basis: being $|\varphi\rangle$ normalized, there exists at least one vector $|e_j\rangle$ such that $|\langle e_j|\varphi\rangle|^2 \geq 1/2^N$: the length of a program that gives $|e_j\rangle$ as output is at most $n$ and the corresponding value for $l(p) + \lceil -\log(|\langle \psi|\varphi\rangle|^2) \rceil$ is exactly $2N$.

The required upper bound for the complexity of a state $|\varphi\rangle$ is thus obtained when the output state $|\psi\rangle$ satisfies the condition $|\langle \psi|\varphi\rangle|^2 \geq 1/2^N$. Such condition can be easily rewritten in terms of a precision parameter $\varepsilon_{\text{Vit}}$ as: $|\langle \psi|\varphi\rangle|^2 \geq 1 - \varepsilon_{\text{Vit}}$ with $\varepsilon_{\text{Vit}} = 1 - 1/2^N$. Rewriting the expression for the upper bound in this particular case (and considering the case $1/2^N \ll 1$) we have:

$$K^{\varepsilon_{\text{Vit}}}(|\varphi\rangle) \leq 2^N \log \varepsilon_{\text{Vit}} + N \simeq N .$$

To find this expression it is necessary to consider the complete expression for the upper bound given in Eq. [3]. We see thus that the definition given by the author satisfies the preliminary condition introduced in section [I].

With this value of the precision parameter $\varepsilon$, for the algorithmic complexity of a state we have:

$$K^{\varepsilon_{\text{Vit}}}(|\varphi\rangle) \leq N^2 2^N \log \varepsilon_{\text{Vit}} \simeq N^2 .$$

This bound is larger than the one that characterizes Vitanyi’s complexity; such a difference is only polynomial.

**Appendix B: Quantum algorithmic complexity of a classical string**

We naturally want to see if our definition is coherent with that of classical Kolmogorov complexity. In order to do this we consider a classical $N$-bit string $\mathbf{x} = x_{i_1}x_{i_2}\cdots x_{i_N}$ (where $x_{i_j} \in \{0,1\}$).

In order to use a procedure built to characterize complexity of quantum state it is necessary to “translate” the classical string into a quantum state; this is easily done simply considering the state $|\mathbf{x}\rangle = |x_{i_1}\rangle|x_{i_2}\rangle\cdots|x_{i_N}\rangle$, with $|x_{i_j}\rangle \in \{|0\},\{1\}\}$.

Considering the nature of this state it is trivial to see that it can be obtained from the initial state $|0\rangle|0\rangle\cdots|0\rangle$ by simply applying one-qubit $\text{NOT}$ gates to the bits corresponding to $|x_{i_j}\rangle = |1\rangle$ and leaving invaried (applying the identity transformation) the bits $|x_{i_j}\rangle = |0\rangle$. 21
To encode these particular circuits, thus, 3 symbols are enough\cite{23}:

\[
I \leftrightarrow \text{identity} \\
N \leftrightarrow \text{NOT gate} \\
L \leftrightarrow \text{NEWLINE}.
\]

Let us consider now a simple example:

Classical string \( x = 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \)

Quantum state \( |x\rangle = |1\rangle \quad |0\rangle \quad |1\rangle \quad |1\rangle \quad |0\rangle \quad |0\rangle \quad |0\rangle \).

<table>
<thead>
<tr>
<th>Qubit</th>
<th>Initial</th>
<th>Transformation</th>
<th>Final</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(</td>
<td>\rangle )</td>
<td>( \text{NOT} )</td>
</tr>
<tr>
<td>2</td>
<td>(</td>
<td>\rangle )</td>
<td>( \text{Id} )</td>
</tr>
<tr>
<td>3</td>
<td>(</td>
<td>\rangle )</td>
<td>( \text{NOT} )</td>
</tr>
<tr>
<td>4</td>
<td>(</td>
<td>\rangle )</td>
<td>( \text{NOT} )</td>
</tr>
<tr>
<td>5</td>
<td>(</td>
<td>\rangle )</td>
<td>( \text{Id} )</td>
</tr>
<tr>
<td>6</td>
<td>(</td>
<td>\rangle )</td>
<td>( \text{NOT} )</td>
</tr>
<tr>
<td>7</td>
<td>(</td>
<td>\rangle )</td>
<td>( \text{Id} )</td>
</tr>
<tr>
<td>8</td>
<td>(</td>
<td>\rangle )</td>
<td>( \text{Id} )</td>
</tr>
</tbody>
</table>

After coding, then, we have the following string representing the circuit:

\[
\omega(C) = N \quad L \quad I \quad L \quad N \quad L \quad N \quad L \quad I \quad L \quad N \quad L \quad I \quad L
\]

(where, to help visualization, we have put larger spaces after line breaks \( L \)). It is evident that there is a direct correspondence between \( \omega(C) \) and the original classical string \( x \): in fact, a program that reproduces \( x \) reproduces also \( \omega(C) \) (with the agreement that \( 0 \leftrightarrow I \) and \( 1 \leftrightarrow N \)) (it will be enough to add a \textit{constant} part that tells the program to insert a line-break after every symbol). Thus it follows immediately that the (classical) Kolmogorov complexity of \( \omega(C) \) coincides with that of \( x \). Considering the definition of the network complexity we have immediately:

\[
K_{\text{Net}}(|x\rangle) \simeq K_{\text{Cl}}(x).
\]  \hspace{1cm} (27)
Naturally the exact procedure followed in the example can be applied to any $N$-bit string $x$ so the result is true in general.


[17] For a review see e. g. the paper by Bennett et al. [2]

[18] This approach, might be closest to the traditional viewpoint -expressed most notably by Niels Bohr- that the quantum state is essentially an expression of an experimental scenario.

[19] We would expect that any notion of complexity should be asymptotically invariant under coarsening of the description of the toolbox (for discussion of this point see Section [3]).

[20] For each $\alpha \in [0,1]$ there exists in fact one (and only one) sequence $\{\alpha_i\}_i$ (with $\alpha_i \in \{0,1\}$) such that $\alpha = \sum_i \alpha_i 2^{-i}$.

[21] This, in fact, would again require to describe arbitrary unitary transformations.

[22] Actually the cardinality of $D$ can be larger than 2, as there will be some words that correspond to operations such as “new line” or similar, but it will always be independent of $N$.

[23] In principle, in this case we need not define a complete gate basis as the only NOT gate is sufficient.