New results for the fully renormalized proton-neutron quasiparticle random phase approximation

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Abstract

A many-body Hamiltonian describing a system of Z protons and N neutrons moving in spherical shell model mean field and interacting among themselves through proton-proton and neutron-neutron pairing and a dipole-dipole proton-neutron interaction of both particle-hole and particle-particle type, is treated within a fully renormalized (FR) pnQRPA approach. Two decoupling schemes are formulated. One of them decouples the equations of motion of particle total number conserving and non-conserving operators. One ends up with two very simple dispersion equations for phonon operators which are formally of Tamm-Dancoff types. For excitations in the (N-1,Z+1) system, Ikeda sum rule is fully satisfied provided the BCS equations are renormalized as well and therefore solved at a time with the FRpnQRPA equations. Next, one constructs two operators \( R_{1\mu}^\dagger \) and \( R_{1,-\mu}(-1)^{1-\mu} \) which commutes with the particle total number conserving operators, \( A_{1\mu}^\dagger \) and \( A_{1,-\mu}(-1)^{1-\mu} \), and moreover could be renormalized so that they become bosons. Then, a phonon operator is built up as a linear combination of these four operators. The FRpnQRPA equations are written down for this complex phonon operator and the ISR is calculated analytically. This formalism allows for an unified description of the dipole excitations in four neighboring nuclei (N-1,Z+1),(N+1,Z-1),(N-1,Z-1),(N+1,Z+1). The phonon vacuum describes the (N,Z) system ground state.
I. INTRODUCTION

Understanding nuclear physics phenomena in terms of single particle motion in a mean field and nucleon-nucleon interaction is one of the most appealing aims of theoretical nuclear physics. Along the latest few decades many progresses have been obtained in the many body treatment of nuclear system. Among the chief achievements one distinguishes the Hartree-Fock, BCS and random phase approximation (RPA)\cite{1,2}. The last mentioned approach can be formulated on the top of either a HF or a BCS ground state. The second version is conventionally called quasiparticle random phase approximation (QRPA). Among many successes of many body theories are the description of the ground state properties, of the excited states, including those of high spin of ground as well as of excited bands, of the electromagnetic transitions, and of the reaction mechanisms.

The proton neutron interaction has been also treated. For example the proton-neutron pairing interaction is currently investigated for nuclei close to the drip line. It is still an open question whether there exists a T=0 proton-neutron pairing phase, although there are some claims that the strong back-bending seen for $^{52}$Fe could be explained by breaking a proton-neutron T=0 pair\cite{3}.

The dipole proton-neutron interaction in the particle-hole ($ph$) as well as the particle-particle ($pp$) channels have been treated within the pnQRPA formalism in order to get quantitative description of the double beta decay with two neutrinos in the final state, $2\nu\beta\beta$\cite{5,6}. Of course, this phenomenon is very interesting by its own, but exhibits a special attraction for theoreticians having in view that it provides a test for the nuclear matrix elements which are also used in the neutrinoless double beta decay ($0\nu\beta\beta$) calculations. Indeed, the discovery of $0\nu\beta\beta$ process would answer a fundamental question whether neutrino is a Majorana or a Dirac particle.

Standard pnQRPA calculations based on $ph$ two body interaction yields a too large rate for the $2\nu\beta\beta$ process. In Ref.\cite{7} it was pointed out that the $\beta^+$ transition matrix element is very sensitive to the $pp$ interaction strength. Since the double beta decay has a branch which could be looked at as the hermitian conjugate matrix element of the $\beta^+$ virtual transition of the daughter nucleus to an dipole $1^+$ state in the intermediate odd-odd nucleus, many groups working on $2\nu\beta\beta$ decay included the $pp$ two body dipole interaction in the pnQRPA calculations. By contrast to the $ph$ interaction, which is repulsive, the $pp$ interaction has an
attractive character. Since such an interaction is not considered in the mean field equations, the approach fails at a critical value of the interaction strength, $g_{pp}$. Before this value is reached the Gamow-Teller transition amplitude, denoted by $M_{GT}$, is rapidly decreasing and after a short interval is becoming equal to zero. The experimental data for the amplitude $M_{GT}$ is met for a value of $g_{pp}$ close to that for which $M_{GT}$ vanishes and moreover close to the critical value of $g_{pp}$ where pnQRPA breaks down. It is obvious that in this region of $g_{pp}$, the pnQRPA results are not stable against adding interaction terms which are not encountered by the adopted many body approach. In order to restore the ground state stability, one needs to make a higher RPA calculation. The first formalism devoted to this feature includes anharmonicities through the boson expansion technique \[8, 9\]. Another method is the renormalized pnQRPA procedure (pnRQRPA) \[10\] which keeps the harmonic picture but renormalizes the bosons by effects coming from some terms of the commutator algebra which are not taken into account in the standard pnQRPA.

In a previous paper \[11\] we have shown that the pnRQRPA does not include the higher RPA effects in a consistent way. Indeed, in Ref. \[10\] only the two quasiparticle dipole operators are renormalized due to the non-vanishing average of the quasiparticle number operators in the renormalized QRPA ground state. In Ref. \[11\] we showed that having non-vanishing proton and neutron quasiparticle numbers, the scattering terms are also renormalized so that they finally satisfy bosonic commutation relations. Thus, a new phonon operator could be defined which includes, in addition, the scattering terms. We baptized the new renormalization formalism as a fully renormalized proton-neutron quasiparticle random phase approximation (FRpnQRPA). The FRpnQRPA equations determining the phonon operator amplitudes have twice as much solutions, $2N_s$, as the standard pnRQRPA equation. For half of these solutions one expects that the amplitude of the scattering terms prevails. In the quoted reference we pointed out which contribution is coming from the $ph$ and which comes from the $pp$ interactions in building up the new modes. It was shown that for charge conserving excitations these modes are spurious \[12\]. In a later publication we treated the scattering terms semi-classically and the harmonic mode defined there describes the wobbling motion of the isospin degrees of freedom \[13\].

It is worth mentioning that both renormalization approaches, pnRQRPA and FRpnQRPA, violates the Ikeda sum rule (ISR) \[14\], saying that the difference of the total $\beta^-$ and $\beta^+$ strengths for a single beta minus emitter is equal to $3(N-Z)$. Aiming at conciliating the
two features, renormalization and having ISR satisfied, recently [15] a new phonon operator was defined which includes also the scattering terms but commutes with the particle total number operator. The resulting formalism is called fully renormalized pnQRPA, i.e. by the name that we adopted four years before. A distinct way to renormalize the RPA equations without modifying the number of solutions and moreover without deriving an explicit expression for the phonon operator was formulated in Ref. [16]. The method is known under the name SCRP A (self consistent RPA).

The present paper continues the investigation we started in Ref [11], addressing the following issues. We want to see to what extent it is possible, in a quasiparticle BCS framework, to separate a harmonic boson operator conserving the particle total number from the components which do not commute with the total number operator. We shall see that the phonon operator conserving the particle total number, treated in a restricted $N_s$ dimensional space, can be embedded in an operator space acting on a $2N_s$ dimensional space and moreover has exactly the same properties as the phonon operator we have previously introduced in Ref. [11]. Also, we aim at treating on an equal footing the particle-hole and deuteron like dipole excitations. For both cases the FRpnQRPA equations are written down explicitly. Analytical expressions for the renormalization factors will be derived. An important question addressed in this paper is whether having a particle total number projected phonon operator assures that the ISR is fulfilled. Another problem treated in the present paper is the following one. In the FRpnQRPA approach, the phonon operator is a linear combination of the renormalized operators (see the notations from the next sections) $A_{\mu}^\dagger(pn), A_{1-\mu}(pn)(-1)^{1-\mu}, B_{1\mu}^\dagger(pn), B_{1-\mu}(pn)(-1)^{1-\mu}$, neglecting the commutators of $A$ and $B$ operators. Here we define two renormalizable operators $R_{\mu}^\dagger(pn), R_{1-\mu}(pn)(-1)^{1-\mu}$ which are exactly commuting with the operators $A_{\mu}^\dagger(pn)$ and $A_{1-\mu}(pn)(-1)^{1-\mu}$, where the latter operators are preserving the total number of particles. The phonon operator is defined as a linear combination of the $A$ and $R$ operators and explicit equations for the phonon amplitudes and energies will be derived.

The objectives sketched above will be accomplished according to the following plan. In Section 2 we briefly review the main ingredients of the FRpnQRPA formulated in Ref. [11]. In Section 3 a consistent decoupling scheme is defined which results in providing two independent sets of Tamm-Dancoff equations describing excitations of the $(N,Z)$ system in the nuclei $(N-1,Z+1)$ and $(N+1,Z+1)$, respectively. For the first type of excitation the Ikeda
II. BRIEF REVIEW OF THE FULLY RENORMALIZED PNQRPA FORMALISM

The single and double beta Gamow-Teller transitions can be described by the following many body model Hamiltonian:

\[
H = \sum_{\tau,jm} (\epsilon_{\tau jm} - \lambda_\tau c_{\tau jm}^\dagger c_{\tau jm} - \sum_{\tau,j,j'} \frac{G_\tau}{4} P_{\tau j}^\dagger P_{\tau j'}) \\
+ 2\chi \sum_{p,n,p',m} \beta_{-\mu}^- (pn) \beta_{+\mu}^+ (p'n') (-\mu) - 2\chi_1 \sum_{p,n,p',m} P_{1\mu}^- (pn) P_{1,-\mu}^+ (p'n') (-\mu).
\]  

(2.1)

where \(c_{\tau jm}^\dagger (c_{\tau jm})\) denotes the creation (annihilation) operator for a nucleon of type \(\tau (=p,n)\) in a spherical shell model \(|nljm\rangle\). The time reversed state corresponding to \(|nljm\rangle\) is \(\tilde{|nljm\rangle} = |nlj - m\rangle (-\mu)\). For the sake of simplifying the notation here only the quantum numbers \(j, m\) are specified. Also, the following notations have been used:

\[
\beta_\mu^- (pn) = \sum (\nu j m |\sigma \mu |n j' m') c_{\nu j m}^\dagger c_{n j' m'} = -\frac{\hat{j}_p}{1} (jp||\sigma||jn) \left[c_{\nu j m}^\dagger c_{n j' m'}\right]_{1\mu}, \\
P_{1\mu}^- (pn) = \sum (\nu j m |\sigma \mu |n j' m') c_{\nu j m}^\dagger c_{n j' m'} = \frac{\hat{j}_p}{1} (jp||\sigma||jn) \left[c_{\nu j m}^\dagger c_{n j' m'}\right]_{1\mu}, \\
\beta_\mu^+ (pn) = (\beta_{-\mu}^- (pn))^\dagger (-\mu), \quad P_{1\mu}^+ (pn) = (P_{1,-\mu}^- (pn))^\dagger (-\mu).
\]  

(2.2)

In what follows, sometimes the notations are simplified even more and the set of quantum numbers for a single \(\tau\) particle state will be denoted by \(\tau\), the other quantum numbers being mentioned only if necessary.

The Hamiltonian \(H\) is treated first by the BCS formalism which defines the quasiparticle representation through the Bogoliubov-Valatin (BV) transformation:

\[
a_{\tau jm}^\dagger = U_j c_{\tau jm}^\dagger - s_{jm} V_j c_{\tau j - m} \\
U_j^2 + V_j^2 = 1, \quad s_{jm} = (-)^{j - m}, \quad \tau = p, n
\]  

(2.3)
The images of the operators $\beta^\pm_\mu$ and $P^\pm_1\mu$ due to the BV transformations are:
\begin{align*}
\beta^-_\mu(k) &= \xi_k A^\dagger_1\mu(k) + \bar{\xi}_k A_{1,-\mu}(k)(-)^{1-\mu} + \eta_k B^\dagger_1mu - \bar{\eta}_k B_{1,-\mu}(k)(-)^{1-\mu}, \\
\beta^+_\mu(k) &= - \left[ \xi_k A^\dagger_1\mu(k) + \bar{\xi}_k A_{1,-\mu}(k)(-)^{1-\mu} - \eta_k B^\dagger_1mu + \bar{\eta}_k B_{1,-\mu}(k)(-)^{1-\mu} \right], \\
P^-_\mu(k) &= \eta_k A^\dagger_1\mu(k) - \bar{\eta}_k A_{1,-\mu}(k)(-)^{1-\mu} - \xi_k B^\dagger_1mu - \bar{\xi}_k B_{1,-\mu}(k)(-)^{1-\mu}, \\
P^+_\mu(k) &= \left[ - \bar{\eta}_k A^\dagger_1\mu(k) + \eta_k A_{1,-\mu}(k)(-)^{1-\mu} - \xi_k B^\dagger_1mu - \bar{\xi}_k B_{1,-\mu}(k)(-)^{1-\mu} \right].
\end{align*}
where the operators $A^\dagger, A, B^\dagger, B$, are the two quasiparticle and quasiparticle density dipole operators, respectively:
\begin{align*}
A^\dagger_1\mu(pn) &= \sum C^j_{m_\mu m_{n_\mu}} a^j_{p_{\mu}m_\mu} a^\dagger_{p_{\mu}m_{n_\mu}}, \quad A_1\mu(pn) = \left( A^\dagger_1\mu(pn) \right)^\dagger, \\
B^\dagger_1\mu(pn) &= \sum C^j_{m_\mu m_{n_\mu}} a^j_{p_{\mu}m_\mu} a_{n_{\mu}m_{\mu}}, \quad B_{1\mu}(pn) = \left( B^\dagger_1\mu(pn) \right)^\dagger. (2.4)
\end{align*}
The factors $\xi, \bar{\xi}, \eta, \bar{\eta}$ have the expressions:
\begin{align*}
\xi_k &= \hat{j}_p \langle j_p | \sigma | j_n \rangle U_{jp} V_{jn}, \quad \bar{\xi}_k = \hat{j}_p \langle j_p | \sigma | j_n \rangle V_{jp} U_{jn}, \quad \hat{j} = \sqrt{2j + 1}, \\
\eta_k &= \hat{j}_p \langle j_p | \sigma | j_n \rangle U_{jp} U_{jn}, \quad \bar{\eta}_k = \hat{j}_p \langle j_p | \sigma | j_n \rangle V_{jp} V_{jn}. (2.5)
\end{align*}
It is worth mentioning that throughout this paper the Rose convention for the reduced matrix element is adopted [17]. Within the proton-neutron quasiparticle random phase approximation (pnQRPA), one assumes that the two quasiparticle dipole operators $A^\dagger_1\mu, A_{1\mu'}$ satisfy quasi-bosonic commutation relations, while the quasiparticle density dipole operator commute with each other. Also, the $A$ operators commute with any of $B^\dagger$ and $B$ operators. The renormalized pnQRPA approximate the commutator $[A_{1\mu}, A^\dagger_{1\mu'}]$ by keeping from its exact expression only the $C$ number term and the monopole term. Moreover, replacing the adopted expression for the commutator by its average on the pnQRPA ground state, unknown yet, the operators $A$ and $A^\dagger$ can be renormalized so that the new operators obey bosonic commutation relation. Note that other commutator equations, are kept as in the standard pnQRPA, i.e. they are taken equal to zero.

The idea advanced in Refs. [11] was that also the proton-neutron quasiparticle density dipole operators can be renormalized so that finally they are gifted with boson properties. Therefore, the mutual commutation relations assumed in Ref. [11] are:
\begin{align*}
[A_{1\mu}(k), A^\dagger_{1\mu'}(k')] &\approx \delta_{k,k'} \delta_{\mu,\mu'} \left( 1 - \frac{\hat{N}_n}{J_n^2} - \frac{\hat{N}_p}{J_p^2} \right),
\end{align*}
\[
\begin{align*}
\left[ B_1^\dagger(k), A_1^\dagger(k') \right] & \approx \left[ B_1^\dagger(k), A_{1\mu'}(k') \right] \approx 0, \\
\left[ B_1^\dagger(k), B_1^\dagger(k') \right] & \approx \delta_{k,k'} \delta_{\mu,\mu'} \left[ \hat{N}_n - \frac{\hat{N}_n}{\hat{J}_n^2} - \frac{\hat{N}_p}{\hat{J}_p^2} \right], \quad k = (j_p, j_n). \tag{2.7}
\end{align*}
\]

Let us denote by \( C_{j_p,j_n}^{(1)} \) and \( C_{j_p,j_n}^{(2)} \) the averages of the r.h.s. of the first and third commutation relations (2.7) on the \( \nu \) vacuum state \( |0\rangle \), respectively. Then, the renormalized operators are:

\[
\begin{align*}
\bar{A}_1^\dagger(k) & = \frac{1}{\sqrt{C_k^{(1)}}} A_1^\dagger(k), \quad \bar{A}_1(k) = \frac{1}{\sqrt{C_k^{(1)}}} A_1(k), \\
\bar{B}_1^\dagger(k) & = \frac{1}{\sqrt{|C_k^{(2)}|}} B_1^\dagger(k), \quad \bar{B}_1(k) = \frac{1}{\sqrt{|C_k^{(2)}|}} B_1(k). \tag{2.8}
\end{align*}
\]

Recalling the specific RPA convention to replace the operator commutators by their average on the ground state, one readily obtains that the renormalized operators defined above satisfy boson like commutation relations:

\[
\begin{align*}
\left[ \bar{A}_1(k), \bar{A}_1^\dagger(k') \right] & = \delta_{k,k'} \delta_{\mu,\mu'}, \\
\left[ \bar{B}_1(k), \bar{B}_1^\dagger(k') \right] & = \delta_{k,k'} \delta_{\mu,\mu'} f_k, \quad f_k = \text{sign}(C_k^{(2)}). \tag{2.9}
\end{align*}
\]

Further, the renormalized operators are used in order to define the \( \nu \) phonon operator

\[
\Gamma_1^\dagger = \sum_k \left[ X(k) \bar{A}_1^\dagger(k) + Z(k) \bar{B}_1^\dagger(k) - Y(k) \bar{A}_{1-\mu}(k) (-1)^{1-\mu} - W(k) \bar{B}_{1-\mu}(k) (-1)^{1-\mu} \right]. \tag{2.10}
\]

where \( \bar{D}_1^\dagger(k) \) stands for either \( \bar{B}_1^\dagger(k) \) or \( \bar{B}_1(k) \) depending on whether \( f_k \) is equal to +1 or −1. The amplitudes \( X, Z, Y, W \) are determined by the fully FRpnQRPA equations provided by the operator equations:

\[
\begin{align*}
\left[ H, \Gamma_1^\dagger \right] & = \omega \Gamma_1^\dagger, \\
\left[ \Gamma_1^\dagger, \Gamma_1^\dagger \right] & = \delta_{\mu,\mu'}. \tag{2.11}
\end{align*}
\]

The number of FRpnQRPA equations is double the number of standard \( \nu \)QRPA equations. They have to be solved at a time with the equations defining the constants \( C^{(1)} \) and \( C^{(2)} \). The phonon vacuum is the ground state of the system which has the property that the corresponding average of the quasiparticle number operator is non-vanishing. Solving these equations, one finds out that half of the solutions have the amplitude \( X \) dominant while for the remaining ones, the amplitudes \( Z \) prevail. The latter solutions have been separately studied in Ref. [13]. It has been proven that such solutions describe a wobbling motion of the isospin degrees of freedom.
III. RESTORING THE GAUGE SYMMETRY FOR THE FULLY RENORMALIZED PNQRPA SOLUTIONS

The major component of vacuum state \( |0 \rangle \) is a state characterizing the even-even system (N,Z) under consideration. Considering the inverse BV transformation for the dipole operators \( A^\dagger, A, B^\dagger, B \) one can easily check that one phonon states are mixtures of components describing the neighboring nuclei (N-1,Z+1),(N+1,Z-1),(N+1,Z+1),(N-1,Z-1). The first two components preserve the total number of nucleons but violate the third isospin component \( T_3 \) while that remaining components violate the total number of nucleons but preserve \( T_3 \). The latter two components mentioned above are the ones which are responsible for the violation of Ikeda sum rule (ISR), which is valid for single beta decays of the (N,Z) nucleus. Aiming at getting a suitable structure for one phonon state, so that ISR is obeyed, it is desirable to start with linear combinations of the basic operators \( A^\dagger, A, B^\dagger, B \) which excite the (N,Z) nucleus to the nuclei (N-1,Z+1),(N+1,Z-1),(N+1,Z+1),(N-1,Z-1), respectively. One can check that such operators are:

\[
\begin{align*}
A_{1\mu}^\dagger (pn) &= U_p V_n A_{1\mu}^\dagger (pn) + U_n V_p A_{1,-\mu} (pn) (-1)^{-\mu} + U_p U_n B_{1\mu}^\dagger (pn) - V_p V_n B_{1,-\mu} (pn) (-1)^{-\mu}, \\
A_{1\mu} (pn) &= U_p V_n A_{1\mu} (pn) + U_n V_p A_{1,-\mu} (pn) (-1)^{-\mu} + U_p U_n B_{1\mu} (pn) - V_p V_n B_{1,-\mu} (pn) (-1)^{-\mu}, \\
A_{1\mu}^\dagger (pn) &= U_p U_n A_{1\mu}^\dagger (pn) - V_p V_n A_{1,-\mu} (pn) (-1)^{-\mu} - U_p V_n B_{1\mu}^\dagger (pn) - V_p U_n B_{1,-\mu} (pn) (-1)^{-\mu}, \\
A_{1\mu} (pn) &= U_p U_n A_{1\mu} (pn) - V_p V_n A_{1,-\mu} (pn) (-1)^{-\mu} - U_p V_n B_{1\mu} (pn) - V_p U_n B_{1,-\mu} (pn) (-1)^{-\mu}.
\end{align*}
\] (3.1)

Indeed, expressed in terms of creation and annihilation particle operators the above operators are:

\[
\begin{align*}
A_{1\mu}^\dagger (pn) &= - \left[ c_p^\dagger c_n^\dagger \right]_{1\mu}, \\
A_{1\mu} (pn) &= - \left[ c_p^\dagger c_n \right]_{1\mu}, \\
A_{1\mu}^\dagger (pn) &= \left[ c_p^\dagger c_n^\dagger \right]_{1\mu}, \\
A_{1\mu} (pn) &= \left[ c_p^\dagger c_n \right]_{1\mu}.
\end{align*}
\] (3.2)

In terms of the new dipole operators, the previously introduced one body operators become:

\[
\begin{align*}
\beta^-_\mu (pn) &= \frac{\hat{\jmath}_p}{1} \langle j p | | \sigma \rangle | j n \rangle A_{1\mu}^\dagger (pn), \\
\beta^+_\mu (pn) &= \frac{\hat{\jmath}_p}{1} \langle j p | | \sigma \rangle | j n \rangle A_{1,-\mu} (pn) (-)^\mu, \\
P^-_{1\mu} (pn) &= \frac{\hat{\jmath}_p}{1} \langle j p | | \sigma \rangle | j n \rangle A_{1\mu}^\dagger (pn), \\
P^+_{1\mu} (pn) &= \frac{\hat{\jmath}_p}{1} \langle j p | | \sigma \rangle | j n \rangle A_{1,-\mu} (pn) (-)^\mu.
\end{align*}
\] (3.3)

The model Hamiltonian can be written as:

\[
H = \sum_{\tau} E_{\tau} \alpha_{\tau jm}^\dagger \alpha_{\tau jm} + 2 \chi \sum_{\sigma p_{m}, p'_{n'}} \sigma_{pn, p'n'} A_{1\mu}^\dagger (pn) A_{1\mu} (p'n') - 2 \chi \sum_{\sigma p_{m}, p'_{n'}} \sigma_{pn, p'n'} A_{1\mu}^\dagger (pn) A_{1\mu} (p'n'),
\]
In order to study the harmonic modes which might be defined with the operators \( A_{1\mu}^\dagger (pn) \), \( A_{1\mu} (pn) \), \( A_{1\mu}^\dagger (pn) \), \( A_{1\mu} (pn) \) we need their mutual commutation relations. These are given in Appendix A. Following the prescription of the fully renormalized pnQRPA, these commutators can be approximated as:

\[
\begin{align*}
[A_{1\mu}^\dagger (pn), A_{\alpha\mu'}^\dagger (p' n')] &\approx \delta_{\mu,\mu'} \delta_{j_p,j_{\mu'}} \delta_{j_n,j_{\mu'}},
\left( U_p^2 - U_n^2 + \frac{U_n^2 - V_n^2}{j_n^2} \hat{N}_n - \frac{U_p^2 - V_p^2}{j_p^2} \hat{N}_p \right), \\
[A_{1\mu} (pn), A_{\alpha\mu'}^\dagger (p' n')] &\approx \delta_{\mu,\mu'} \delta_{j_p,j_{\mu'}} \delta_{j_n,j_{\mu'}},
\left( U_p^2 - V_n^2 - \frac{U_n^2 - V_n^2}{j_n^2} \hat{N}_n - \frac{U_p^2 - V_p^2}{j_p^2} \hat{N}_p \right), \\
[A_{1\mu} (pn), A_{\alpha\mu'} (p' n')] &\approx \delta_{\mu,\mu'} \delta_{j_p,j_{\mu'}} \delta_{j_n,j_{\mu'}},
U_n V_n \left[ 1 - 2 \frac{\hat{N}_n}{j_n^2} \right], \\
[A_{1\mu}^\dagger (pn), A_{1\mu'}^\dagger (p' n')(\cdot)^{1-\mu'}] &\approx \delta_{\mu,\mu'} \delta_{j_p,j_{\mu'}} \delta_{j_n,j_{\mu'}},
U_p V_p \left[ 1 - 2 \frac{\hat{N}_p}{j_p^2} \right].
\end{align*}
\]

(3.5)

Here, \( \hat{N}_p \) and \( \hat{N}_n \) stand for the proton and neutron quasiparticle number operators, respectively:

\[
\hat{N}_\tau = \sum_{jm} a_{\tau jm}^\dagger a_{\tau jm}; \tau = p, n.
\]

(3.6)

For what follows it is useful to introduce the notations:

\[
\begin{align*}
D_1(pn) &= U_p^2 - U_n^2 + \frac{1}{2j_n + 1} (U_n^2 - V_n^2) \langle \hat{N}_n \rangle - \frac{1}{2j_p + 1} (U_p^2 - V_p^2) \langle \hat{N}_p \rangle,
D_2(pn) &= U_p^2 - V_n^2 - \frac{1}{2j_n + 1} (U_n^2 - V_n^2) \langle \hat{N}_n \rangle - \frac{1}{2j_p + 1} (U_p^2 - V_p^2) \langle \hat{N}_p \rangle,
D_n &= U_n V_n \left[ 1 - \frac{2}{2j_n + 1} \langle \hat{N}_n \rangle \right], \\
D_p &= U_p V_p \left[ 1 - \frac{2}{2j_p + 1} \langle \hat{N}_p \rangle \right].
\end{align*}
\]

(3.7)

Here the average value of an operator \( \hat{O} \) on the renormalized phonon operator vacuum state, undefined for the time being, is denoted by \( \langle \hat{O} \rangle \). According to Eq. (A.1) the quantities \( D_n \) and \( D_p \) represent to some extent the overlap of the ground state of the \((N,Z)\) system with states describing the \((N+2,Z)\) and \((N,Z+2)\) systems, respectively. The latter states are obtained by adding one pair of neutrons and one pair of protons to the \((Z,N)\) ground state, respectively. Both pairs, of neutrons and protons, have a vanishing total angular momentum.
Note that for the single particle states involved in single $\beta^-$ transitions, $D_1(pn)$ is a positive quantity, while $D_2(pn)$ might be either positive or negative, depending on the relative values of the occupation probabilities for protons and neutrons in the states $p$ and $n$, respectively. Let us denote by $j_c$ the critical angular momentum having the property:

If $\epsilon_{pj} \leq \epsilon_{pj_c}$, then $D_2(pn) \leq 0$,
If $\epsilon_{pj} > \epsilon_{pj_c}$, then $D_2(pn) > 0$.  \(3.8\)

It is useful to introduce the functions:

$$\theta_{pn} = \begin{cases} 1 & \text{if } D_2(pn) > 0, \\ 0 & \text{if } D_2(pn) \leq 0, \end{cases}$$  \(3.9\)

$$\bar{\theta}_{pn} = \begin{cases} 1 & \text{if } D_2(pn) \leq 0, \\ 0 & \text{if } D_2(pn) > 0. \end{cases}$$

Then the operators $\bar{A}_{1\mu}^\dagger, A_{1\mu}^\dagger$ can be renormalized as:

$$\bar{A}_{1\mu}^\dagger(pn) = \frac{1}{\sqrt{D_1(pn)}} A_{1\mu}^\dagger,$$

$$\bar{A}_{1\mu}^\dagger(pn) = \frac{1}{\sqrt{|D_2(pn)|}} A_{1\mu}^\dagger. \tag{3.10}$$

One can certainly introduce new operators having proper boson like normalization:

$$\bar{B}_{1\mu}^\dagger(pn) = \theta_{pn} \bar{A}_{1\mu}^\dagger + \bar{\theta}_{pn} \bar{A}_{1,\mu}(-)^{1-\mu},$$

$$\bar{B}_{1\mu}(pn) = \theta_{pn} A_{1\mu} + \bar{\theta}_{pn} A_{1,\mu}(-)^{1-\mu}, \tag{3.11}$$

The result is that we have two pairs of operators satisfying boson like commutation relations:

$$[\bar{A}_{1\mu}(pn), \bar{A}_{1\mu}^\dagger(p'n')] = \delta_{\mu,\mu'} \delta_{j_p,j_{p'}} \delta_{j_n,j_{n'}},$$

$$[\bar{B}_{1\mu}(pn), \bar{B}_{1\mu}^\dagger(p'n')] = \delta_{\mu,\mu'} \delta_{j_p,j_{p'}} \delta_{j_n,j_{n'}}. \tag{3.12}$$

In this Section we suppose that $D_p$ and $D_n$ are negligible small. Also, we ignore the contribution of the pairing interaction coupling the equations of motion associated to the operators $\bar{A}_{1\mu}^\dagger$ and $A_{1\mu}^\dagger$. Under these circumstances the two boson operators are independent from each other,

$$[\bar{B}_{1\mu}(pn), \bar{A}_{1\mu}^\dagger(p'n')] = 0. \tag{3.13}$$
and consequently the equations of motion for the quoted operators are decoupled.

\[
[H, \hat{A}^{\dagger}_{1\mu}(pn)] = \left[ E_p(U_p^2 - V_p^2) + E_n(V_n^2 - U_n^2) \right] \hat{A}^{\dagger}_{1\mu}(pn) + 2\chi \sum \sigma_{p_{m},n_{1}}^{(1)} \hat{A}^{\dagger}_{1\mu}(p_{1}n_{1}), \quad (3.14)
\]

\[
[H, \hat{A}_{1\mu}(pn)] = \left[ E_p(U_p^2 - V_p^2) + E_n(V_n^2 - U_n^2) \right] \hat{A}_{1\mu}(pn) - 2\chi_1 \varphi_{pn} \sum \sigma_{p_{m},n_{1}}^{(2)} \hat{A}^{\dagger}_{1\mu}(p_{1}n_{1}).
\]

Here the following notations have been used:

\[
\sigma_{p_{m},n_{1}}^{(k)} = \frac{\gamma_p}{1} (p||\sigma||n)|D_k(p, n)|^{1/2} \frac{\gamma_p}{1} (p_{1}||\sigma||n_1)|D_k(p_{1}, n_1)|^{1/2}, \quad k = 1, 2
\]

\[
\varphi_{pn} = D_2(pn) \frac{D_2(pn)}{|D_2(pn)|}. \quad (3.15)
\]

We look for the linear combination operators

\[
\Gamma^{\dagger}_{1\mu} = \sum_{p, n} \chi(pn) \hat{A}^{\dagger}_{1\mu}(pn), \quad \mathcal{G}^{\dagger}_{1\mu} = \sum_{p, n} \chi(pn) \hat{A}^{\dagger}_{1\mu}(pn)
\]

with the properties:

\[
[H, \Gamma^{\dagger}_{1\mu}] = \omega \Gamma^{\dagger}_{1\mu}, \quad [\Gamma_{1\mu}, \Gamma^{\dagger}_{1\mu}'] = \delta_{\mu,\mu'}, \quad [H, \mathcal{G}^{\dagger}_{1\mu}] = \Omega \mathcal{G}^{\dagger}_{1\mu}, \quad [\mathcal{G}_{1\mu}, \mathcal{G}^{\dagger}_{1\mu'}] = \delta_{\mu,\mu'}. \quad (3.17)
\]

These equations provide homogeneous systems of linear equations for the amplitudes \( \chi \) and \( \chi' \), respectively. The compatibility condition yields for \( \omega \) and \( \Omega \) the dispersion equations:

\[
2\chi \sum_{p, n} \frac{2\gamma_p}{3} \frac{(p || \sigma || n)^2}{E_p(U_p^2 - V_p^2) + E_n(V_n^2 - U_n^2) - \omega} D_1(pn) + 1 = 0,
\]

\[
2\chi_1 \sum_{p, n} \frac{2\gamma_p}{3} \frac{(p || \sigma || n)^2}{E_p(U_p^2 - V_p^2) + E_n(U_n^2 - V_n^2) - \Omega} D_2(pn) - 1 = 0. \quad (3.18)
\]

The distinct feature of these equations consists of that the poles of the functions involved in the l.h.s. of Eq. (3.18) are simple. Thus, while in the standard pnQRPA the dispersion equation depends on the squared energy, here the dependence is simply on energy. The important consequence is that the equations derived above do not have vanishing solutions.

The states created by acting with the phonon operators \( \Gamma^{\dagger} \) and \( \mathcal{G}^{\dagger} \) on the ground state of an \((Z,N)\) system are dipole states \( 1^+ \) in the neighboring odd-odd \((Z+1,N-1)\) and \((Z+1,N+1)\) nuclei. While the first state has a particle-hole \((ph)\) character, the second one is a particle-particle \((pp)\) like excitation. The phonon amplitudes have the expressions:

\[
\chi(pn) = -2\chi \frac{\frac{\gamma_p}{1} (p || \sigma || n)}{E_p(U_p^2 - V_p^2) + E_n(V_n^2 - U_n^2) - \omega} S,
\]

\[
\chi(pn) = 2\chi_1 \frac{\varphi_{pn}}{E_p(U_p^2 - V_p^2) + E_n(U_n^2 - V_n^2) - \Omega} S. \quad (3.19)
\]
The factors $S$ and $S$ are determined by the second equation from (3.17), requiring that the phonon operators are normalized to unity. The final results are:

$$S^{-1} = 2\chi \left[ \sum_{p,n} \frac{2^{j_p+1} (\langle p || \sigma || n \rangle)^2 D_1(pn)}{\left( E_p(U_p^2 - V_p^2) + E_n(V_n^2 - U_n^2) - \omega \right)^2} \right]^{1/2},$$

$$S^{-1} = 2\chi_1 \left[ \sum_{p,n} \frac{2^{j_p+1} (\langle p || \sigma || n \rangle)^2 D_2(pn)}{\left( E_p(U_p^2 - V_p^2) + E_n(U_n^2 - V_n^2) - \Omega \right)^2} \right]^{1/2}. \quad (3.20)$$

Note that both the dispersion equations and the phonon amplitudes depend on the renormalization factors $D$, which at their turn depend on the quoted amplitudes. Therefore, the FRpnQRPA equations and the defining equation (3.7) should be solved at a time. In order to do that we have to provide a recipe for how to calculate the averages for the quasiparticle number operators involved in Eq. (3.7). In what follows, we shall describe separately the results for the phonon operators $\Gamma^{\dagger}$ and $G^{\dagger}$. As we proceeded in Ref. [11], we seek for a boson representation of the quasiparticle number operators, determined so that the their commutation relations with the phonon operators are preserved.

$$\hat{N}_p = \sum_{k,\mu} C_k(p) \Gamma_{1\mu}(k) \Gamma_{1\mu}^{\dagger}(k),$$

$$\hat{N}_n = \sum_{k,\mu} C_k(n) \Gamma_{1\mu}(k) \Gamma_{1\mu}^{\dagger}(k). \quad (3.21)$$

The expansion coefficients have the following expressions:

$$C_k(\tau) = -\langle 0 | \left[ [\hat{N}_\tau, \Gamma_{1\mu}(k)], \Gamma_{1\mu}^{\dagger}(k) \right] | 0 \rangle, \quad \tau = p, n. \quad (3.22)$$

Here $| 0 \rangle$ stands for the FRpnQRPA vacuum. Calculating the commutators involved in the above equations, one finds:

$$\langle \hat{N}_p \rangle = \sum_k C_k(p)$$

$$= \sum_{k,n} \frac{|X_k(pn)|^2}{D_1(pn)} \left( U_p^2 V_n^2 + U_n^2 V_p^2 + \frac{\langle \hat{N}_n \rangle}{j_n^2} (U_n^2 - V_n^2)(U_p^2 - V_p^2) - \frac{\langle \hat{N}_p \rangle}{j_p^2} \right),$$

$$\langle \hat{N}_n \rangle = \sum_k C_k(n)$$

$$= -\sum_{k,p} \frac{|X_k(pn)|^2}{D_1(pn)} \left( U_p^2 V_n^2 + U_n^2 V_p^2 + \frac{\langle \hat{N}_n \rangle}{j_n^2} (U_n^2 - V_n^2)(U_p^2 - V_p^2) - \frac{\langle \hat{N}_p \rangle}{j_p^2} \right). \quad (3.23)$$

These equations allow to express the averages $\langle \hat{N}_p \rangle$ and $\langle \hat{N}_n \rangle$ as functions of $D_1(pn)$. Inserting the results in (3.7), a set of equations for the renormalization factors is obtained. Since
the averaged quasiparticle numbers depend on the $X$ amplitudes, so do the renormalization factors $D_1(pn)$.

As for the deuteron like phonon operator the results are:

$$\hat{N}_p = \sum_{k,\mu} F_k(p) \mathcal{G}_1\mu(k) \mathcal{G}^\dagger_1\mu(k),$$

$$\hat{N}_n = \sum_{k,\mu} F_k(n) \mathcal{G}_1\mu(k) \mathcal{G}^\dagger_1\mu(k),$$

(3.24)

where

$$F_k(\tau) = -\langle 0 | \left[ [\hat{N}_{\tau}, \mathcal{G}_1\mu(k)], \mathcal{G}^\dagger_1\mu(k) \right] | 0 \rangle, \quad \tau = p, n.$$  

(3.25)

Using the final results for the coefficients $F$, one obtains:

$$\langle \hat{N}_p \rangle = \sum_k F_k(p)$$

$$= \sum_{k,n} \frac{|X_k(pm)|^2 \phi(pm)}{D_2(pm)} \left( U_p^2 U_n^2 + V_p^2 V_n^2 - \frac{\langle \hat{N}_n \rangle}{j_n} (U_n^2 - V_n^2) (U_p^2 - V_p^2) - \frac{\langle \hat{N}_p \rangle}{j_p} \right),$$

$$\langle \hat{N}_n \rangle = \sum_k F_k(n)$$

$$= -\sum_{k,p} \frac{|X_k(pm)|^2 \phi(pm)}{D_2(pm)} \left( U_p^2 U_n^2 + V_p^2 V_n^2 - \frac{\langle \hat{N}_n \rangle}{j_n} (U_n^2 - V_n^2) (U_p^2 - V_p^2) - \frac{\langle \hat{N}_p \rangle}{j_p} \right).$$

(3.26)

These can be viewed as a system of nonlinear equations for $\langle \hat{N}_p \rangle$ and $\langle \hat{N}_n \rangle$ for a given set of amplitudes $X$. Solving these equations and inserting the solutions in Eq.(3.7) one obtains the values of $D_2(pm)$. An easier way would be to express $\langle \hat{N}_p \rangle$ and $\langle \hat{N}_n \rangle$ as function of $D_2(pm)$ and then Eq.(3.7) becomes a nonlinear equation for $D_2(pm)$ that should be solved at a time with the equations (3.18) and (3.19).

Consider, again, the equations associated to the operators $A$ and $A$. The quantities $D_1(pn)$ should fulfill an additional consistency equation caused by the fact that the phonon operator commutes with the quasiparticle total number operator. Requiring that also $T_3$ is preserved in the average, one obtains that the numbers of protons and neutrons are separately preserved. The average values of the proton and neutron number operators corresponding to the renormalized pnQRPA ground state have the expressions:

$$Z = \sum_p \frac{j_p^2}{j_p^2} \left( U_p^2 V_p^2 - \frac{j_p^2}{j_n^2} \langle \hat{N}_p \rangle \right),$$

$$N = \sum_p \frac{j_n^2}{j_n^2} \left( U_n^2 V_n^2 - \frac{j_n^2}{j_p^2} \langle \hat{N}_n \rangle \right),$$

(3.27)
These equations suggest that in order to have a pnQRPA ground state with non-vanishing quasiparticle numbers and with good proton and neutron numbers it is necessary to renormalize the BCS equations for chemical potentials. In this way the BCS and fully FRpnQRPA equations are coupled to each other. It should be mentioned that the above equations have been derived by assuming that the averages of the quasiparticle pair operators are vanishing. There is still an alternative option, namely to keep the standard BCS equations as they are but enforce the consistency restrictions:

\[
\sum_p U_p V_p \hat{\jmath}_p \langle [a^\dagger_p a^\dagger_{\tilde{p}}]_{00} + [a_p a_p]_{00} \rangle = \sum_p (V_p^2 - U_p^2) \langle \hat{N}_p \rangle,
\]

\[
\sum_n U_n V_n \hat{\jmath}_n \langle [a^\dagger_n a^\dagger_{\tilde{n}}]_{00} + [a_n a_n]_{00} \rangle = \sum_n (V_n^2 - U_n^2) \langle \hat{N}_n \rangle,
\]

(3.28)

As we shall see a bit later, following the second path sketched before, the ISR is violated.

Despite the fact the phonon operator defined before is of a Tamman-Dancoff type, it might be viewed as a FRpnQRPA phonon operator acting in the extended 4\(N_s\) dimensional space. Therefore, the present results reproduce a set of \(N_s\) solutions, with \(N_s\) denoting the number of proton-neutron states (p,n) which can couple at an angular momentum equal to unity, out of 2\(N_s\) positive solutions provided by FRpnQRPA.

Indeed, denoting by

\[
D_A(pn) = 1 - \frac{1}{2j_n+1} \langle \hat{N}_n \rangle - \frac{1}{2j_p+1} \langle \hat{N}_p \rangle,
\]

\[
D_B(pn) = \frac{1}{2j_n+1} \langle \hat{N}_n \rangle - \frac{1}{2j_p+1} \langle \hat{N}_p \rangle,
\]

(3.29)

the phonon operator can be expressed in terms of the renormalized operators:

\[
\tilde{A}_{1\mu}^\dagger(pn) = \frac{1}{\sqrt{|D_A(pn)|}} A_{1\mu}^\dagger(pn), \quad \tilde{B}_{1\mu}^\dagger(pn) = \frac{1}{\sqrt{|D_B(pn)|}} B_{1\mu}^\dagger(pn).
\]

(3.30)

\[
\Gamma_{1\mu}^\dagger = \sum_{p,n} X(pn) D_A^{1/2}(pn) \left( U_p V_p D_A^{1/2} \tilde{A}_{1\mu}^\dagger(pn) + U_n V_n D_A^{1/2} \tilde{A}_{1,-\mu}(pn)(-1)^{1-\mu} \right) + U_p U_n |D_B|^{1/2} \tilde{B}_{1\mu}^\dagger(pn) - V_n V_p |D_B|^{1/2} \tilde{B}_{1,-\mu}(pn)(-1)^{1-\mu} \]

\[
\equiv \sum \left[ X(pn) \tilde{A}_{1\mu}^\dagger(pn) - Y(pn) \tilde{A}_{1,-\mu}(pn)(-1)^{1-\mu} + Z(pn) \tilde{B}_{1\mu}^\dagger(pn) - W(pn) \tilde{B}_{1,-\mu}(pn)(-1)^{1-\mu} \right].
\]

(3.31)

The norm for the extended phonon is:

\[
\sum_{p,n} \left( X^2(pn) - Y^2(pn) + Z^2(pn) - W^2(pn) \right)
\]
\[
\sum_{p,n} X^2(pn) \frac{1}{D_1(pn)} \left[ (U^2_{p} V^2_{n} - U^2_{n} V^2_{p}) \left( 1 - \frac{\langle \hat{N}_n \rangle}{j^2_n} - \frac{\langle \hat{N}_p \rangle}{j^2_p} \right) + (U^2_{p} U^2_{n} - V^2_{p} V^2_{n}) \left( \frac{\langle \hat{N}_n \rangle}{j^2_n} - \frac{\langle \hat{N}_p \rangle}{j^2_p} \right) \right]
\]

\[
= \sum_{p,n} X^2(pn) \frac{1}{D_1(pn)} \left[ U^2_{p} - V^2_{n} + (U^2_{n} - V^2_{p}) \frac{\langle \hat{N}_n \rangle}{j^2_n} - (U^2_{p} - V^2_{p}) \frac{\langle \hat{N}_p \rangle}{j^2_p} \right] = \sum_{p,n} X^2(pn) = 1. \quad (3.32)
\]

Thus, the Tamm-Dancoff phonon operator can be embedded in the operator space acting on the fully renormalized pnQRPA states. Reciprocally, Eq. (3.32) tells us how to obtain the Tamm-Dancoff phonon amplitudes as functions of the fully renormalized pnQRPA phonon amplitudes.

Note that, although the operator \( A \) is a particle hole operator in particle representation, the Tamm-Dancoff equations accounts for quasiparticle correlations and therefore describe the system’s small oscillations around a static BCS ground state.

Let us check now whether the Ikeda sum rule is obeyed by the ground state corresponding to the Tamm-Dancoff phonon \( \Gamma^\dagger \). ISR is generated by the identity:

\[
\sum_{\mu} \left[ \beta^\dagger_{-\mu}, \beta^-_{\mu} \right] (-)^{\mu} = 3 \left[ \hat{N}_n - \hat{N}_p \right], \quad (3.33)
\]

where \( \hat{N}_\tau (\tau = p, n) \) denotes the \( \tau \)-particle number operator. Averaging this equation on the renormalized ground state and then inserting between the two operators \( \beta^\pm \) the unity operator, one obtains:

\[
S_I = \sum_{\mu,k} \langle 0 | \beta^\dagger_{-\mu} | 1_k \rangle \langle 1_k | \beta^-_{\mu} | 0 \rangle
\]

\[
= \sum_{p,n,p',n':\mu} \frac{\hat{j}_{p}}{1} \langle j_{p} | | j_{n} \rangle \langle 0 | A_{\mu}(pn) | 1_k \rangle \langle 1_k | A_{\mu}^\dagger (p'n') | 0 \rangle \frac{\hat{j}_{p'}}{1} \langle j_{p'} | | j_{n'} \rangle \]

\[
= \sum_{p,n,p',n':k} \frac{\hat{j}_{p}}{1} \langle j_{p} | | j_{n} \rangle \frac{\hat{j}_{p'}}{1} \langle j_{p'} | | j_{n'} \rangle X_k(pn) X_k(p'n')3
\]

\[
= \sum_{p,n} (2j_{p} + 1) \langle j_{p} | | j_{n} \rangle^2 D_1(pn). \quad (3.34)
\]

Here the index \( k \) is an ordering label for the roots of the dispersion equation (3.18). Note that \( D_1(pn) \) is a sum of terms depending exclusively on \( p \) and \( n \) respectively, and moreover

\[
\sum_{p} |\langle j_{p} | | j_{n} \rangle|^2 = 3,
\]

\[
\sum_{n} |\langle j_{n} | | j_{p} \rangle|^2 = 3. \quad (3.35)
\]
Taking now into account the result from Eq. (3.27) one obtains:

\[ S_I = 3(N - Z). \]  \hspace{1cm} (3.36)

We may conclude that within the present formalism the Ikeda sum rule is satisfied. A peculiar feature of the present result is that ISR is satisfied although the strength of the \( \beta^+ \) transition is vanishing. Therefore, projecting out the components of good particle total number from the phonon operator, one obtains a phonon operator which is formally of Tamm-Dancoff type and describes a subset of \( pn \) excitations obtainable due to the unprojected operator. Moreover, ignoring the terms coupling the equations of motion of the particle total number conserving operators with those of operators non-conserving the particle total number, the two body \( pp \) interaction is ruled out. We have seen that assuming a consistent decoupling scheme, two independent excitations appear, one describing the \((N-1,Z+1)\) system while the other one the \((N+1,Z+1)\) nucleus. The first excitations are suitable for describing the rates of the single \( \beta^- \) process while the other type of states are deuteron like states and might be populated in a transfer reaction process. Similar formalism can be immediately written down for the states in the neighboring \((N+1,Z-1)\), \((N-1,Z-1)\) nuclei obtained in a \( \beta^+ \) process and by removing a dipole deuteron from the mother nucleus, respectively.

IV. A POSSIBLE IMPROVEMENT OF THE FRPNQRPA APPROACH

In the previous section we have assumed that the commutators of the operators \( A^{\dagger} \) and \( A^{\dagger} \) are negligible small. Let us now investigate whether it is possible to define a linear combination

\[ C^{\dagger}_{1\mu} = X(k)A^{\dagger}_{1\mu}(k) + Y(k)A_{1\mu}(k)(-1)^{1-\mu}, \]  \hspace{1cm} (4.1)

which commute with both \( A^{\dagger}_{1\mu} \) and \( A \), irrespective of \( D_p \) and \( D_n \) magnitudes. One finds out that a necessary condition is that \( D_p(k) = D_n(k) \) which results in having either \( X(k) = Y(k) \) or \( X(k) = -Y(k) \). A similar result is obtained for a linear combination of \( A^{\dagger} \) and \( A \) required to commute with \( A^{\dagger} \) and \( A \). One may conclude that, rigorously speaking, it is not possible to define non-spurious \( pnQRPA \) phonon operators as linear superposition of \( A^{\dagger} \) and \( A \) which are fully decoupled of the phonon operators built up with \( A^{\dagger} \) and \( A \). In this context one
may state that the linear combination of $A^\dagger$ and $A$ used as phonon operator, in Ref. [15], which commutes with the particle total number operator is not justified.

However, it can be checked that there is a linear combination operator

$$\mathcal{R}_{1\mu}^\dagger (pn) = a A_{1\mu}^\dagger (pn) + b A_{1,-\mu} (pn) - 1 - \mu (pn) + c A_{1\mu}^\dagger (pn) + z A_{1,-\mu} (pn).$$

(4.2)

corresponding to a specific set of coefficients

$$a = \frac{U_n^2 - V_n^2}{2U_n V_n (U_p^2 - U_n^2)}, \quad b = \frac{U_p^2 - V_p^2}{2U_p V_p (U_p^2 - U_n^2)}, \quad c = \frac{1}{U_p^2 - U_n^2},$$

(4.3)

$$z = \frac{1}{2D_1 (U_p^2 - U_n^2)} \left( \frac{U_n^2 - V_n^2}{U_n V_n} D_n - \frac{U_p^2 - V_p^2}{U_p V_p} D_p \right),$$

which commute with $A_{1\mu}^\dagger$ and $A_{1,-\mu} (pn)$. Note the the new proton-neutron operators can also be renormalized,

$$\mathcal{R}_{1\mu}^\dagger (pn) = \frac{1}{\sqrt{|D_3 (pn)|}} \mathcal{R}_{1\mu}^\dagger (pn) \mathcal{R}_{1,-\mu} (pn) = \frac{1}{\sqrt{|D_3 (pn)|}} \mathcal{R}_{1,-\mu} (pn) (pn).$$

(4.4)

where

$$D_3 (pn) = (c^2 - z^2) D_1 (pn) + (a^2 - b^2) D_2 (pn) + 2 (az - bc) D_p + 2 (ac - bz) D_n,$$

(4.5)

so that the new operators satisfy boson like commutation relations:

$$[\mathcal{R}_{1\mu}^\dagger (pn), \mathcal{R}_{1\mu}^\dagger (pn') \mathcal{R}_{1,-\mu} (pn) \mathcal{R}_{1,-\mu} (pn') = \delta_{p,n} \delta_{n,n'} \delta_{\mu,\mu'}.$$ (4.6)

After some laborious, otherwise elementary, calculations one finds the equations of motion for the renormalized operators $A^\dagger, A_{1,-\mu} (pn); R_{1\mu}^\dagger, R_{1,-\mu} (pn)$:

$$[H, A_{1\mu}^\dagger (p n)] = T_{11}(p n) A_{1\mu}^\dagger (p n) + T_{12}(p n) A_{1,-\mu} (p n) + T_{13}(p n) \mathcal{R}_{1\mu}^\dagger (p n) + T_{14}(p n) \mathcal{R}_{1,-\mu} (p n),$$

$$[H, A_{1,-\mu} (p n)] = T_{21}(p n) A_{1\mu}^\dagger (p n) + T_{22}(p n) A_{1,-\mu} (p n) + T_{23}(p n) \mathcal{R}_{1\mu}^\dagger (p n) + T_{24}(p n) \mathcal{R}_{1,-\mu} (p n),$$

$$[H, \mathcal{R}_{1\mu}^\dagger (p n)] = T_{31}(p n) A_{1\mu}^\dagger (p n) + T_{32}(p n) A_{1,-\mu} (p n) + T_{33}(p n) \mathcal{R}_{1\mu}^\dagger (p n) + T_{34}(p n) \mathcal{R}_{1,-\mu} (p n),$$

$$[H, \mathcal{R}_{1,-\mu} (p n)] = T_{41}(p n) A_{1\mu}^\dagger (p n) + T_{42}(p n) A_{1,-\mu} (p n) + T_{43}(p n) \mathcal{R}_{1\mu}^\dagger (p n) + T_{44}(p n) \mathcal{R}_{1,-\mu} (p n).$$ (4.7)
Analytical expressions for the coefficients $T_{ik}$ involved in Eq. (4.7) are given in Appendix B. At this stage a phonon operator can be defined:

$$\Gamma^\dagger = \sum_{k=(p,n)} \left[ X(k)\bar{A}_{1,\mu}^\dagger(k) + Z(k)\bar{R}_{1\mu}^\dagger(k) - Y(k)\bar{A}_{1,-\mu}(k)(-1)^{1-\mu} - W(k)\bar{R}_{1,-\mu}(k)(-1)^{1-\mu} \right],$$

(4.8)

by requiring that the following equations are obeyed:

$$[H, \Gamma^\dagger_{1\mu}] = \omega \Gamma^\dagger_{1\mu}, \quad [\Gamma_{1\mu}, \Gamma^\dagger_{1\mu'}] = \delta_{\mu\mu'}. \quad (4.9)$$

The above operator equations provide a system of linear and homogeneous equations for the phonon amplitudes:

$$\begin{pmatrix} A & B \\ -B^* & -A^* \end{pmatrix} \begin{pmatrix} X \\ Z \\ Y \\ W \end{pmatrix} = \omega \begin{pmatrix} X \\ Z \\ Y \\ W \end{pmatrix}, \quad (4.10)$$

where the sub-matrices $A$ and $B$ have the following expressions in terms of the matrices $T$ involved in the equations of motion written above:

$$A = \begin{pmatrix} \bar{T}_{11} \\ \bar{T}_{13} \\ \bar{T}_{33} \end{pmatrix}, \quad (4.11)$$

$$B = \begin{pmatrix} \bar{T}_{12} \\ \bar{T}_{14} \\ \bar{T}_{32} \\ \bar{T}_{34} \end{pmatrix}. \quad (4.12)$$

Here $\bar{T}_{ik}$ denotes the transposed matrix of $T_{ik}$:

$$\bar{T}_{ik}(p_1n_1, pn) = T_{ik}(pn, p_1n_1). \quad (4.13)$$

The second equation (4.9) yields the normalization equation for the phonon amplitudes:

$$\sum_k \left[ |X(k)|^2 + |Z(k)|^2 - |Y(k)|^2 - |W(k)|^2 \right] = 1. \quad (4.14)$$

The results for the normalization factor $D_3(pn)$ are as follows. First one determines the boson expansions:

$$\hat{N}_p = \sum_{k,\mu} G_k(p)\Gamma_{1\mu}(k)\Gamma^\dagger_{1\mu}(k),$$

$$\hat{N}_n = \sum_{k,\mu} G_k(n)\Gamma_{1\mu}(k)\Gamma^\dagger_{1\mu}(k), \quad (4.15)$$
The final result for the average values are:

\[
\langle \hat{N}_p \rangle = \sum_{n,k} \left[ (\bar{X}_k^2(pn) + \bar{Y}_k^2(pn)) + \frac{2\langle \hat{N}_n \rangle}{j^2} (\bar{Z}_k^2(pn) + \bar{W}_k^2(pn) - \bar{X}_k^2(pn) - \bar{Y}_k^2(pn)) \right] \\
- \frac{\langle \hat{N}_p \rangle}{j^2} \left( \bar{Z}_k^2(pn) + \bar{W}_k^2(pn) + \bar{X}_k^2(pn) + \bar{Y}_k^2(pn) \right), \\
\langle \hat{N}_n \rangle = -\sum_{p,k} \left[ (\bar{X}_k^2(pn) + \bar{Y}_k^2(pn)) + \frac{2\langle \hat{N}_n \rangle}{j^2} (\bar{Z}_k^2(pn) + \bar{W}_k^2(pn) - \bar{X}_k^2(pn) - \bar{Y}_k^2(pn)) \right] \\
- \frac{\langle \hat{N}_p \rangle}{j^2} \left( \bar{Z}_k^2(pn) + \bar{W}_k^2(pn) + \bar{X}_k^2(pn) + \bar{Y}_k^2(pn) \right). 
\]  

(4.17)

\( \bar{X}, \bar{Y}, \bar{Z}, \bar{W} \) are the amplitudes of the phonon operator \( \Gamma^\dagger \) when it is expressed in terms of the primary operators \( A^\dagger, A, B^\dagger, B \). Their analytical expressions are given in Appendix C.

From Eq. (4.17) one determines the averages \( \langle \hat{N}_p \rangle \) and \( \langle \hat{N}_n \rangle \) and then by means of (3.7) and (4.5) the expressions of \( D_1(pn) \) and \( D_3(pn) \) are readily obtained.

Following a similar procedure as in the previous section, the Ikeda sum rule can be easily calculated. The result is:

\[
S_I = \sum_{p,n,p',n'} \hat{\gamma}_p \hat{\gamma}_{p'} \langle j_p | j_n \rangle \langle j_{p'} | j_{n'} \rangle D_1^{1/2}(pn)D_1^{1/2}(p'n') (X_k(pn)X_k(p'n') - Y_k(pn)Y_k(p'n')), 
\]  

(4.18)

where the low index \( k \) distinguishes the FRpnQRPA solutions.

If the components \( Z \) and \( W \) of the FRpnQRPA mode are ignored then the summation over \( k \) in the above equation would give \( \delta_{pp'}\delta_{nn'} \) and

\[
S_I = \sum_p (2j_p + 1) \langle j_p | \sigma || j_n \rangle^2 D_1(pn). 
\]  

(4.19)

From this stage on one may follow the path sketched in the previous Section to prove that

\[
S_I = 3(N - Z). 
\]  

(4.20)

However such a picture is not fully consistent. Indeed, ignoring the amplitudes \( Z \) and \( W \) means to neglect the \( T_{13}, T_{14}, T_{23}, T_{24} \) terms from equation of motion. To be consistent, one has to neglect also the \( \chi_1 \) terms form \( T_{11} \) and \( T_{12} \) which would lead to a phonon operator to which the two body \( pp \) interaction does not contribute at all. Moreover, recall the fact that
The commutator $[A^\dagger, A]$ was neglected arguing that factors like $U_p V_p$ and $U_n V_n$ are small. Based on similar arguments, the $UV$ terms from $T_{11}$ and $T_{12}$ have also to be left out. Then one arrives at the Tamm-Dancoff structure for the phonon operator as we obtained in the previous Section.

Coming back to Ikeda sum rule, it is obvious that the summation over $k$ of

$$(X_k(pn)X_k(p'n') - Y_k(pn)Y_k(p'n'))$$

is an under-unity but close to unity quantity for $p = p'$ and $n = n'$, and close to zero when $(p', n') \neq (pn)$. It results that ISR is slightly underestimated by the fully renormalized and consistent pnQRPA approach.

V. SUMMARY AND CONCLUSIONS

In the previous sections we analyzed two distinct ways of rearranging the two quasiparticle and quasiparticle density dipole operators inside the FRpmQRPA phonon operators, determined by the decoupling hypothesis adopted. In the first scheme we supposed that the operators $A^\dagger$ and $A$ commuting with the particle total number operator, are fully decoupled from those which miss this property, i.e. $A^\dagger, A$. Also, we neglected the contribution to the equations of motions for the above quoted operators, generated by the pairing interaction which are comparable small with the small terms mentioned before. The result is that one obtains two decoupled sets of equations of motion, both of Tamm-Dancoff type. One describes a particle-hole dipole excitation in the $(N-1,Z+1)$ nucleus while the other one is associated to a deuteron dipole excitation in the neighboring $(N+1,Z+1)$ nucleus. For the first case the ISR is fully satisfied provided the BCS equations are renormalized as well [18, 19, 20]. It is proved that, rigorously speaking, to define a phonon operator mixing both $A^\dagger$ and $A$ and keeping at a time the decoupling picture of the operators breaking the gauge symmetry associated with the particle total number, is not possible. However it is possible to define a linear combination of $A^\dagger, A, A^\dagger$ and $A$ denoted by $R^\dagger$ which commutes simultaneously with $A^\dagger$ and $A$. Finally, the phonon operator is built up based on the independent operators $A^\dagger, A, R^\dagger$ and $R$. We should mention that the original FRpnQRPA is lacking this feature. Due to the fact that in the previous publication the effects generated by non-commutativity of the component operators of the FRpnQRPA phonon operators were thrown out, the results of the present paper constitute a step forward for the renormalization formalisms. It is concluded that the ISR is underestimated by the general scheme, but has
the advantage that keeps the standard BCS unmodified. One hopes, however, that adding a boson expansion (BE) on the top of the FRpnQRPA formalism the ISR will be further improved. This expectation is supported by the results of a previous work \cite{21} where a BE was performed in terms of the partially renormalized phonon operators (where the scattering terms are ignored). The result is determined by the fact that while the renormalized pnQRPA underestimates the ISR, the BE has an opposite effect.

VI. APPENDIX A

Here we list the exact expressions for the commutation relations of the operators $A_{1\mu}^\dagger (pn), A_{1\mu} (pn), A_{1\mu}^\dagger (pn), A_{1\mu} (pn):$

$$[A_{1\mu} (pn), A_{1\mu}^\dagger (p'n')] = \sum \hat{f} C_{\mu}^{1} f_{m, \mu'}^1 (-)^f W(1_{j, p} f_{j,n'}; j, n) [c_{n'} c_{n}]_{F_{m, f}} \delta_{p,p'},$$

$$[A_{1\mu}^\dagger (pn), A_{1, -\mu} (p'n')(-)^{1-\mu'}] = \sum 3(-)^f C_{\mu}^{1} f_{-m, \mu'}^1 W(j, n, p f_{j, p'}; 1_{p}) [c_{p'} c_{p}]_{f, -m, f} \delta_{n,n'},$$

$$[A_{1\mu} (pn), A_{1\mu}^\dagger (p'n')] = \sum \hat{f} C_{\mu}^{1} f_{m, \mu'}^1 (-)^{f+1} W(1_{j, p} f_{j,n'}; j, n) [c_{n'} c_{n}]_{F_{m, f}} \delta_{p,p'} + W(1_{j, f} f_{j,n'}; j, n) [c_{p'} c_{p}]_{F_{m, f}} \delta_{n,n'},$$

$$[A_{1\mu} (pn), A_{1\mu}^\dagger (p'n')] = \sum \hat{f} C_{\mu}^{1} f_{m, \mu'}^1 (-)^{f+j_{n'}+1} W(1_{j, p} f_{j,n'}; j, n) [c_{n'} c_{n}]_{F_{m, f}} \delta_{p,p'} - W(1_{j, f} f_{j,n'}; j, n) [c_{p'} c_{p}]_{F_{m, f}} \delta_{n,n'}. \quad (6.1)$$

VII. APPENDIX B

Here we give the explicit expressions for the coefficients $T_{ik}$ involved in Eq.4.7

$$T_{11}(p_1 n_1; pn) = \left[ E_{p_1} \left( U_{p_1}^2 - V_{p_1}^2 - 2U_{p_1} V_{p_1} \frac{b_1 c_1 - a_1 z_1}{a_1^2 - b_1^2} \right) E_{n_1} \left( U_{n_1}^2 - V_{n_1}^2 - 2U_{n_1} V_{n_1} \frac{b_1 z_1 - a_1 c_1}{a_1^2 - b_1^2} \right) \right] \times \delta_{p,p_1} \delta_{n,n_1} + 2\sigma^{(1)}_{p_1 n_1; pn} \left[ \chi - \chi_1 \sigma^{(1)}_{p_1 n_1; pn} \frac{(b_1 z_1 - a_1 c_1)(bc - az) + (b_1 c_1 - a_1 z_1)(bc - az)}{(a_1^2 - b_1^2)(a_2^2 - b_2^2)} \right],$$

$$T_{12}(p_1 n_1; pn) = \left[ -2E_{p_1} U_{p_1} V_{p_1} \frac{b_1 z_1 - a_1 c_1}{a_1^2 - b_1^2} + 2E_{n_1} U_{n_1} V_{n_1} \frac{b_1 c_1 - a_1 z_1}{a_1^2 - b_1^2} \right] \delta_{p,p_1} \delta_{n,n_1} - 2\chi_1 \sigma^{(1)}_{p_1 n_1; pn} \frac{(b_1 z_1 - a_1 c_1)(bc - az) + (b_1 c_1 - a_1 z_1)(bc - az)}{(a_1^2 - b_1^2)(a_2^2 - b_2^2)},$$

$$T_{13}(p_1 n_1; pn) = -2\chi_1 \sigma^{(1)}_{p_1 n_1; pn} \frac{(b_1 z_1 - a_1 c_1)a - (b_1 c_1 - a_1 z_1)b}{(a_1^2 - b_1^2)(a_2^2 - b_2^2)}. \quad 21$$
\begin{align}
T_{14}(p_1 n_1; p n) &= -2 \chi_1 \sigma_{p_1 n_1; p n}^{(13)} \frac{-(b_1 z_1 - a_1 c_1)b + (b_1 c_1 - a_1 z_1)a}{(a^2 - b^2)(a_1^2 - b_1^2)}, \\
T_{21}(p_1 n_1, p n) &= -T_{12}^*(p_1 n_1, p n), \\
T_{22}(p_1 n_1, p n) &= -T_{11}^*(p_1 n_1, p n), \\
T_{23}(p_1 n_1, p n) &= -T_{14}^*(p_1 n_1, p n), \\
T_{24}(p_1 n_1, p n) &= -T_{13}^*(p_1 n_1, p n), \\
T_{31}(p_1 n_1, p n) &= \frac{|D_3(p_1 n_1)|^{-1/2} D_1^{1/2}(p n)}{a_1^2 - b_1^2} \left[ E_{p_1} \left( \left( U_{p_1}^2 - V_{p_1}^2 \right) c_1 + 2 U_{p_1} V_{p_1} b_1 \right) \left( a_1^2 - b_1^2 \right) \\
&+ (b_1 z_1 - a_1 c_1) \left( U_{p_1}^2 - V_{p_1}^2 \right) a_1 - 2 U_{p_1} V_{p_1} z_1 \right) \right] + (b_1 c_1 - a_1 z_1) \left( U_{p_1}^2 - V_{p_1}^2 \right) b_1 - 2 U_{p_1} V_{p_1} c_1 \right] \\
&+ E_{n_1} \left( (U_{n_1}^2 - V_{n_1}^2) c_1 + 2 U_{n_1} V_{n_1} a_1 \right) \left( a_1^2 - b_1^2 \right) + (b_1 z_1 - a_1 c_1) \left( U_{n_1}^2 - V_{n_1}^2 \right) a_1 + 2 U_{n_1} V_{n_1} c_1 \\
&+ (b_1 c_1 - a_1 z_1) \left( U_{n_1}^2 - V_{n_1}^2 \right) b_1 + 2 U_{n_1} V_{n_1} z_1 \right] \delta_{p_1, p} \delta_{n_1, n} \\
&- 2 \chi_1 \sigma_{p_1 n_1; p n}^{(33)} \frac{(b z - a) a_1 - (a c - a z)b_1}{(a^2 - b^2)(a_1^2 - b_1^2)}, \\
T_{32}(p_1 n_1, p n) &= \frac{|D_3(p_1 n_1)|^{-1/2} D_1^{1/2}(p n)}{a_1^2 - b_1^2} \left[ E_{p_1} \left( \left( U_{p_1}^2 - V_{p_1}^2 \right) z_1 + 2 U_{p_1} V_{p_1} a_1 \right) \left( a_1^2 - b_1^2 \right) \\
&+ (b_1 z_1 - a_1 c_1) \left( U_{p_1}^2 - V_{p_1}^2 \right) b_1 - 2 U_{p_1} V_{p_1} z_1 \right) \right] + (b_1 c_1 - a_1 z_1) \left( U_{p_1}^2 - V_{p_1}^2 \right) a_1 - 2 U_{p_1} V_{p_1} c_1 \right] \\
&+ E_{n_1} \left( (U_{n_1}^2 - V_{n_1}^2) z_1 + 2 U_{n_1} V_{n_1} b_1 \right) \left( a_1^2 - b_1^2 \right) + (b_1 z_1 - a_1 c_1) \left( U_{n_1}^2 - V_{n_1}^2 \right) b_1 + 2 U_{n_1} V_{n_1} z_1 \\
&+ (b_1 c_1 - a_1 z_1) \left( U_{n_1}^2 - V_{n_1}^2 \right) a_1 + 2 U_{n_1} V_{n_1} c_1 \right] \delta_{p_1, p} \delta_{n_1, n} \\
&- 2 \chi_1 \sigma_{p_1 n_1; p n}^{(33)} \frac{(b c - a) a_1 - (b z - a c)b_1}{(a^2 - b^2)(a_1^2 - b_1^2)}, \\
T_{33}(p_1 n_1, p n) &= \frac{\delta_{p_1, p} \delta_{n_1, n}}{a_1^2 - b_1^2} \left[ E_{p_1} \left( \left( U_{p_1}^2 - V_{p_1}^2 \right) (a_1^2 - b_1^2) - 2 U_{p_1} V_{p_1} (a_1 z_1 - b_1 c_1) \right) \\
&+ E_{n_1} \left( \left( U_{n_1}^2 - V_{n_1}^2 \right) (a_1^2 - b_1^2) + 2 U_{n_1} V_{n_1} (a_1 c_1 - b_1 z_1) \right) \right] \\
&- 2 \chi_1 \sigma_{p_1 n_1; p n}^{(33)} \frac{a a_1 + b b_1}{(a^2 - b^2)(a_1^2 - b_1^2)}, \\
T_{34}(p_1 n_1, p n) &= \frac{\delta_{p_1, p} \delta_{n_1, n}}{a_1^2 - b_1^2} \left[ 2 E_{p_1} U_{p_1} V_{p_1} (b_1 z_1 - a_1 c_1) + 2 E_{n_1} U_{n_1} V_{n_1} (a_1 z_1 - b_1 c_1) \right] \\
&+ 2 \chi_1 \sigma_{p_1 n_1; p n}^{(33)} \frac{b a_1 + a b_1}{(a^2 - b^2)(a_1^2 - b_1^2)}, \\
T_{41}(p_1 n_1, p n) &= -T_{32}^*(p_1 n_1, p n), \\
T_{42}(p_1 n_1, p n) &= -T_{31}^*(p_1 n_1, p n), \\
T_{43}(p_1 n_1, p n) &= -T_{34}^*(p_1 n_1, p n), \\
T_{44}(p_1 n_1, p n) &= -T_{33}^*(p_1 n_1, p n). 
\end{align}
The expressions for $a, b, c, z$ have been defined before by Eq.(4.3). The coefficients $a_1, b_1, c_1, z_1$ are obtained from Eq.(4.3) by replacing the indices $p$ and $n$ by $p_1$ and $n_1$, respectively. The index $^*$ stands for the complex conjugation. Several notations have been used:

\[
\sigma^{(1)}(p_1n_1, pn) = D_1^{1/2}(p_1n_1)\sigma_{p_1n_1,pn}D_1^{1/2}(pn), \\
\sigma^{(3)}(p_1n_1, pn) = \epsilon_{p_1n_1}|D_3(p_1n_1)|^{1/2}\sigma_{p_1n_1,pn}|D_3(pn)|^{1/2}, \\
\sigma^{(13)}(p_1n_1, pn) = D_1^{1/2}(p_1n_1)\sigma_{p_1n_1,pn}|D_3(pn)|^{1/2}, \\
\sigma^{(31)}(p_1n_1, pn) = \epsilon_{p_1n_1}|D_3(p_1n_1)|^{1/2}\sigma_{p_1n_1,pn}D_1^{1/2}(pn), \\
\epsilon_{p_1n_1} = \frac{D_3(p_1n_1)}{|D_3(p_1n_1)|}. \tag{7.3}
\]

VIII. APPENDIX C

Here we give the explicit expressions for the bar amplitudes involved in the equations for the quasiparticle number operators averages (4.11). The results are:

\[
\begin{align*}
\bar{X}_k(pn) &= \frac{1}{\sqrt{D_1(pn)}}(U_pV_nX_k(pn) - U_nV_pY_k(pn)) + \frac{1}{\sqrt{|D_3(pn)|}}
\left[Z_k(pn) \left(aU_pU_n - bV_pV_n + cU_pV_n + zU_nV_p\right) - W_k(pn) \left(-aV_pV_n + bU_pU_n + cU_nV_p + zU_pV_n\right)\right], \\
\bar{Y}_k(pn) &= \frac{1}{\sqrt{D_1(pn)}}(U_nV_pX_k(pn) - U_pV_nY_k(pn)) + \frac{1}{\sqrt{|D_3(pn)|}}
\left[Z_k(pn) \left(-aV_pV_n + bU_pU_n + cU_nV_p + zU_pV_n\right) - W_k(pn) \left(aU_pU_n - bV_pV_n + cU_pV_n + zU_nV_p\right)\right], \\
\bar{Z}_k(pn) &= \frac{1}{\sqrt{D_1(pn)}}(U_pU_nX_k(pn) + V_nV_pY_k(pn)) + \frac{1}{\sqrt{|D_3(pn)|}}
\left[Z_k(pn) \left(-aU_pV_n - bV_pU_n + cU_pU_n - zV_nV_p\right) - W_k(pn) \left(-aV_pU_n - bU_pV_n - cV_nV_p + zU_pU_n\right)\right], \\
\bar{W}_k(pn) &= \frac{1}{\sqrt{D_1(pn)}}(\sqrt{p_n}V_pX_k(pn) - U_pU_nY_k(pn)) + \frac{1}{\sqrt{|D_3(pn)|}}
\left[Z_k(pn) \left(-aV_pU_n - bU_pV_n - cV_nV_p + zU_pU_n\right) - W_k(pn) \left(-aU_pV_n - bV_pU_n + cU_pU_n - zV_nV_p\right)\right]. \tag{8.1}
\end{align*}
\]