Codimension-two branes in six-dimensional supergravity and the cosmological constant problem

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We investigate in detail recent suggestions that codimension-two braneworlds in six dimensional supergravity might circumvent Weinberg’s no-go theorem for self-tuning of the cosmological constant. The branes are given finite thickness in order to regularize mild singularities in their vicinity, and we allow them to have an arbitrary equation of state. We study perturbatively the time evolution of the solutions by solving the equations of motion linearized around a static background. Even allowing for the most general possibility of warping and nonconical singularities, the geometry does not relax to a static solution when the brane stress-energies are perturbed. Rather, both the internal and external geometries become time-dependent, and the system does not exhibit any self-tuning behavior.

I. INTRODUCTION

There has been considerable recent interest in the possibility that braneworld constructions give new possibilities for solving the cosmological constant problem [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. One rationale is that intrinsically extra-dimensional effects might provide a loophole to Weinberg’s no-go theorem against self-tuning mechanisms, which is formulated in 4D [13]. Such attempts first arose using 5D models, where solutions could be constructed with the property that, regardless of the 4D cosmological constant (brane tension), the universe was static [1, 2, 3]. Unfortunately it was shown that these solutions had singularities in the bulk which when regularized corresponded to the presence of an additional fine-tuned brane; moreover these solutions had flat directions which would give rise to nonstatic solu-

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tions when the branes were perturbed away from their fine-tuned values. Other attempts involving black holes in the bulk were also plagued by instabilities [14, 15, 16, 17].

The case of a 6D bulk with (codimension-two) 3-branes has proved to be somewhat more subtle [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47]. The conical singularity induced in the 6D Ricci scalar by the codimension-two source exactly cancels the contribution of the delta-function source itself when the 6D action is dimensionally reduced. The contribution of the brane tension to the 4D cosmological constant thus disappears automatically; the 4D effective action does not see it—it is as though it has been fine-tuned out of the theory, but in fact it was the dynamics of 6D general relativity which canceled it. Superficially, this looks exactly like the kind of inherently extra-dimensional effect which might well circumvent Weinberg’s theorem. In practice however, the idea has not yet been shown to work. The problem is that the tension through its gravitational couplings affects other terms in the effective action besides the bare delta function source. In the models that have been studied so far, the sensitivity of these extra terms to the brane tension spoils the hoped-for self-tuning effect that would manifest itself when the tension undergoes a dynamical change, as during a phase transition.

The failure of self-tuning in the simplest model, where the extra dimensions are compactified by the flux of a U(1) gauge field, was demonstrated in references [12] and [41]. In [41], it was also argued that the same observations rule out self-tuning in supergravity versions of the model. We disagree with this point of view, as explained below. Conflicting arguments in the literature hold out the hope that self-tuning could still work in 6D SUGRA models [43, 44], so it is worthwhile to investigate in detail, given the notorious intractability of the cosmological constant problem.

We review the formalism of codimension-two braneworld models in section II, with and without supersymmetry, and the details of arguments for and against self-tuning which have been made for the SUSY case. We also explain why it is useful to consider finite-thickness branes. In section III we set up and solve the leading-order equations of motion for the SUSY case, treating the time-dependent brane sources as perturbations. Section IV focuses on a particular class of warped background solutions which is crucial to settling the self-tuning issue; we demonstrate that these solutions are adequately described by our perturbative treatment so that self-tuning, if present, would be visible within our calculational framework.
In Section V we impose the boundary conditions at the branes, which reduces the general solution of section III to the particular ones of cosmological interest, in the case of time-dependent change of brane tension. We show that the system does not relax back to a static solution, but rather evolves like a universe with nonvanishing cosmological constant—there is no self-tuning. These results are discussed in section VI, and technical details are given in the appendix.

II. REVIEW OF CODIMENSION-TWO BRANEWORLDS

In this section we will briefly review how self-tuning is known to fail in the nonsupersymmetric 6D models, and the issues surrounding whether it can work in SUGRA versions of the models.

A. Einstein-Hilbert gravity

The simplest model which allows for compactification of the two extra dimensions has the action

\[ S = \int d^6x \sqrt{-g} \left[ \frac{M_4^4}{2} R - \frac{1}{4} F_{MN} F^{MN} - \Lambda_6 \right] + S_{\text{branes}}, \]  

(1)

where

\[ S_{\text{branes}} = -\int d^4x d^2y \sqrt{-g} \sum_i \tau_i \frac{\delta^{(2)}(\vec{y} - \vec{y}_i)}{\sqrt{g_2}} \]  

(2)

The interplay between a positive bulk cosmological constant and a gauge field flux \( F_{r\theta} \) gives an energetically preferred size for the compact space. (It should be noted that this type of model was initially studied in the mid 1980’s, before the concept of branes was introduced [48, 49, 50, 51, 52]).

Solutions exist with static extra dimensions and which are maximally symmetric in the large four dimensions, [53]

\[ ds^2 = -dt^2 + e^{2Ht} dx^2 + k^{-2} \left( dr^2 + \sin^2 r d\theta^2 \right) \]

\[ A_\theta = \frac{\beta}{k^2} (1 \pm \cos r) \]  

(3)

with \( k^2 = M_6^{-4}(\Lambda_6/2 + 3\beta^2/4) \), \( H^2 = M_6^{-4}(\Lambda_6/6 - \beta^2/12) \), the coordinates \( r \) and \( \theta \) range
over \((0, \pi)\) and \((0, 2\pi)\) respectively, and where the 6D Ricci scalar has the form
\[
R = \frac{2}{M_6^4} \left( \sum_i \tau_i \delta^{(2)}(\vec{y} - \vec{y}_i) \frac{(\vec{y} - \vec{y}_i)}{\sqrt{g_2}} + 6H^2 + k^2 \right)
\]

Thus, the singular parts of the action cancel exactly when evaluated for the classical solutions, and the effective 4D cosmological constant is determined entirely by the smooth parts:
\[
\Lambda_4 = V_2 \left( \frac{1}{2} \Lambda_6 - \frac{1}{4} \beta^2 \right),
\]

where \(V_2\) is the volume of the compact manifold,
\[
V_2 = 2 (2\pi - \delta) k^{-2},
\]

and \(\delta\) is the deficit angle associated with conical singularities induced by 3-branes, so that in the presence of brane tension, the coordinate \(\theta\) defined above ranges from \(0\) to \(2\pi - \delta\).

It requires a fine tuning of the bulk quantities \(\Lambda_6\) and \(\beta\) to have a vanishing cosmological constant, so the cancellation of the singular parts would not in itself explain the smallness of \(\Lambda_4\). Nevertheless it is suggestive; for example if supersymmetry in the bulk caused (5) to vanish, then SUSY could be broken on the branes without affecting \(\Lambda_4\).

However this seemingly elegant idea does not work, as was explicitly shown in [12]. The reason can be understood intuitively as follows [41]. The simplest version of the model has two branes with equal tension \(\tau\) at antipodal points of the spherical extra dimensions, which produces conical defects and a deficit angle \(\delta = \tau/M_6^4\). The volume of the extra dimensions depends linearly on the deficit angle, eq. (6). The flux of the gauge field in the extra dimensions is proportional to the volume, \(\Phi = F_{\tau\theta} V_2\). This flux is conserved, so if one tries to change the brane tensions, the field strength has to change to compensate for the change in volume. Any such change in \(F_{\tau\theta}\) spoils the tuning needed to make \(\Lambda_4\) vanish in (5). Thus \(\Lambda_4\) is linearly dependent on the brane tensions—not through the delta function terms in the original action, but as a consequence of the conical defect which they produce on the bulk geometry.
B. Supergravity

A supergravity version of the above model involving chiral fields $\Phi_a$ and a dilaton $\phi$ can be derived from the action

$$S = \int d^6 x \sqrt{-g} \left[ \frac{M_4^2}{2} \left( R - \partial_a \phi \partial^a \phi \right) - \frac{1}{4} e^{-\phi} F_{MN} F^{MN} - \frac{1}{2} h(\Phi)_{ab} \partial_a \Phi^a \partial_b \Phi^b - e^\phi v(\Phi) \right] + S_{\text{branes}},$$

(7)

One can consistently assume that the potential $v(\Phi)$ is minimized when the fields $\Phi_a = 0$, and thus ignore these fields and treat $v(\Phi)$ as the bulk cosmological constant, $\Lambda_6$.

Comparing with the nonSUSY model, it is easy to see that there exist solutions with constant $\phi$, where everywhere we have replaced $\Lambda_6 \rightarrow \Lambda_6 e^\phi$ and $\beta^2 \rightarrow \beta^2 e^{-\phi}$, provided that $\phi$ takes the value $\phi_0$ such that its induced potential is stationary,

$$V' (\phi_0) = -\frac{1}{2} \beta^2 e^{-\phi_0} + \Lambda_6 e^{\phi_0} \equiv 0 \quad (8)$$

Comparing to (5), this is precisely the value needed to make the effective cosmological constant vanish,

$$\Lambda_4 \rightarrow V_2 \left( \frac{1}{2} \Lambda_6 e^{\phi_0} - \frac{1}{4} \beta^2 e^{-\phi_0} \right) = 0 \quad (9)$$

This seemingly automatic adjustment of $\Lambda_4$ to zero is one reason that previous authors [9, 31, 36, 42, 43, 44] have been encouraged to seek a self-tuning mechanism in this model.

More generally, there are three classes of solutions for this model whose 4D metric is maximally symmetric [30], and all of them have vanishing vacuum energy. The solution displayed above is the simplest one, which corresponds to putting branes of equal tension at the antipodes of the the compact space. A second class of solutions has branes of unequal tension, in which case the metric no longer factorizes, but instead is warped. The third class of solutions, which are also warped, is more exotic: it has nonconical singularities at the antipodal points whose stress energy does not correspond to simple brane sources. In particular, the $T_{rr}$ and $T_{\theta\theta}$ stress tensor components are nonvanishing for these backgrounds.

1. The argument against self-tuning

In ref. [41], it was argued that, rather than displaying self-tuning, the SUGRA model is a clear example of Weinberg’s no-go theorem. Ref. [41] dimensionally reduces the model with
two equal-tension branes to 4D gravity coupled to the dilaton \( \phi \) and the radion \( \psi \), which enters the metric ansatz in the form

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu + M_6^{-2} e^{-2\psi} (dr^2 + \sin^2 r \, d\theta^2)
\]

(10)

The potential which results for the two scalar fields is found to have the form

\[
V(\psi, \phi) = M_6^{-4} e^{\sigma_2} \left( \frac{\beta^2}{2\alpha^2} e^{-2\sigma_1} - 2M_6^6 e^{-\sigma_1} + 2\Lambda_6 \right)
\]

(11)

where \( \sigma_2 = 2\psi - \phi \), \( \sigma_1 = 2\psi + \phi \), and \( \alpha \) is related to the deficit angle induced by the branes, \( \alpha = 1 - \tau/(2\pi M_6^4) \). This potential looks different from the dilaton potential in (9) because in the latter, the Einstein equations have already been imposed, which fixes \( \psi \) in terms of \( \beta^2 \) and \( \Lambda_6 \), whereas this has not been done in (11). Ref. [41] goes on to observe that, starting from some configuration in which \( V(\psi, \phi) = 0 \), any change in the brane tensions, hence in the value of \( \alpha \), will make \( V(\psi, \phi) \) nonzero, and induce a runaway potential for the field \( \sigma_2 \).

If the 4D potential (11) really captures the essential features of the model, then clearly there is no self-tuning; however this ansatz describes only the unwarped solutions. If self-tuning occurs, it might be through a transition from an unwarped to a warped solution, especially an exotic nonconically singular one, which cannot be described by the ansatz (10).

More generally, the proposal of [9, 31, 36, 42, 43, 44] involves large extra dimensions, with \( M_6^{-1} \sim 0.1 \) mm, where the Kaluza-Klein mass gap is of the same order of magnitude as the mass of the stable direction \( \sigma_1 \) in the potential (11). The argument of [41] has thus been called into question, on the basis that there is no justification for ignoring the role of light KK modes while keeping other modes that are of the same energy in the effective 4D description. This objection is similar in spirit to the previous one, in that mild warping of the background metric can be thought of as a coherent state of KK modes.

2. The argument for self-tuning

One reason for suspecting a self-tuning mechanism in the Salam-Sezgin model is the observation by [30] that the only solutions with maximal symmetry in the 4D part of the metric are those with vanishing curvature in 4D, that is, Minkowski space. These solutions fall into two classes, one of which has only conical singularities (or possibly no singularities).
It has the warped line element and dilaton profile

\[
 ds^2 = \sqrt{\frac{\hat{r}^2 + \hat{r}_1^2}{\hat{r}^2 + \hat{r}_0^2}} \left( dx^2 + \frac{d\hat{r}^2}{(1 + \hat{r}^2/\hat{r}_0^2)^2} + \frac{\hat{r}^2 d\theta^2}{(1 + \hat{r}^2/\hat{r}_1^2)^2} \right)
\]

\[
 \phi(\hat{r}) = \phi_0 + \ln \left( \frac{\hat{r}^2 + \hat{r}_1^2}{\hat{r}^2 + \hat{r}_0^2} \right)
\]

(12)

where

\[
 \hat{r}_0 = \frac{4e^{-\phi_0}}{\Lambda_6}; \quad \hat{r}_1 = \frac{8e^{\phi_0}}{\beta^2}
\]

(13)

In the special case where \( \hat{r}_0^2 = \hat{r}_1^2 \) (i.e., \( \Lambda_6 e^{\phi_0} = \beta^2 e^{-\phi_0}/2 \)), we recover the unwarped spherical football-shaped solutions, which have branes of equal tension at the antipodal points. For \( \hat{r}_0^2 \neq \hat{r}_1^2 \), the tensions of the branes are related to each other by

\[
 \left( 1 - 4\tau_+/M_6^4 \right) \left( 1 - 4\tau_-/M_6^4 \right) = \frac{\hat{r}_1^2}{\hat{r}_0^2} = N^2
\]

(14)

where \( N \) is an integer, related to the quantization of gauge field flux on the compact manifold.

The most general solutions are more complicated; they are given by

\[
 ds^2 = W^2 dx^2 + a^2 \left( W^8 d\eta^2 + d\theta^2 \right)
\]

\[
 \phi(\eta) = \phi_0 + 4 \ln W + 2\lambda_3 \eta
\]

(15)

with

\[
 W^4 = \frac{\lambda_2}{\lambda_1} \sqrt{\frac{e^{\phi_0}\lambda_1}{2\Lambda_6} \cosh \lambda_1 (\eta - \eta_1)} \cosh \lambda_2 (\eta - \eta_2)
\]

\[
 a^{-4} = \sqrt{\frac{\Lambda_6}{8} \beta^3 \lambda_1^{-3} \lambda_2^{-1} e^{-2\lambda_3 \eta} \cosh^3 \lambda_1 (\eta - \eta_1) \cosh \lambda_2 (\eta - \eta_2)}
\]

(16)

and \( \lambda_3 = \sqrt{\lambda_2^2 - \lambda_1^2} \). When \( \lambda_3 = 0 \), this reduces to the special solutions above.

One can see the nonconical nature of the singularities when \( \lambda_3 \neq 0 \) by performing a coordinate redefinition [43]. Writing \( dr = aW^4 d\eta \), in the limit \( \eta \to \pm \infty \)

\[
 ds_2^2 = dr^2 + r^{2\alpha_\pm} d\theta^2
\]

(17)

where

\[
 \alpha_\pm = \frac{\pm 2\lambda_3 + \lambda_2 + 3\lambda_1}{\pm 2\lambda_3 + 5\lambda_2 - \lambda_1}
\]

(18)

As soon as \( \lambda_3 \neq 0 \), one finds that \( \alpha_\pm \neq 1 \) and the metric near the branes no longer corresponds to that of a simple conical singularity. Rather, we find a geometry more like
that of a trumpet, where the circumference of closed circles around the singular points is not linearly proportional to their radius. It is the existence of such solutions which has given hope of circumventing the argument of [41]. Indeed it is not excluded that under a perturbation to the branes’ tensions, the system would relax to a static solution with nonzero $\lambda_3$ rather than move to a runaway solution with purely conical singularities. It is the presence of such behaviour that we will be looking to confirm or negate in the analysis that follows.

C. The need for thick branes

The solutions we have summarized above only concern pure tension branes and are therefore not suited for the analysis we wish to carry out, where the energy density on the branes is time dependent, and its equation of state must be different from $\rho = -p$. In order to study this more general case, it is necessary to regularize the singularity at the brane positions by giving the branes finite thickness. The reasons for this were explained in detail in [12]. We will only recall here the main line of reasoning.

For a metric of the form

$$ds^2 = -n^2(\bar{r}, t)dt^2 + a^2(\bar{r}, t)d\bar{x}^2 + f^2(\bar{r}, t)(d\bar{r}^2 + \bar{r}^2d\theta^2)$$ (19)

codimension-two branes will appear in the stress energy tensor $T^A_B$ as $(\rho, p) \times \delta(\bar{r})/[2\pi f^2(\bar{r}, t)\bar{r}]$. There must be terms in the Einstein tensor that provide the necessary delta functions to match the ones from the stress-energy tensor. Since the two dimensional delta function is given by

$$\delta(\bar{r}) = 2\pi\bar{r}\nabla^2\ln(\bar{r})$$ (20)

the delta function terms in the Einstein tensor will come from terms like $\nabla^2\ln(n)$, $\nabla^2\ln(a)$ and $\nabla^2\ln(f)$. In order to accommodate general equations of state, the first two of these must be nonzero near the branes, which means that the warp factor must scale like $\bar{r}^\alpha(t)$ as $\bar{r} \to 0$. As was done in [12] to treat the Einstein-Hilbert case, we will be smoothing out this mildly singular behaviour by giving the brane finite thickness, allowing us to find well-behaved solutions with arbitrary equations of state, and thus arbitrary time dependence of the stress-energy tensor, on the branes.
III. THICK CODIMENSION-TWO BRANEWORLD IN 6D SUPERGRAVITY

For the reasons that have just been exposed, we need to regularize the branes by giving them nonzero thickness in order to consistently study their cosmology in situations where their stress-energy is not simply constant. We will be following the formalism of [12], where such a construction was used to conclusively rule out the presence of self-tuning in the context of six dimensional Einstein-Hilbert gravity. The only new ingredient here will be the presence of a dilaton field.

We start with the most general dynamical metric having axial symmetry in the extra dimensions

$$ds^2 = -n(r,t)^2 dt^2 + a(r,t)^2 d\vec{x}^2 + b(r,t)^2 dr^2 + c(r,t)^2 d\theta^2 + 2E(r,t) dr dt$$

and perturb its components in the following manner

$$n(r,t) = e^{A_0(r)+N_1(r,t)}; \quad a(r,t) = a_0(t) e^{A_0(r)+A_1(r,t)}; \quad b(r,t) = b_0(t) e^{B_0(r)+B_1(r,t)};$$
$$c(r,t) = c_0(t) e^{C_0(r)+C_1(r,t)}; \quad E(r,t) = E_1(r,t).$$

The zeroth order solutions are for simplicity taken to be ones corresponding to zero-tension branes, and the perturbations represent the effect of small time-dependent brane stress-energies. We also expand the dilaton and gauge potential as

$$e^{\phi(r,t)} = \varphi_0(t) e^{\phi_0(r)+\phi_1(r,t)}; \quad A_\theta(r,t) = A_\theta^{(0)}(r) + A_\theta^{(1)}(r,t).$$

The stress energy tensor is given as

$$T^a_b = t^a_b + s^a_b + s^{a*}_b$$

where $t^a_b$ represents contributions from all bulk fields, and $s^a_b$ and $s^{a*}_b$ represent the brane content. Since we are assuming that the branes are only present at the perturbative level, their stress energy can be written as

$$s^i_r = -\rho \theta(r_0 - r) - \delta(r_0 - r) F_0(t); \quad s^i_t = p \theta(r_0 - r) - \delta(r_0 - r) F_0(t);$$
$$s^r_\theta = p^r_r \theta(r_0 - r); \quad s^0_\theta = p^0_\theta \theta(r_0 - r); \quad s^i_\theta = p^i_\theta \theta(r_0 - r); \quad s^i_r = p^i_r \theta(r_0 - r);$$
$$s^r_s = -\rho_s \theta(r - r_s) - \delta(r - r_s) F_s(t); \quad s^i_s = p_s \theta(r - r_s) - \delta(r - r_s) F_s(t);$$
$$s^r_r = p^r_r \theta(r - r_s); \quad s^{0*}_\theta = p^{0*}_\theta \theta(r - r_s); \quad s^{i*}_s = p^{i*}_s \theta(r - r_s); \quad s^r_s = p^r_s \theta(r - r_s);$$

$$(25)$$
where \( \theta(r) \) are Heaviside step functions. Our branes are therefore represented by cores of finite radii \( r_0 \) and \( r_* \) located around the poles of an axially symmetric compact internal space. For simplicity, we will assume \( r_* \) and \( r_0 \) are such that both branes have the same thickness at the level of the unperturbed background geometry.

Notice the inclusion of one-dimensional delta function terms in the brane stress-energy. In [12], these came from expanding the step function in a Taylor series, treating the time dependence of the thickness as a perturbation. Here, since the background tension is assumed to vanish, such terms do not appear unless we put them in by hand. The end result however can be shown to be exactly the same: the one-dimensional delta functions effectively encode the time dependence of the brane thickness. (See [39] and [45] for discussions on this point). These functions are not assumed to be the same at both branes (i.e. we will not be demanding that \( F_0 = F_* \)), so that the above assumption that the background thicknesses are equal is not overly constraining.

Finally, we will assume that time derivatives of \( O(\rho) \) perturbations are of \( O(\rho^{3/2}) \), which is implied by the usual law for conservation of energy \( \dot{\rho} \sim (\dot{a}/a)\rho \sim \rho^{3/2} \).

A. Background solutions

With the above ansatz for the metric and matter content, working in coordinates where \( B_0(r) = 0 \) one can show that

\[
\frac{\partial A_0^{(0)}}{\partial r} = -\beta e^{-4A_0(r)+C_0(r)+\phi_0(r)}. \tag{26}
\]

Since at this level there is no stress-energy on the brane to induce warping or singularities, the only possible solution is the perfectly spherical static “football” solution \( \Box \), where

\[
A_0(r) = 0 \tag{27}
\]
\[
\phi_0(r) = \phi_0 \tag{28}
\]
\[
e^{C_0(r)} = \frac{\sin(kr)}{k}. \tag{29}
\]

(Note that the coordinate \( r \) defined here differs from the one in section IIA by the rescaling \( r \rightarrow kr \). In these coordinates, \( \theta \) ranges over \((0, 2\pi)\) even in the presence of a deficit angle, while \( r \) of course ranges over \((0, \pi/k)\).) Substituting these into the background equations of
motion, one finds
\[ \varphi_0(t)v(\Phi_0) = \frac{\beta^2}{2\varphi_0(t)c_0(t)b_0(t)^2} \] (30)
\[ \frac{k^2 M_6^4}{b_0(t)^2} = \varphi_0(t)v(\Phi_0) + \frac{\beta^2}{2\varphi_0(t)c_0(t)^2b_0(t)^2}. \] (31)
One can readily check that these imply that
\[ \beta^2 = \frac{k^4 M_6^8}{2v(\Phi_0)}. \] (32)
and that the time dependent functions \( b_0(t), c_0(t) \) and \( \varphi_0(t) \) obey
\[ b_0(t) = c_0(t) \] (33)
\[ \varphi_0(t) = \sqrt{2v(\Phi_0)c_0(t)^2}. \] (34)
Notice here a key difference with the Einstein-Hilbert case, where there is no dilaton. In this case, we would have \( \varphi_0(t)e^{\varphi_0}v(\Phi_0) \to \Lambda_6 \) and \( \beta^2/[\varphi_0(t)e^{\varphi_0}] \to \beta^2 \), and we would instead find that \( b_0(t) \) and \( c_0(t) \) must be constants. This difference can be understood as follows. In the Einstein-Hilbert case, since the radion is stabilized we find in our perturbative analysis that the scale factor of the internal space is static. In the SUSY case, there are two degrees of freedom: the radion and the dilaton. Only one linear combination of the two is stable, while the other corresponds to a flat direction. It is this mode which appears here as a new dynamical entity.

B. Perturbed equations of motion

Before writing down the equations of motion to linear order in the perturbations, it is useful as was done in [12] to find a set of variables which are invariant under the gauge transformation
\[ t \to f(r,t); \quad r \to g(r,t). \] (35)
One convenient set is
\[ Z = N'_1 - A'_1; \quad W = 3A'_1 + N'_1; \quad X = \frac{C'_1}{C'_0} - B_1 - \frac{C''_0}{C'_0^2}C_1; \]
\[ Y = A_0^{(1)' - A_0^{(0)'}(B_1 + C_1); \quad U = \tilde{A}_0^{(1)' - \frac{\tilde{A}_0^{(0)'}}{C_0'}\tilde{C}_1}; \]
\[ \tilde{p} = -s^r_t - s^r_{*t}; \quad \tilde{\rho} = s^t_i + s^t_{*i}; \quad \tilde{p}_t = s^r_t + s^r_{*t}; \quad \tilde{p}_r = s^t_r + s^t_{*r} + s^t_{*r}; \]
\[ \tilde{p}_5 = s^r_r + s^r_{*r}; \quad \tilde{p}_0 = s^\theta_0 - s^r_r + s^\theta_{*r} - s^r_{*r}, \] (36)
in terms of which the perturbed equations of motion become

\[ \begin{align*}
W' - C_0'W &= c_0(t)^2 \frac{\tilde{p}_6}{M_6^4}, \\
\frac{Z'}{c_0(t)^2} + \frac{C_0'Z}{c_0(t)^2} &= 2 \left[ \frac{\tilde{a}_0}{a_0} \left( \frac{\tilde{a}_0}{a_0} \right)^2 - \frac{\tilde{a}_0}{a_0} \frac{\tilde{c}_0}{c_0} + 2 \left( \frac{\tilde{c}_0}{c_0} \right)^2 + \frac{1}{M_6^4} (\tilde{\rho} + \tilde{p}) \right] \\
Y' - C_0'Y &= e^{\phi_0} \beta e^{C_0(r)} (W - \phi'_1) = 0 \\
\frac{C_0'W}{c_0(t)^2} + \sqrt{2v(\Phi_0)} e^{-C_0(r)} Y + \beta \sqrt{2v(\Phi_0)} e^{\phi_0} &= 3 \left[ \frac{\tilde{a}_0}{a_0} + \left( \frac{\tilde{a}_0}{a_0} \right)^2 \right] + \frac{\tilde{c}_0}{c_0} + 2 \left( \frac{\tilde{c}_0}{c_0} \right)^2 + \frac{1}{M_6^4} \tilde{p}_5 \\
\dot{\tilde{p}}_5 - C_0' \tilde{p}_6 &= 0 \\
U' - \dot{Y} - \beta e^{\phi_0} e^{C_0(r)} \dot{X} &= 0 \\
\tilde{p}_5 &= -c_0(t)^2 \tilde{p}_t \\
3 \frac{\dot{\tilde{a}}_0}{a_0} Z - C_0' \dot{X} + \frac{3}{4} (\dot{Z} - \dot{W}) &= \frac{c_0(t)^2}{M_6^4} \tilde{p}_t - \frac{\sqrt{2v(\Phi_0)} e^{-C_0(r)}}{M_6^4} U - \frac{\tilde{c}_0}{c_0} (W + 2\phi'_1) \\
\dot{\tilde{p}}_t + 3 \frac{\dot{a}_0}{a_0} (\tilde{\rho} + \tilde{p}) + \frac{\tilde{c}_0}{c_0} (2\tilde{\rho} + 2\tilde{p}_5 + \tilde{p}_6) &= \tilde{p}_t' + C_0' \tilde{p}_t \\
\frac{\phi''_1 + C_0' \phi'_1 + C_0' W}{3 c_0(t)^2} &= \left[ \frac{\dot{\tilde{a}}_0}{a_0} + \left( \frac{\dot{a}_0}{a_0} \right)^2 - \frac{1}{3} \frac{\dot{c}_0}{c_0} - \frac{\dot{a}_0}{a_0} \frac{\dot{c}_0}{c_0} + \frac{\tilde{p}_5}{3 M_6^4} \right] \\
\frac{C_0' X'}{c_0(t)^2} - \frac{2 X \beta \sqrt{2v(\Phi_0)} e^{\phi_0}}{M_6^4 c_0(t)^2} + \frac{3 C_0' W}{4 c_0(t)^2} - \frac{\sqrt{2v(\Phi_0)} e^{-C_0(r)}}{c_0(t)^2 M_6^4} Y &= -\frac{1}{4 M_6^4} (\tilde{\rho} - 3 \tilde{p} + 3 \tilde{p}_6) + 3 \left[ \frac{\dot{\tilde{a}}_0}{a_0} + \left( \frac{\dot{a}_0}{a_0} \right)^2 + \frac{\dot{c}_0}{c_0} + \frac{3}{2} \frac{\dot{a}_0}{a_0} \frac{\dot{c}_0}{c_0} + \frac{4}{3} \left( \frac{\dot{c}_0}{c_0} \right)^2 \right] \\
\end{align*} \]

As was done in [12], we assume all the \( \rho \)'s and \( p \)'s are functions of time only. Also, in order to simplify calculations and because in the absence of a specific microscopic model for the brane matter content we are free to choose \( \tilde{p}_6 \) as we wish, we will assume that

\[ \tilde{p}_6(r, t) = \theta(r_0 - r) e^{2C_0(r)} P_6(t) + \theta(r - (\pi/k - r_0)) e^{2C_0(r)} P_{s6}(t). \]

C. Solutions to the perturbed equations of motion

The preceding system of equations may seem daunting, but is actually quite straightforward to solve. One simply uses eqs. (38), (37) and (41) to solve for \( Z(r, t), W(r, t) \) and \( \dot{\tilde{p}}_5(r, t) \) respectively. Once these solutions are found, one can use eq. (46) to solve for \( \phi_1(r, t) \).
and then either eq. (40) or eq. (39) to solve for $Y(r,t)$. Then $X(r,t)$ can be solved using eq. (47), while eq. (45) solves for $\tilde{p}(r,t)$ and either eq. (42) or eq. (44) solve for $U(r,t)$.

One must then impose appropriate boundary conditions, namely that all solutions are regular at the poles, and smooth across the core/bulk boundary except where indicated otherwise by the presence of the one-dimensional delta function terms in the stress-energy tensor.

In the interest of brevity we will only present here the general solutions for $W(r,t)$ and $\phi_1(r,t)$ in the bulk, which will be required for our argument in the following section.

\[ W^{(\text{bulk})}(r,t) = \mathcal{F}_2^{(\text{bulk})}(t) \sin(k r) \]  
\[ \phi_1^{(\text{bulk})}(r,t) = \frac{1}{2k} \left[ \mathcal{F}_2^{(\text{bulk})}(t) + \mathcal{F}_4^{(\text{bulk})}(t) \right] \ln(1 - \cos(k r)) + \mathcal{F}_5^{(\text{bulk})}(t) + \frac{\cos(k r)}{2k} \mathcal{F}_2^{(\text{bulk})}(t) \]

where $\mathcal{F}_i(t)$ are constants of integration set by boundary conditions. The interested reader will find solutions to the other gauge invariant variables in the appendix.

**IV. PRESENCE OF THE $\lambda_3 \neq 0$ SOLUTIONS**

As we have explained above, the hope is that starting from a static $\lambda_3 = 0$ solution the system will naturally evolve to a static $\lambda_3 \neq 0$ solution when the brane tensions are perturbed. In order for our results to confirm or rule out this idea, we must verify that our perturbative solutions actually include the $\lambda_3 \neq 0$ case. We will now show this explicitly.

We work in the convenient gauge where $B_1(r,t) = 0$, which means that effectively $g_{rr} = 1$. In this gauge we can write the metric as

\[ ds^2 = -n(r)dx^\mu dx_\mu + dr^2 + c(r)^2 d\theta^2 \]  
(51)

and it was shown in [30] that

\[ \lambda_3 = \frac{1}{2} c(r) \left[ 2n(r)^4 \phi(r)' + 4n(r)^3 n(r)' \right]. \]

(52)

In our perturbative language, the background solution has a static dilaton and no warping so that $\lambda_3$ vanishes identically at this order. If we wish to prove that our perturbative results
include the nonconical solutions with \( \lambda_3 \neq 0 \), we have to show that at \( \mathcal{O}(\rho) \), the combination on the right hand side of eq. (52) does not vanish. For static perturbations, we can write eq. (52) to \( \mathcal{O}(\rho) \) as

\[
\delta \lambda_3 = \frac{1}{2} e^{C_0(r)} \left[ 2\phi_1(r)' + W(r) \right].
\]  

(53)

Substituting our bulk solutions into this expression and assuming staticity (i.e. \( a_0(t) \equiv 1, c_0(t) \equiv 1, \rho = -p \) and \( \rho_* = -p_* \)), we find that

\[
\delta \lambda_3 = \frac{1}{12k^4 M_6^4} (1 - \cos(kr_0))^3 \left( \mathcal{P}_6(t) - \mathcal{P}_{*6}(t) \right).
\]  

(54)

This result tells us that the solutions with nonconical singularities at the poles are related to the presence on the branes of stress-energy along the extra dimensions (recall that we defined \( s_\theta - s_r^r = \theta(r_0 - r)p_6(r, t) \equiv \theta(r_0 - r)e^{2C_0(r)}\mathcal{P}_6(t) \) and \( s_*^\theta - s_*^r = \theta(r - r_*)p_*6(r, t) \equiv \theta(r_0 - r_* e^{2C_0(r)}\mathcal{P}_{*6}(t) \)). This result is compatible with the discussions of [43] and [45].

Having shown that our solutions include the case of interest, we can now address the question of whether the model can solve the cosmological constant problem through self-tuning.

V. COSMOLOGY AND THE QUESTION OF SELF-TUNING

A. Boundary conditions and the Friedmann equations

In braneworld models, one derives the Friedmann equations on the brane(s) by imposing appropriate boundary conditions. In the model we are studying, these imply that all functions are regular at the origin, and smooth across the brane/bulk boundaries. Since all the gauge invariant quantities we have defined only appear as first \( r \) derivatives in the equations of motion, the functions themselves must match, but not their first derivatives. The only exception is \( \phi_1(r, t) \), for which both the function itself and its first derivative must be matched across the boundaries at \( r = r_0 \) and \( r = \pi/k - r_0 \). One must also take into account the one-dimensional delta functions in the brane stress-energy tensor, which affect the following junction conditions

\[
\lim_{\epsilon \to 0} X(r_0 - \epsilon, t) - X(r_0 + \epsilon, t) = \frac{\tan(kr_0)}{kM_6^4} c_0(t)^2 \mathcal{F}_0(t)
\]  

(55)
\[
\lim_{\epsilon \to 0} X(r_* + \epsilon, t) - X(r_* - \epsilon, t) = \frac{\tan(kr_0)}{kM_6^4} c_0(t)^2 \mathcal{F}_*(t) \tag{56}
\]

\[
p_t(r_0, t) = -\dot{\mathcal{F}}_0(t) - 2\frac{\dot{c}_0}{c_0} \mathcal{F}_0(t) \tag{57}
\]

\[
p_t^*(r_*, t) = \dot{\mathcal{F}}_*(t) + 2\frac{\dot{c}_0}{c_0} \mathcal{F}_*(t) \tag{58}
\]

Solving for all variables and boundary conditions, one is led to the Friedmann equations, and conservation of energy

\[
\left(\frac{\dot{a}_0}{a_0}\right)^2 = \frac{(1 - \cos(kr_0))}{8M_6^4} (\rho + p + \rho_* + p_* - \frac{(1 - \cos(kr_0))^3}{72k^2M_6^4}(P_6 + P_{*6})
\]

\[\begin{aligned}
&+ \frac{2\dot{c}_0}{3c_0} + \left(\frac{\dot{c}_0}{c_0}\right)^2 \tag{59}
\end{aligned}
\]

\[\begin{aligned}
\frac{\ddot{a}_0}{a_0} - \left(\frac{\dot{a}_0}{a_0}\right)^2 &= \frac{(1 - \cos(kr_0))}{4M_6^4} (\rho + p + \rho_* + p_* - \frac{\dot{c}_0}{c_0} - \frac{2(\dot{c}_0)^2}{c_0} + \frac{\dot{a}_0}{a_0} \frac{\dot{c}_0}{c_0}) \tag{60}
\end{aligned}
\]

\[\begin{aligned}
\dot{\rho} + 3\frac{\dot{a}_0}{a_0}(\rho + p) &= \frac{k \sin(kr_0)}{1 - \cos(kr_0)} \left(\dot{\mathcal{F}}_0 + 2\frac{\dot{c}_0}{c_0} \mathcal{F}_0\right) - \frac{\dot{c}_0}{c_0} \left(2\rho + \frac{(1 - \cos(kr_0))^2}{3k^2} P_6\right) \tag{61}
\end{aligned}
\]

\[\begin{aligned}
\dot{\rho_*} + 3\frac{\dot{a}_0}{a_0}(\rho_* + p_*) &= \frac{k \sin(kr_0)}{1 - \cos(kr_0)} \left(\dot{\mathcal{F}}_* + 2\frac{\dot{c}_0}{c_0} \mathcal{F}_*\right) - \frac{\dot{c}_0}{c_0} \left(2\rho_* + \frac{(1 - \cos(kr_0))^2}{3k^2} P_{*6}\right) \tag{62}
\end{aligned}
\]

Consistency between these four equations implies that

\[
\mathcal{F}_0 + \mathcal{F}_* = -\frac{(1 - \cos(kr_0))}{4k \sin(kr_0)} (\rho - 3p + \rho_* - 3p_*)
\]

\[\begin{aligned}
&+ \frac{15 \cos(kr_0) - 6 \cos(2kr_0) - 10 + \cos(3kr_0)}{48k^3 \sin(kr_0)} (P_6 + P_{*6}) + \frac{Q_1}{c_0(t)^2} \tag{63}
\end{aligned}
\]

Note that the constant of integration $Q_1$ which appears in eq. (63) can be shown to be proportional to the perturbation to the flux.

### B. Effective four dimensional quantities

As was discussed in [12], the quantities we have used so far are not the ones a four dimensional observer on the brane would identify as the energy density and pressure. Indeed, to define the effective four dimensional quantities, we must integrate the 6D quantities over the thickness of the brane,

\[
S^{(4)a}_b = 2\pi \int_0^{r_0} dr \, b(r, t) \, c(r, t) \, S^{(6)a}_b \tag{64}
\]
which with our choice of gauge \((g_{rr} \equiv 1)\) leads to

\[
\begin{align*}
\rho^{(4)}(t) & = 2\pi \int_0^{r_0} c_0(t)^2 e^{C_0(r)} \rho(t) dr + 2\pi c_0(t)^2 e^{C_0(r_0)} F_0(t) \\
p^{(4)}(t) & = 2\pi \int_0^{r_0} c_0(t)^2 e^{C_0(r)} p(t) dr - 2\pi c_0(t)^2 e^{C_0(r_0)} F_0(t) \\
\rho^*(t) & = 2\pi \int_{\pi/k-r_0}^{\pi/k} c_0(t)^2 e^{C_0(r)} \rho^*(t) dr + 2\pi c_0(t)^2 e^{C_0(r_*)} F^*(t) \\
p^*(t) & = 2\pi \int_{\pi/k-r_0}^{\pi/k} c_0(t)^2 e^{C_0(r)} p^*(t) dr - 2\pi c_0(t)^2 e^{C_0(r_*)} F^*(t).
\end{align*}
\]

We also define the 4d Newton constant as

\[
\frac{1}{8\pi G_4(t)} = M_4^2 = 2\pi \int_0^{\pi/k} c_0(t)^2 e^{C_0(r)} M_6^4 dr
\]

\[
\Rightarrow G_4(t) = \frac{k^2}{32\pi^2 M_6^4 c_0(t)^2}.
\]

We come now to the main result of this paper, the Friedmann equations on general codimension-two branes in six-dimensional supergravity. Writing these in terms of the effective four dimensional quantities defined above, we find

\[
\begin{align*}
\left(\frac{\dot{a}_0}{a_0}\right)^2 & = \frac{8\pi G_4(t)}{3} (\rho^{(4)} + p^{(4)}) - \frac{16\pi^2 \sin(kr_0)}{3k} G_4(t) Q_1 + \frac{1}{3} \left(\frac{\dot{c}_0}{c_0}\right)^2 \\
& - \frac{\dot{a}_0 \dot{c}_0}{a_0 c_0}
\end{align*}
\]

\[
(71)
\]

\[
\frac{\ddot{a}_0}{a_0} - \left(\frac{\dot{a}_0}{a_0}\right)^2 = -4\pi G_4(t) (\rho^{(4)} + p^{(4)} + \rho^*(4) + p^*(4)) - \frac{\ddot{c}_0}{c_0} - 2 \left(\frac{\dot{c}_0}{c_0}\right)^2 \\
& + \frac{\dot{a}_0 \dot{c}_0}{a_0 c_0}
\]

\[
(72)
\]

\[
\begin{align*}
\dot{\rho}^{(4)} & = -3 \frac{\dot{a}_0}{a_0} (\rho^{(4)} + p^{(4)}) - \frac{2\pi (1 - \cos(kr_0))^3 c_0(t) \dot{c}_0(t)}{3k^4} P_6 \\
\dot{\rho}^*(4) & = -3 \frac{\dot{a}_0}{a_0} (\rho^*(4) + p^*(4)) - \frac{2\pi (1 - \cos(kr_0))^3 c_0(t) \dot{c}_0(t)}{3k^4} P^*_6
\end{align*}
\]

\[
(73)
\]

\[
\begin{align*}
\dot{c}_0 + \left(\frac{\dot{c}_0}{c_0}\right)^2 & + 3 \frac{\dot{a}_0 \dot{c}_0}{a_0 c_0} = \pi G_4(t) \left(\rho^{(4)} - 3p^{(4)} + \rho^*(4) - 3p^*(4)\right) \\
& + \frac{2\pi^2 c_0(t)^2 (1 - \cos(kr_0))^3}{3k^4} G_4(t) (P_6 + P^*_6) - \frac{8\pi^2 \sin(kr_0)}{k} G_4(t) Q_1.
\end{align*}
\]

\[
(74)
\]

\[
(75)
\]

We see a major difference with the Einstein-Hilbert case studied in [12]: here the 4d Newton constant is time-dependent at first order in the perturbations, whereas in the Einstein-Hilbert case such corrections would only appear at \(O(\rho^2)\). Although we will not focus on this point for the rest of the paper, we could use results of fifth force experiments and measures
of the time variation of Newton’s constant to strongly constrain the dynamics of the variable $c_0(t)$. In fact, one can show from the equations presented above that in the $P_6 = P_{*6} = 0$ limit, the equations can be derived from the effective action

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{16\pi\bar{G}} \left[ \zeta R - \frac{1}{2\zeta} \partial_\mu \zeta \partial^\mu \zeta - V(\zeta) \right] + \mathcal{L}_m \right)$$

(76)

by making the identifications $c_0(t) = \sqrt{\zeta}, V(\zeta) = -32\pi^2 \sin(kr_0)\bar{G}Q/k$ and $\bar{G}/\zeta(t) = G_4(t)$. In other words, the model in that case corresponds to Brans-Dicke gravity with $\omega = 1/2$ and a flat potential, a theory which is clearly ruled out by experiment. Note that this is consistent with the arguments of [41], since this is precisely the limit where non-conical singularities are absent, and corresponds to the case that was studied in that work.

When one does include non-zero $P_6$ and $P_{*6}$ terms, we see from eqs. (73,74) that conservation of energy is violated from a four dimensional perspective, and it is thus hard to imagine how one might write down an effective four dimensional action. It should be noted that this should not come as a surprise, since it is precisely this type of intrinsically extra-dimensional effect which motivated this approach as a plausible way to circumvent Weinberg’s no-go theorem and arrive at a self-tuning solution to the cosmological constant problem.

C. Absence of self-tuning

Using eqs. (71)-(75), we can now categorically answer in the negative the question of whether or not the model self-tunes to a static solution under perturbations to the brane tensions.

Indeed, it was already known that in the absence of the solutions with nonconical singularities, one needs to fine-tune the brane tensions with the flux from the bulk gauge field in order to obtain static solutions [9, 30, 31, 41, 43]. The hope, as explained above, was that including the solutions with nonconical singularities would allow the model to naturally evolve from a static solution with conical singularities to a static solution with nonconical singularities when one perturbs the tensions. We see clearly now that this cannot happen, for two reasons.

First, nonconical singularities are only present when the functions $P_6$ and $P_{*6}$ are nonzero. However, these have no dynamical equation of motion; they are free parameters in this model. Therefore, even if it were possible to use them in order to cancel a change in the
brane tensions, this would have to be done by hand and would thus represent an arbitrary fine-tuning of the model’s parameters. This would be made all the more unnatural by the fact that it would have to be a time-dependent fine-tuning if it were to hold across multiple phase transitions on either brane.

Second, it is not even possible to use the nonconical singularities to cancel perturbations to the brane tensions and get new static solutions. This can be seen by setting $\rho^{(4)} = -p^{(4)}$ and $\rho^{(4)*} = -p^{(4)*}$ in the above equations, and setting all time derivatives to zero. We then find that eq. (71) necessitates a relation between $\rho^{(4)}$, $\rho^{(4)*}$ and $Q_1$ that, when plugged into eq. (76) tells us that $P_6 = -P_{*6}$. In other words, one must tune the brane tensions in order to obtain a static solution regardless of whether we have conical or nonconical singularities. Moreover, if we do have nonconical singularities, we must perform an additional fine-tuning of the functions $P_6$ and $P_{*6}$ sourcing them in order to keep the solutions static. Therefore, we have conclusively shown that no self-tuning mechanism for the cosmological constant is present in the type of six dimensional supergravity we have considered here.

We should note that although this final result agrees with the conclusions of [41], the analysis we have performed here was far from superfluous, since we have shown that self-tuning fails not only for the simple case considered in that work, but in the much more general case which includes warping and nonconical singularities. Indeed, the solutions we find are qualitatively different from the ones in [41] when $P_6$ and $P_{*6}$ are nonzero. One cannot simply derive them from an effective four dimensional action since they correspond to a theory where the usual law of conservation of energy does not hold (see eqs. (73,74)).

One final point of interest is the fact that one can see from our results that we cannot set only one of $a_0$ and $c_0$ constant without imposing unnatural constraints on the brane stress-energy tensor components (see eq. (75) in the case $c_0 = cst$ for example). They must either both be constant (which requires the fine-tuning mentioned in the previous paragraphs), or both be time dependent. This explains why previous solutions always singled out solutions with Minkowski 4-space over de Sitter or anti-de Sitter space: the initial assumption was always made that the internal space was static! Indeed, rather than saying that once the radion and dilaton are stabilized, solutions with a static external space are required by the equations of motion, a more correct statement would be that choosing the large dimensions to be Minkowski is actually part of the tuning one has to do in order to obtain a static radion and dilaton.
D. Dropping axial symmetry: a loophole?

Although our results appear quite conclusive regarding the absence of a self-tuning mechanism for the cosmological constant in six dimensional supergravity, one should be keep in mind which assumptions were made in deriving this result, to see if relaxing one of them might in fact lead to a different conclusion.

The solutions we have found are more general than those that had so far been presented in the literature because they allow for general equations of state for the brane content and they include the possibility of warping and of nonconical singularities in a fully dynamical context. There is however one restriction we have imposed on our metric ansatz which could be relaxed: axial symmetry of the internal space.

In a recent paper [46], explicit solutions have been found with an arbitrary number of 3-branes on a compact two-dimensional internal space by dropping the requirement of axial symmetry. The analysis in [46] concludes that the only solution with two branes is one with equal tension branes located at the poles of a spherical internal space, but this result is derived by excluding the possibility of warping and is restricted to Einstein-Hilbert gravity. It is interesting to ask what might happen in the supersymmetric case if we were to add the possibility of breaking axial symmetry in response to a change in the brane tensions, in addition to the possibility of nonconical singularities and warping. While there is no compelling reason to believe that allowing the branes to shift their positions would lead to self-tuning, we feel it is nonetheless an avenue worth investigating in future work, since it could conceivably lead to interesting and perhaps unexpected effects.

VI. CONCLUSION AND OUTLOOK

In this paper, we have studied the cosmology on a codimension-two brane in six-dimensional supergravity. We achieved this by solving the dynamical field equations linearized around a static background. In order to deal with the mildly singular behaviour of the warp factor at the position of the branes when the equation of state of their matter content is different from that of pure tension, we regularized the branes by giving them a nonzero thickness.

Our results show that there is no self-tuning of the cosmological constant to zero in such
a setup. Rather, any change to the branes’ tensions leads to expansion in both the external space and the internal space. In fact, we find that the internal and external space must always either both be static (which necessitates fine-tuning) or both be time dependent, but that we cannot have one be static while the other evolves without imposing unnatural relations between the brane stress-energy tensor components. This explains why previous solutions, which always assumed a static internal space from the start, singled out static solutions in the large dimensions.

Our solutions include both nonconical singularities and warping of the metric, and we emphasize that this represents a new result, since no dynamical treatment of such a model including both these possibilities had been carried out to date. The only assumption we made is that of axial symmetry of the internal space, and although it would be interesting to look for solutions where this assumption is dropped, we see no reason to believe that it would lead to self-tuning.

Although we do not believe that 6D supergravity theories offer a solution to the cosmological constant problem, such models with large extra dimensions can nevertheless provide a novel source of dynamical dark energy (whose size is naturally of the right magnitude once the cosmological constant is set to zero) as well as accelerator signals which would be visible at the LHC [44], and they therefore merit more study along these lines.

VII. ACKNOWLEDGMENTS

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APPENDIX A: GENERAL SOLUTIONS TO THE PERTURBED EQUATIONS OF MOTION

We present here the general solutions to the perturbed equations of motion (37)-(47). We only present the solutions in the core located in the region $0 \leq r \leq r_0$. Bulk solutions have the same form, without the terms coming from the brane stress-energy, while solutions in the other core have the same form with the terms from the brane stress-energy replaced by
the corresponding terms for this second brane ($\rho_s$ instead of $\rho$ for example). The solutions for the integration “constants” $\mathcal{F}_i(t)$ will not be given explicitly, but can be found in each region of the model through the imposition of boundary conditions, as explained in the main text.

\[
Z(r, t) = \frac{\cos(k r) c_0(t)^2}{k \sin(k r)} \left[ \frac{2}{k} \left( \frac{\ddot{a}_0}{a_0} \right)^2 - 2 \frac{\dddot{a}_0}{a_0} - 2 \frac{\ddot{a}_0}{a_0} \frac{\ddot{c}_0}{c_0} + 2 \frac{\ddot{a}_0}{a_0} \right] - \frac{1}{M_6^4} (\rho(t) + p(t))
\]

\[
W(r, t) = - \frac{\cos(k r) \sin(k r) c_0(t)^2}{k^3 M_6^4} \mathcal{P}_6(t) + \sin(k r) \mathcal{F}_2(t)
\]

\[
p^r_i(r, t) = - \frac{\cos(2 k r)}{4k^2} \mathcal{P}_6(t) + \mathcal{F}_3(t)
\]

\[
\phi_1(r, t) = \ln(\sin(k r)) \frac{c_0(t)^2}{k^2} \left[ -\frac{3}{k} \left( \frac{\ddot{a}_0}{a_0} \right)^2 + 3 \frac{\dddot{a}_0}{a_0} - 3 \frac{\ddot{a}_0}{a_0} \frac{\ddot{c}_0}{c_0} - \frac{k \mathcal{F}_2(t)}{2} \right] - \frac{c_0(t)^2 \mathcal{P}_6(t)}{12k^2 M_6^4} \cos(k r) - \frac{5 \mathcal{P}_6(t)}{12k^2 M_6^4} \cos(k r)^2 + \mathcal{F}_5(t)
\]

\[
Y(r, t) = \sin(k r) \ln(\sin(k r)) \beta e^{\phi_0} \frac{c_0(t)^2}{12k^3} \left[ \frac{36}{a_0} \frac{\ddot{a}_0}{a_0} + 36 \left( \frac{\ddot{a}_0}{a_0} \right)^2 - 36 \frac{\dddot{a}_0}{a_0} \frac{\ddot{c}_0}{c_0} - 12 \frac{12}{a_0} \frac{\ddot{c}_0}{c_0} + \frac{12 \mathcal{F}_3(t)}{M_6^4} \right]
\]

\[
X(r, t) = \tan(k r)^2 \ln(\sin(k r)) \frac{c_0(t)^2}{24k^2} \left[ \frac{36}{a_0} \frac{\ddot{a}_0}{a_0} + 36 \left( \frac{\ddot{a}_0}{a_0} \right)^2 - 36 \frac{\dddot{a}_0}{a_0} \frac{\ddot{c}_0}{c_0} - 12 \frac{12}{a_0} \frac{\ddot{c}_0}{c_0} + \frac{12 \mathcal{F}_3(t)}{M_6^4} \right]
\]
\[ -\frac{4}{3} \left( \frac{c_0}{c_0} \right)^2 - \frac{\mathcal{F}_9(t)}{6M_6^4} + \frac{k^2\mathcal{F}_5(t)}{3c_0(t)^2} + \frac{17\mathcal{P}_6(t)}{72k^2M_6^4} + \frac{1}{M_6^4}(\rho(t) - 3\rho(t)) \]  

(A6)

\[ p^r_t(r, t) = \frac{\mathcal{P}_6(t)c_0}{3k^3}\frac{c_0}{c_0}\cot(kr)(2\cos(kr)^2 - 1) - \frac{\cot(kr)}{k} \left[ \dot{\rho} + 3\frac{\dot{a}_0}{a_0}(\rho(t) + p(t)) \right] + \frac{\dot{c}_0}{c_0} \left( 2\rho(t) + 2\mathcal{F}_3(t) + \frac{7\mathcal{P}_6(t)}{6k^2} \right) + \frac{\mathcal{F}_7(t)}{\sin(kr)} \]  

(A7)

We have not written down \( U(r, t) \) explicitly since it can be obtained trivially from the algebraic equation \((44)\) once all other functions are given.


[53] solutions with $H^2 < 0$ can be analytically continued to 4D AdS.