DIFFERENTIAL FORM OF THE COLLISION INTEGRAL
FOR A RELATIVISTIC PLASMA

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The differential formulation of the Landau-Fokker-Planck collision integral is developed for
the case of relativistic electromagnetic interactions.

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Kinetic theory is founded upon the Boltzmann equation, which is a conservation equa-
tion for the phase-space distribution function of each species in an ensemble of interacting
particles. For the case of Coulomb interactions, Landau\(^1\) expressed the collision term in the
Fokker-Planck form. This mixed integro-differential representation was extended to rela-
tivistic electromagnetic interactions by Beliaev and Budker.\(^2\) For the nonrelativistic case,
it was shown by Rosenbluth et al.\(^3\) and by Trubnikov\(^4\) that the integrals appearing in the
collision term can be expressed in terms of the solution of a pair of differential equations.
The present work extends that formulation to the relativistic collision integral. Using an
expansion in spherical harmonics the relativistic differential formulation is then applied to
calculate the scattering and slowing down of fast particles in a relativistic equilibrium back-
ground plasma. Our work is relevant to the study of high temperature plasma in fusion
energy research and in astrophysics.

In the work of Landau\(^1\) and that of Beliaev and Budker,\(^2,5\) the collision term that
occurs on the right-hand side of the Boltzmann equation for species \(a\) and describes the
effect of collisions with species \(b\) is written in the Fokker-Planck form,
\[
C_{ab} = \frac{\partial}{\partial u} \left( \mathbf{D}_{ab} \cdot \frac{\partial f_a}{\partial u} - \mathbf{F}_{ab} f_a \right),
\]
in which the coefficients \(\mathbf{D}_{ab}\) and \(\mathbf{F}_{ab}\) are defined by
\[
\begin{align*}
\mathbf{D}_{ab}(u) &= \frac{q_a q_b^2}{8 \pi \epsilon_0^2 m_a} \log \Lambda_{ab} \int \mathbf{U}(u, u') f_b(u') \, d^3 u', \\
\mathbf{F}_{ab}(u) &= -\frac{q_a q_b^2}{8 \pi \epsilon_0^2 m_a m_b} \log \Lambda_{ab} \int \left( \frac{\partial}{\partial u'} \cdot \mathbf{U}(u, u') \right) f_b(u') \, d^3 u'.
\end{align*}
\]
Here, \(f_a\) and \(f_b\) are the distribution functions for the two species, \(u\) is the ratio of momentum
to species mass, \(q_a\) and \(q_b\) are the species charge, \(m_a\) and \(m_b\) are the species mass, \(\epsilon_0\) is
the vacuum dielectric permittivity, and \(\log \Lambda_{ab}\) is the Coulomb logarithm. The kernel \(\mathbf{U}\)
is specified below. This form of the collision operator is only approximate because of the
introduction of cutoffs in the collision integral. More accurate operators that take into
account Debye shielding at large impact parameters and large-angle scattering and quantum
effects at small impact parameters have been derived.\(^6,7\) The purpose of this letter is to
present a differential formulation for the integral transforms that occur in Eqs. (2). To avoid
unnecessary clutter we discard the factor that depends only on the species properties, drop
the species subscript, and consider the transforms
\[
\begin{align*}
\mathbf{D}(u) &= \frac{1}{8 \pi} \int \mathbf{U}(u, u') f(u') \, d^3 u', \\
\mathbf{F}(u) &= -\frac{1}{8 \pi} \int \left( \frac{\partial}{\partial u'} \cdot \mathbf{U}(u, u') \right) f(u') \, d^3 u'.
\end{align*}
\]
For guidance, let us recall briefly the nonrelativistic theory. In that case the momentum-to-mass ratios \( u \) and \( u' \) reduce to the velocities \( v \) and \( v' \), and the collision kernel is the one given by Landau,\(^1\) \( \mathbf{U} = (|s|^2 I - s s^\perp)/|s|^3 \), where \( s = v - v' \). It may be seen that \( \mathbf{U} = \partial^2|s|/\partial v_i \partial v_j \) and \( (\partial/\partial v^i) \mathbf{U} = -20|s|^{-1}/\partial v^i \). To obtain the differential formulation, these representations are inserted into Eqs. (3), and the differentiation with respect to \( v \) is moved outside the integration over \( v' \). Defining the potentials \( h(v) = -(1/8\pi) \int |s| f dv^i \) and \( g(v) = -(1/4\pi) \int |s|^{-1} f dv^i \), we have \( \mathbf{D} = -\partial^2 h/\partial v_i \partial v_j \) and \( \mathbf{F} = -\partial g/\partial v_i \partial v_j \). Furthermore, from \( \Delta|s| = 2|s|^{-1} \) and \( \Delta|s|^{-1} = -4\pi\delta(s) \) it follows that \( h \) and \( g \) obey the equations \( \Delta h = g \) and \( \Delta g = f \). (\( \Delta \) denotes the Laplacian with respect to the variable \( v \).) These equations provide the differential formulation of the collision term in the nonrelativistic case.

The Landau collision kernel was obtained in a semi-relativistic fashion, assuming Coulomb collisions and relativistic particle kinematics. It is a good approximation to the differential form of Rosenbluth and Trubnikov relies on the stronger assumptions \( |v|^2 \ll c^2 \) and \( |v'|^2 \ll c^2 \), and is therefore entirely nonrelativistic. A differential formulation that is exactly equivalent to the Landau collision integral was given by Franz.\(^8\)

We turn now to the differential formulation of the relativistic collision integral due to Beliaev and Budker.\(^2,5,6\) They obtained the expression

\[
\mathbf{U}(u, u') = \frac{r^2/(\gamma \gamma')}{(r^2 - 1)^{3/2}}((r^2 - 1) I - uu - uu' + r(uu' + uu')), \tag{4a}
\]

in which \( \gamma = \sqrt{1 + |u|^2} \), \( \gamma' = \sqrt{1 + |u'|^2} \), and \( r = \gamma \gamma' - uu' \). (We set \( c = 1 \) in this part of the paper.) One finds

\[
\frac{\partial}{\partial u_i} \cdot \mathbf{U}(u, u') = \frac{2r^2/(\gamma \gamma')}{(r^2 - 1)^{3/2}}(ru - u'). \tag{4b}
\]

Notice that \( r \) is the relativistic correction factor for the relative velocity between the two particles (i.e., for the velocity of one particle in the rest frame of the other). Conversely, this relative velocity is given by \( r^{-1}(r^2 - 1)^{1/2} \).

In developing a differential formulation for the collision term based on the Beliaev and Budker kernel, it is helpful to work in terms of relativistically covariant quantities. The expression \( \gamma \gamma' \mathbf{U} \) is equal to the space part of a four-tensor \( W \) that depends on the four-vectors \( u = (\gamma, u) \) and \( u' = (\gamma', u') \),

\[
W^{ij}(u, u') = \frac{r^2}{(r^2 - 1)^{3/2}}((r^2 - 1) g^{ij} - u^i u^j - u'^i u'^j + r(u^i u'^j + u'^i u^j)), \tag{5a}
\]

where \( g^{ij} \) is the metric tensor, with signature \( -+++ \). (\( r = -u_i u'^i \) is clearly a four-scalar.) The tensor \( W \) is symmetric (\( W^{ij} = W^{ji} \)), symmetric in \( u \) and \( u' \), satisfies \( u_i W^{ij} = 0 \), and satisfies \( W_i = 2r^2(r^2 - 1)^{-1/2} \). Likewise \( \gamma \gamma'(\partial/\partial u_i) \cdot \mathbf{U} \) is the space part of the four-vector \( V \), where

\[
V^i(u, u') = \frac{2r^2}{(r^2 - 1)^{3/2}}(ru^i - u'^i). \tag{5b}
\]

If the relativistic differential formulation is to parallel most closely the nonrelativistic formulation, then one should find a representation of the form \( W^{ij} = H^{ij} \psi \) and \( V^i = -2G^i \varphi \), where \( \psi \) and \( \varphi \) are four-scalars depending on \( u \) and \( u' \), and \( H^{ij} \) and \( G^i \) are covariant differential operators acting on the variable \( u \). In the nonrelativistic limit, \( \psi \) should reduce to \( |v - v'| \) and \( \varphi \) should reduce to \( |v - v'|^{-1} \). It should be possible to transform \( \psi \) and \( \varphi \) to delta functions by a sequence of second-order differential operators. The potentials
would be defined as \( h = -(1/8\pi) \int (\psi f' / \gamma') \, d^3u' \) and \( g = -(1/4\pi) \int (\varphi f / \gamma)^2 \, d^3u' \); these expressions define four-scalars (cf. Ref. 5). The differential equations satisfied by \( h \) and \( g \) follow immediately from those satisfied by \( \psi \) and \( \varphi \). Finally, \( D \) would be obtained as the space part of \( -\gamma^{-1}\partial_\gamma h \) and \( F \) as the space part of \( -\gamma^{-1}\partial_\gamma g \). In fact, it will turn out that the relativistic formulation has to be somewhat more complicated, but not fundamentally different from the outline just sketched.

A function of the four-vectors \( u \) and \( u' \) that is a four-scalar must be a function of \( r = -u.u' \) alone. The form of the differential operators \( H^i \) and \( G^i \) is restricted because these should be interior operators on the surface \( u^2 = -1 \) in four-space. In addition, it is required that \( H^i = H^i \) and \( u_i H^i = 0 \). Under those restrictions it is found that the most general form of \( H^i \) and \( G^i \), up to a multiplicative constant, is

\[
\begin{align*}
\mathcal{L}^j \chi &= (g^{ik} + u^i u^k) \frac{\partial^2 \chi}{\partial u^k \partial u^i} + (g^{ij} + u^i u^j) u^m \frac{\partial \chi}{\partial u^m}, \\
K^i \chi &= (g^{ik} + u^i u^k) \frac{\partial \chi}{\partial u^k}.
\end{align*}
\]

The spatial part of \( \mathcal{L}^j \chi \) is \( L \chi \) and that of \( K^i \chi \) is \( K \chi \) where

\[
\begin{align*}
L \chi &= \gamma^{-2} \frac{\partial^2 \chi}{\partial \vec{v} \partial \vec{v}} - \vec{v} \frac{\partial \chi}{\partial \vec{v}} - \frac{\partial \chi}{\partial \vec{v}}, \\
K \chi &= \gamma^{-1} \frac{\partial \chi}{\partial \vec{v}},
\end{align*}
\]

in which \( \vec{v} = u / \gamma \), and \( \partial / \partial \vec{v} = \gamma (\vec{l} + \vec{u} \vec{u}) \cdot \partial / \partial \vec{u} \). If \( \chi \) is a function of \( r \) alone then

\[
\begin{align*}
\mathcal{L}^j \chi &= \frac{d^2 \chi}{dr^2} (ru^i - u^i) (ru^j - u^j) + r \frac{d \chi}{dr} (g^{ij} + u^i u^j) \\
K^i \chi &= (d \chi / dr) (ru^i - u^i).
\end{align*}
\]

One is thereby led to the representations

\[
\begin{align*}
W^j &= \left[ \mathcal{L}^j + g^{ij} + u^i u^j \right] \sqrt{r^2 - 1} \\
&\quad - \left[ \mathcal{L}^j - g^{ij} + u^i u^j \right] \left( r \cosh^{-1} r - \sqrt{r^2 - 1} \right), \\
V^i &= -2K^i \left( r(r^2 - 1)^{-1/2} - \cosh^{-1} r \right).
\end{align*}
\]

These representations for \( W \) and \( V \) are only suitable for constructing a differential formulation of the collision term if the functions that occur on the right-hand sides can be reduced to delta functions by some sequence of differential operators. For that purpose the contraction \( L = L^i_i \) is needed; in terms of the three-space variables it is

\[
L \chi = (\vec{l} + \vec{u} \vec{u}) : \frac{\partial^2 \chi}{\partial \vec{u} \partial \vec{u}} + 3 \vec{u} \cdot \frac{\partial \chi}{\partial \vec{u}}.
\]
The explicit form of the differential representation of Eqs. (3) based on the Beliaev and Budker collision kernel follows: The potentials are

\[
\begin{align*}
    h_0 &= -(1/4\pi) \int (r^2 - 1)^{-1/2} f(u')/\gamma' \, d^3 u', \\
    h_1 &= -(1/8\pi) \int \sqrt{r^2 - 1} f(u')/\gamma' \, d^3 u', \\
    h_2 &= -(1/32\pi) \int (r \cosh^{-1} r - \sqrt{r^2 - 1}) f(u')/\gamma' \, d^3 u', \\
    g_0 &= -(1/4\pi) \int r(r^2 - 1)^{-1/2} f(u')/\gamma' \, d^3 u', \\
    g_1 &= -(1/8\pi) \int \cosh^{-1} r f(u')/\gamma' \, d^3 u'.
\end{align*}
\]

These potentials satisfy the differential equations

\[
\begin{align*}
    [L + 1] h_0 &= f, \\
    [L - 3] h_1 &= h_0, \\
    [L - 3] h_2 &= h_1, \\
    L g_0 &= f, \\
    L g_1 &= g_0.
\end{align*}
\]

Finally one obtains \( D \) and \( F \) as

\[
\begin{align*}
    D(u) &= -\gamma^{-1} [L + \mathbf{l} + uu'] h_1 + 4\gamma^{-1} [L - \mathbf{l} - uu'] h_2, \\
    F(u) &= -\gamma^{-1} K(g_0 - 2g_1).
\end{align*}
\]

Equations (11–12) together with the definitions, Eqs. (7) and (9), provide the differential formulation in the relativistic case.

In order to proceed further analytically, it is useful to decompose the distribution function and the potentials in spherical harmonics, e.g.,

\[
f(u, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_{nm}(u) P_n^m(\cos \theta) \exp(i m \phi).
\]

Here \( u = |u| \) (different from the convention used earlier), \( \theta \) is the polar angle, and \( \phi \) is the azimuthal angle. The equation \([L - \alpha] g = f\) is equivalent to the system of separated equations \([L_n - \alpha] g_{nm} = f_{nm}\), where

\[
[L_n - \alpha] y = (1 + u^2) \frac{d^2 y}{du^2} + (2u^{-1} + 3u) \frac{dy}{du} - \left( \frac{n(n + 1)}{u^2} + \alpha \right) y.
\]

After the change of variable \( x = \sinh^{-1} u \) and the change of unknown \( z = (\sinh x)^{-n} y \), then the equation \([L_n - \alpha] y = w\) transforms to \([D_n - a^2] z = (\sinh x)^{-n} w\), where \( a^2 = \alpha + 1 \) and

\[
[D_n - a^2] z = \frac{d^2 z}{dx^2} + 2(n + 1)(\coth x) \frac{dz}{dx} + \left( (n + 1)^2 - a^2 \right) z.
\]

The solution to the homogeneous equation \([D_n - a^2] z = 0\) is required in order to construct a Green’s function for the problem. To obtain this solution we note the following recurrence: If \( z_{n-1,a} \) solves \([D_{n-1} - a^2] z = 0\), then \( z_{n,a} = (\sinh x)^{-1}(d/dx)z_{n-1,a} \) solves \([D_n - a^2] z = 0\). Furthermore, for \( n = -1 \) the homogeneous equation is trivial to solve. However, the recurrence breaks down in the case that \( a \) is an integer. If \( a = n \), then \( z_{n-1,a} = 1 \) solves
\[ [D_{n-1} - a^2]z = 0, \]  
and differentiation produces the null solution to \([D_n - a^2]z = 0\). The recurrence must then be restarted from the general solution to \([D_n - n^2]z = 0\), which is

\[ z_{n,n} = (\sinh x)^{-2n-1} \left( C_1 + C_2 \int_0^x (\sinh x')^{2n} dx' \right). \]

The integral that occurs here can be expressed in closed form.

The Green’s function allows us to reduce the separated ordinary differential equations to quadrature. An important special application for these results is in the treatment of collisions off an equilibrium background distribution. Assuming that \(f_b\) is a stationary Maxwellian with density \(n_b\) and temperature \(T_b\) and that the energy of the colliding particles greatly exceeds \(T_b\), we obtain

\[
D_{uu} = \Gamma_{ab} \frac{K_1}{K_2} \frac{u_{\theta b}^2}{v_b^2} \left( 1 - \frac{K_0}{K_1} \frac{u_{\theta b}^2}{\gamma^2 c^2} \right),
\]

\[
D_{\theta\theta} = \Gamma_{ab} \frac{1}{2v} \left[ 1 - \frac{K_1}{K_2} \left( \frac{u_{\theta b}^2}{u^2} + \frac{u_{\theta b}^2}{\gamma^2 c^2} \right) + \frac{K_0}{K_2} \frac{u_{\theta b}^2}{u^2} \frac{u_{\theta b}^2}{\gamma^2 c^2} \right],
\]

and \(F_u = -(m_a v/T_b) D_{uu}\). (The other components of \(D\) and \(F\) vanish.) Here we have put the expressions for \(D\) and \(F\) into dimensional form as in Eqs. (2). \(K_n\) is the \(n\)th-order Bessel function of the second kind, the argument for the Bessel functions is \(m_b c^2/T_b\), \(u_{\theta b}^2 = T_b/m_b\), and \(\Gamma_{ab} = n_b a^2 \nu_b^2 \log \Lambda_{ab}/(4\pi \varepsilon_0 m_a^2)\). The errors are exponentially small in \(u/u_{\theta b}\).

To conclude, we have presented a differential formulation for the Beliaev and Budker\(^2\) relativistic collision integral. This permits the rapid numerical evaluation of the collision term. A decomposition into spherical harmonics is useful in carrying out analytical work. It also provides a convenient method for calculating the boundary conditions for the potentials.

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