Ultraviolet Behavior of the Gluon Propagator in the Maximal Abelian Gauge

S. M. Morozov\textsuperscript{a}, R. N. Rogalyov\textsuperscript{b}

\textsuperscript{a} Institute of Theoretical and Experimental Physics, Moscow, 117259, Russia
\textsuperscript{b} Institute for High Energy Physics, Protvino, Moscow region, 142281, Russia

Abstract

The ultraviolet asymptotic behavior of the gluon propagator is evaluated in the maximal Abelian gauge in the SU(2) gauge theory on the basis of the renormalization-group improved perturbation theory at the one-loop level. Square-root singularities obtained in the Euclidean domain are attributed to artifacts of the one-loop approximation in the maximal Abelian gauge and the standard normalization condition for the propagator used in our study. It is argued that this gauge is essentially nonperturbative.

1 Introduction

Maximal Abelian Gauge (MAG) \cite{Mag} offers one of the best instruments for theoretical studies of dynamics of the gauge fields in QCD at a large scale (\sim 1 fm). In this gauge, the boson fields can be naturally divided into the "Abelian" and "non-Abelian" (or, in other words, "diagonal" and "off-diagonal") components. Assuming that the "diagonal" components give a leading contribution to the path integral,\textsuperscript{1} the Yang–Mills theory on a lattice can be reduced to the dual Abelian Higgs model. In so doing, confinement of quarks and gluons comes about from monopole condensation in the dual Abelian Higgs model.

Therefore, propagators of "diagonal" and "off-diagonal" gluons are of particular interest because, from a naive point of view, their ratio indicates whether the Abelian dominance hypothesis is true or not. Recently \cite{Prop}, the behavior of the "diagonal" and "off-diagonal" gluon propagators was estimated numerically in the framework of lattice gauge theory. At small momenta it was found that the propagator of the "Abelian" field components dominates. In this work, we study asymptotic behavior of the propagators of the "diagonal" and "off-diagonal" gluon at large momenta in the framework of perturbation theory in the continuum field theory.\textsuperscript{2} It should be noted that the interest in this gauge was quickened by the study \cite{Ren}, where it was demonstrated that the SU(2) Yang-Mills theory can be renormalized in this gauge.

\textsuperscript{1}This assumption is referred to as the hypothesis of Abelian dominance \cite{Hyp}.
\textsuperscript{2}We work in the Euclidean space, throughout this paper, \( p^2 = p_0^2 + p_1^2 + p_2^2 + p_3^2 \), that is, Euclidean momenta are positive.
In practical calculations in a lattice Yang–Mills theory, the maximal Abelian gauge is fixed by finding a stationary point of the functional

\[ \Phi = \sum_{x,\mu} \text{Tr} \left( U_{x,\mu}^\dagger \sigma_3 U_{x,\mu} \sigma_3 \right) \quad (1) \]

with respect to the gauge transformations of the type

\[ \Lambda : \quad U_{x,\mu} \rightarrow \Lambda_x \dagger U_{x,\mu} \Lambda_x + \hat{\mu}, \quad (2) \]

where \( U_{x,\mu} \in SU(2) \) are the link variables and \( \Lambda_x \) is a \( SU(2) \)-valued function defined at sites of a lattice. In other words, the gauge condition is fixed by the relations

\[ \frac{\delta \Phi}{\delta \Lambda} = 0. \quad (3) \]

At infinitesimal gauge transformations,

\[ \Lambda = \exp(-i\omega_a \sigma_a / 2) = \cos \frac{\vec{\omega}}{2} - i \frac{\omega_a \sigma_a}{|\vec{\omega}|} \sin \frac{\vec{\omega}}{2}, \quad (4) \]

the variation of the functional (1) takes the form

\[ \delta \Phi = \text{Tr} \left[ U_{x,\mu} (\sigma_3 + M_1(x + \hat{\mu}) + M_2(x + \hat{\mu})) U_{x,\mu}^\dagger (\sigma_3 + M_1(x) + M_2(x)) \right] \]

\[ - \text{Tr} [U_{x,\mu} \sigma_3 U_{x,\mu}^\dagger \sigma_3], \]

where

\[ M_1(x) = -\epsilon^{abc} \omega_a(x) \sigma_b, \quad M_2(x) = \frac{1}{2} (\delta^{ac} \delta^{b3} - \delta^{ab} \delta^{c3}) \omega_a \omega_b \sigma_c. \quad (5) \]

Using the decomposition

\[ U_{x,\mu} = u_0 + i \sum_{n=1}^3 u_n \sigma_n = \begin{pmatrix} u_0 + i u_3 & u_2 + i u_1 \\ -u_2 + i u_1 & u_0 - i u_3 \end{pmatrix}, \]

we define the vector potentials \( A^a_\mu \) by the formula \( u^a_\mu = -\frac{g a}{2} A^a_\mu \) \((a = 1, 2, 3)\). In this case, the gauge condition has the form

\[ a \nabla_\mu^B ((u^3_\mu - i u^0_\mu)(u^1_\mu - i u^2_\mu)) + 2u^3_\mu(u^1_\mu - i u^2_\mu) = 0, \quad (6) \]

\[ a \nabla_\mu^B ((u^3_\mu + i u^0_\mu)(u^1_\mu + i u^2_\mu)) + 2u^3_\mu(u^1_\mu + i u^2_\mu) = 0, \]

where \( \nabla_\mu^B \) is the operator of backward derivative on a lattice: \( \nabla_\mu^B u^\nu_\mu(x) = \frac{u^\nu_\mu(x) - u^\nu_\mu(x - \hat{\mu})}{a} \).

Having regard to the relation \( u^0_\mu = 1 + O(g^2 a^2) \), we find that, in the leading order in the lattice spacing, the gauge condition (6) has the form

\[ (\partial_\mu + ig A^3_\mu)(A^1_\mu + i A^2_\mu) = 0. \quad (7) \]

\[ (\partial_\mu - ig A^3_\mu)(A^1_\mu - i A^2_\mu) = 0. \quad (8) \]

\[ 2 \]
Here and below, this naive limit of the above gauge conditions is named maximal Abelian
gauge in the continuum theory. The continuum theory in this gauge can conveniently be
quantized in the BRST formalism. To do this, we should introduce the gauge-fixing terms
as follows:
\[ \Delta L_{GF} = -\frac{1}{2\alpha} \left| (\partial_\mu + ig A_\mu^3) (A_\mu^1 + i A_\mu^2) \right|^2 - \frac{1}{2\beta} \left( \partial_\mu A_\mu^3 \right)^2 , \]
where \( \alpha \) and \( \beta \) are the gauge parameters. The former term is responsible for the gauge
condition (7), the latter term is needed to fix the remaining \( U(1) \) gauge arbitrariness. The
case \( \alpha = \beta = 1 \) corresponds to the case \( \xi = 1 \) from article [4].

The BRST invariant Lagrangian of the Yang–Mills field in the maximal Abelian gauge
has the form
\[ \mathcal{L} = \frac{1}{4} f_{\mu\nu} f_{\mu\nu} + \frac{1}{2\beta} (\partial_\mu a_\mu)^2 + \frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{1}{2\alpha} (\partial_\mu A_\mu^a)^2 \\
- i \kappa \bar{C}^3 \partial^2 C^3 - i \bar{C}^a \partial^2 C^a - \frac{\alpha}{4} g^2 \varepsilon^{ab} \varepsilon^{cd} \bar{C}^a C^b C^c C^d \\
+ \frac{1}{2} g \varepsilon^{ab} (a_\mu A^b_\mu \partial_\nu A^a_\mu - a_\mu A^a_\mu \partial_\nu A^b_\mu + A^a_\mu \partial_\nu A^b_\mu a_\nu - a_\mu A^b_\mu \partial_\nu A^a_\mu \\
+ a_\mu A^a_\mu \partial_\nu A^b_\mu + \frac{1}{\alpha} a_\mu A^a_\mu \partial_\nu A^b_\mu + \frac{1}{\alpha} a_\mu A^b_\mu \partial_\nu A^a_\mu) \\
- ig \varepsilon^{ab} a_\mu (C^b \partial_\mu C^a - \bar{C}^a \partial_\mu C^b) - i g \kappa \varepsilon^{ab} \bar{C}^3 \partial_\mu (A^a_\mu C^b) \\
+ \frac{1}{4} g^2 \delta^{ab} \left( 2 \delta_{\mu\nu} \delta_{\rho\sigma} - (1 - \frac{1}{\alpha}) (\delta_{\mu\rho} \delta_{\nu\sigma} + \delta_{\mu\sigma} \delta_{\nu\rho}) \right) A^a_\mu A^b_\nu a_\rho a_\sigma \\
+ \frac{1}{4} g^2 (\delta_{\mu\nu} \delta_{\rho\sigma} - \delta_{\mu\rho} \delta_{\nu\sigma}) A^a_\mu A^b_\nu A^a_\rho A^b_\sigma \\
+ ig \varepsilon^{ab} \partial_\mu C^a C^b a_\mu a_\nu + ig^2 (\varepsilon^{ad} \varepsilon^{cb} + \varepsilon^{ac} \varepsilon^{db}) \bar{C}^a C^b A^c_\mu A^d_\mu , \]
where the gluon field \( A^a_\mu \) is divided into the ”Abelian” and ”non-Abelian” components as follows:
\[ A^a_\mu T^A = A^a_\mu T^a + a_\mu T^3 , \]
where \( T^A \) are the \( SU(2) \) generators; indices denoted by capital letters are assigned one of
the integers 1, 2, 3 and indices denoted by small letters—1, 2. The gluon fields with the indices
denoted by small letters are named ”off-diagonal” and the Abelian component \( a_\mu \) defined
by the equation (11) is named the field of the ”diagonal” gluon. Let us also introduce the notation:
\[ \tilde{F}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu , \]
\[ f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu . \]
A detailed discussion of the parameter \( \kappa \) can be found in [6], here we set \( \kappa = 1 \).

3 Feynman rules

The expressions for the gluon and ghost propagators and three- and four-particle vertices
are readily obtained by standard techniques. The results are shown in the table
### Propagators

<table>
<thead>
<tr>
<th>Propagators</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>\langle a_\mu(p)a_\nu(-p) \rangle</td>
<td>\frac{1}{p^2} \left( \delta_{\mu\nu} - \frac{1}{3} \frac{p_\mu p_\nu}{p^2} \right)</td>
</tr>
<tr>
<td>\langle A^a_\mu(p)A^b_\nu(-p) \rangle</td>
<td>\frac{\delta^{ab}}{p^2} \left( \delta_{\mu\nu} - \frac{1}{3} \frac{p_\mu p_\nu}{p^2} \right)</td>
</tr>
<tr>
<td>\langle C^a(p)C^a(-p) \rangle</td>
<td>- \frac{i}{p^2}</td>
</tr>
<tr>
<td>\langle \bar{C}^a(p)C^b(-p) \rangle</td>
<td>-i \frac{\delta^{ab}}{p^2}</td>
</tr>
</tbody>
</table>

### Three-particle vertices

<table>
<thead>
<tr>
<th>Three-particle vertices</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>\langle a_\mu(p)A^a_\rho(q)A^b_\sigma(r) \rangle</td>
<td>-ig \varepsilon^{ab} \left( (q - r)<em>\mu \delta</em>{\rho\sigma} + (r - p + \frac{2}{\alpha})<em>\rho \delta</em>{\sigma\mu}</td>
</tr>
</tbody>
</table><p>ight) |
| \langle a_\mu(p)A^b_\nu(q)A^c_\rho(r) \rangle | \frac{g}{2} \left( \varepsilon^{abc} \varepsilon_{\mu\nu\rho\sigma} + \alpha \delta_{\mu\nu} \delta_{\rho\sigma} \right) |
| \langle C^a(p)C^b(q)a_\mu \rangle | g \left( \varepsilon^{ab} \varepsilon^{cd} \alpha \right) |</p>

### Four-particle vertices

<table>
<thead>
<tr>
<th>Four-particle vertices</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>\langle a_\mu a_\nu A^a_\rho A^b_\sigma \rangle</td>
<td>-ig \varepsilon^{ab} \left( \varepsilon^{cd} \varepsilon_{\mu\nu\rho\sigma} + \varepsilon^{ac} \varepsilon_{\mu\nu\rho} + \varepsilon^{ad} \varepsilon_{\mu\nu\sigma} \right)</td>
</tr>
<tr>
<td>\langle A^a_\mu A^b_\nu A^c_\rho A^d_\sigma \rangle</td>
<td>\frac{g^2}{16} \left( \varepsilon^{abc} \varepsilon_{\mu\nu\rho\sigma} + \alpha \delta_{\mu\nu} \delta_{\rho\sigma} \right)</td>
</tr>
</tbody>
</table>

It should be noted that the "diagonal" ghost \( C^3 \) interacts with no field, whereas the "diagonal" anti-ghost \( \bar{C}^3 \) interacts with some fields. Therefore, the "diagonal" ghosts should be disregarded in the loop expansion of perturbation theory.

## 4 Gluon Propagator to One-Loop

The diagrams contributing to the polarization operator of the "diagonal" gluon are shown in Fig. (1).

\[
\Pi^{33}_{\mu\nu}(p) = \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2} \\
\text{Diagram 3} \\
\text{Diagram 4}
\end{array}
\]

Figure 1: Polarization operator of the "diagonal" gluon.

Our computations are performed using the dimensional regularization techniques with the space-time dimension \( D = 4 - 2\epsilon \); in so doing, the contribution of the tadpole diagrams vanishes.

The contribution of the non-vanishing diagrams to the gluon polarization operator has the form

\[
\Pi^{33}_{\mu\nu} = \frac{g^2}{16\pi^2} \left( \frac{22}{3} L + \left( \frac{205}{18} + 3\alpha + \frac{\alpha^2}{2} \right) \right),
\] (10)
where
\[ L = \frac{1}{\epsilon} - \gamma_E + \ln\left( \frac{4\pi\mu^2}{p^2} \right) \] (11)
and \( \gamma_E \) is the Euler constant.

Performing analogous computations for the ”off-diagonal” gluon (see diagrams in Fig. (2)), we arrive at

\[ \Pi_{\mu\nu}^{ab}(p) = \begin{array}{c}
\begin{array}{c}
\text{Diagram A} \\
\begin{array}{c}
\text{Diagram B} \\
\begin{array}{c}
\text{Diagram C} \\
\text{Diagram D}
\end{array}
\end{array}
\end{array}
\end{array} \]

Figure 2: Polarization operator of the ”off-diagonal” gluon.

\[ \Pi_{\mu\nu}^{ab}(p) = -\delta^{ab} \frac{g^2}{16\pi^2} \left( (\delta_{\mu\nu}p^2 - p_\mu p_\nu)T^{off}(p^2) + p_\mu p_\nu L^{off}(p^2) \right), \] (12)

where
\[ T^{off}(p^2) = \left(-\frac{17}{6} + \beta + \frac{\alpha}{2}\right) L + \left(-\frac{43}{18} + \frac{\beta}{2} - \frac{\alpha\beta}{2} - \frac{\alpha}{2}\right); \] (13)
\[ L^{off}(p^2) = \left(-\frac{1}{2} + \frac{\beta}{\alpha} - \frac{3}{2\alpha} - \frac{3}{\alpha^2}\right) L + \frac{1}{2\alpha} \left(-7 + 3\beta - \frac{3\beta}{\alpha} - \frac{5}{\alpha}\right). \]

The expressions (10) and (12) involve both divergent and finite parts (the divergent part was computed in [6]). The finite part of the longitudinal component of the ”off-diagonal” propagator (12) involves terms proportional to \( \frac{1}{\alpha} \) and \( \frac{1}{\alpha^2} \), which are singular when \( \alpha \to 0 \). However, they give no contribution to the expressions for physical quantities. Their presence in the expression for the propagator is due to the fact that the interaction vertex in this gauge is proportional to \( \frac{1}{\alpha} \), where \( \alpha \) is the gauge parameter that appears in the expression for the unperturbed propagator. For this reason, \( \alpha \to 0 \) corresponds to very dangerous limit and thus it is safe in the perturbation expansion to assume a nonzero value of \( \alpha \).

In the \( \overline{\text{MS}} \) subtraction scheme, the counterterms added to the Lagrangian used to compensate for the divergent terms are as follows:

\[ \Delta L_{\text{counter}} = \frac{(Z_A^{(3)})^2 - 1}{4} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + \frac{X^{(3)}}{2} (\partial_\mu A^a_\mu)^2 + \frac{(Z_A^{(3)})^2 - 1}{4} (\partial_\mu a^a_\nu - \partial_\nu a^a_\mu)^2 + \frac{X^{(3)}}{2} (\partial_\mu a^a_\mu)^2, \] (14)
where
\[ Z_A^{(3)} = 1 + \frac{11}{3} \frac{g^2}{16\pi^2\epsilon'}, \]
\[ Z_A^\text{off} = 1 + \frac{g^2}{16\pi^2\epsilon'} \left( \frac{17}{6} - \frac{\alpha}{2} \right), \]
\[ X^{(3)} = 0, \]
\[ X^\text{off} = \frac{g^2}{16\pi^2\epsilon'} \left( \frac{1}{2} - \frac{\beta}{\alpha} + \frac{3}{\alpha^2} + \frac{3}{2\alpha} \right), \]
and the modified parameter $\epsilon'$ is defined by the formula
\[ \frac{1}{\epsilon'} = \frac{1}{\epsilon} - \gamma_E + \ln(4\pi). \]

5 The renormalization group equation

5.1 The case of Lorentz gauge

First we consider the renormalization group equation in the Lorentz gauge in order to demonstrate some features of its solution in the maximal Abelian gauge. The Lorentz gauge can be specified by the gauge fixing term in the Lagrangian as follows:

\[ \Delta L_{GF} = \frac{1}{2\alpha} (\partial_\mu A^A_\mu)^2. \]

The gluon propagator in this gauge is parametrized by only one scalar function
\[ G^{AB}_{\mu\nu}(p) = \frac{\delta^{AB}}{p^2} \left( p_\mu p_\nu - p^2 \frac{\eta_{\mu\nu}}{p^2} \right) G^{\text{LOR}}(\frac{p^2}{\mu^2}, g, \alpha) + \alpha \frac{p_\mu p_\nu}{p^2}. \]

The renormalization group equation has the form
\[ \left( \mu^2 \frac{\partial}{\partial \mu^2} + \beta(g) \frac{\partial}{\partial g} + \delta(g) \frac{\partial}{\partial \alpha} + 2\gamma(g, \alpha) \right) G^{\text{LOR}}(\mu, \alpha, g) = 0, \]
where
\[ \beta(g) = \mu^2 \frac{\partial g}{\partial \mu^2}, \quad \gamma(g) = \frac{\mu^2}{Z_A} \frac{dZ_A}{\mu^2}, \quad \delta(g) = \mu^2 \frac{\partial \alpha}{\partial \mu^2}. \]

The gauge parameter $\alpha$ is considered as yet another coupling constant.

In the dimensional regularization approach, the renormalization group functions $\beta(g)$ and $\gamma(g)$ are determined from the relations
\[ \beta(g) = \frac{1}{2} \left( g \frac{da_1}{dg} - a_1 \right), \quad \gamma(g) = -\frac{1}{2} g \frac{dc_1}{dg}. \]
where $a_1$ and $c_1$ are the coefficients of $\frac{1}{\epsilon}$ in the $\frac{1}{\epsilon}$-expansion of the renormalization factors $Z_A$ and $Z_g$:

$$Z_g = 1 + \sum_{n=0}^{\infty} \frac{a_n(g)}{\epsilon^n}$$

$$Z_A = 1 + \sum_{n=0}^{\infty} \frac{c_n(g)}{\epsilon^n}.$$  (21)

In the one-loop approximation, the $\beta$ function is independent of gauge and subtraction scheme. For the $SU(2)$ gauge field it is given by

$$\beta(g) = -bg^3 = -\frac{11}{3} \frac{g^3}{16\pi^2}. \quad \text{(22)}$$

In the Lorentz gauge, there are no contributions to the longitudinal component of the gluon polarization operator and there is no counterterms to the gauge fixing term in the Lagrangian. From this observation we obtain the relation between the renormalization factors, $Z_A^2 = Z_\alpha$, that determines the dependence of the gauge parameter $\alpha$ on the normalization point. In the case of the Landau gauge ($\alpha = 0$) one can consider $\delta(g) = 0$ (see [9]); in the case $\alpha \neq 0$, the function $\delta(g)$ is not trivial.

Here we solve the equation (18) for $\gamma = cg^2$, $\beta = -bg^3$. First we note that, when $\gamma(g) = 0$, any function $F(z)$ with $z = \frac{1}{g^2} + 2b \ln \frac{p^2}{\mu^2}$ provides a solution of the equation (18). This function is specified by the initial (or boundary) conditions. That is, if the sought-for function is given over a line transversal to characteristic curves of the equation then the function $F(z)$ can be determined and thus the sought-for function is known over the $(g, \frac{p^2}{\mu^2})$ plane.

A useful method of solution of the equations of the type (18) with $\gamma(g) \neq 0$ is based on a treatment of the solutions of the respective auxiliary equation

$$\left( -\frac{\partial}{\partial \lambda} + \beta(g) \frac{\partial}{\partial g} + \delta(g) \frac{\partial}{\partial \alpha} - 2\gamma(g, \alpha) G^{LOR} \frac{\partial}{\partial G^{LOR}} \right) V(\lambda, \alpha, g, G^{LOR}) = 0, \quad \text{(23)}$$

where $\lambda = \ln \frac{p^2}{\mu^2}$. Then the sought-for function $G^{LOR}(\lambda, g, \alpha)$ is determined from the condition

$$V(\lambda, \alpha, g, G^{LOR}) = 0, \quad \text{(24)}$$

where $V(\lambda, \alpha, g, G^{LOR})$ is any solution of the auxiliary equation (23). The solution of the auxiliary equation (23) is provided by any function which is constant along each characteristic curve. The characteristic curves are defined by the equation

$$\frac{d\lambda}{1} = \frac{dg}{\beta(g)} = \frac{d\alpha}{\delta(g, \alpha)} = -\frac{dG^{LOR}}{2\gamma(g, \alpha)G^{LOR}}. \quad \text{(25)}$$

Integration of these equations gives

$$\frac{1}{g^2} + \frac{11}{24\pi^2} \lambda = C_1, \quad \text{(26)}$$
\[
g^{-\frac{\mu}{1+\mu}} \frac{13-3\alpha}{3\alpha} = C_2, \tag{27}
\]

\[
\frac{G_{\text{LOR}}}{\alpha} = C_3. \tag{28}
\]

Each of these equations defines three-dimensional hypersurface and a characteristic curve is an intersection of these hypersurfaces. Thus the solution of the equation (23) is given by

\[
V(\lambda, \alpha, g, G_{\text{LOR}}) = v\left(\frac{G_{\text{LOR}}}{\alpha}, g^{-\frac{\mu}{1+\mu}} \frac{13-3\alpha}{3\alpha}, \frac{1}{g^2} + \frac{11}{24\pi^2} \lambda\right), \tag{29}
\]

where \(v(x, y, z)\) is an arbitrary sufficiently smooth function.

![Figure 3: Projection of characteristic curves onto the \((\alpha, g)\) plane.](image)

The condition (24) implies that

\[
G_{\text{LOR}} = \alpha \Phi\left(g^{-\frac{\mu}{1+\mu}} \frac{13-3\alpha}{3\alpha}, \frac{1}{g^2} + \frac{11}{24\pi^2} \lambda\right). \tag{30}
\]

Formula (30), where \(\Phi(y, z)\) is an arbitrary function, represents a general solution of the renormalization group equation in the Lorentz gauge. A specific form of the function \(\Phi(y, z)\) (a particular solution) is determined from the boundary conditions; it is natural to fix them at the hyperplane \(\lambda = 0\). The behavior of the characteristic curves makes it possible to determine the asymptotic behavior of the propagator for \(p^2 \to \infty\) and some fixed values of \(g\) and \(\alpha\) provided that its behavior for \(g \to 0\), \(\alpha \to 13/3\), and some fixed value of \(\mu\) (say, \(\mu^2 = p^2\)) is known. In the latter domain, perturbation theory works and thus the condition needed for the evaluation of the propagator at large momenta is fulfilled. In the leading order of perturbation theory, \(G = 1\) and higher-order corrections are small. For \(\lambda = 0\), we obtain the equation

\[
\Phi(y, z) = \frac{1}{\alpha^y}, \tag{31}
\]

which can be used for the determination of the dependence of \(\Phi\) on \(y\) and \(z\). On this hyperplane, \(y = g^{-\frac{\mu}{1+\mu}} \frac{13-3\alpha}{3\alpha}\) and \(z = \frac{1}{g^2}\) and, therefore, \(\alpha = \frac{13}{3 \left(1 + yz^{-\frac{\mu}{1+\mu}}\right)}\).
Thus we arrive at
\[ \Phi(y, z) = \frac{3}{13} \left( 1 + y z \left( -\frac{13}{22} \right) \right), \quad (32) \]
\[ G^{\text{LOR}} \left( \frac{p^2}{\mu^2}, g, \alpha \right) = \frac{3\alpha}{13} + \left( 1 - \frac{3\alpha}{13} \right) \left( 1 + \frac{11g^2}{24\pi^2 \ln \left( \frac{p^2}{\mu^2} \right)} \right) \left( -\frac{13}{22} \right). \quad (33) \]

Now we should take into account the scale dependence of the renormalized quantities \( \alpha \) and \( g \). The expression for the running coupling has the form
\[ g^2 = \frac{24\pi^2}{11 \ln \left( \frac{\mu^2}{\Lambda^2} \right)}. \quad (34) \]

In the Lorentz gauge, the scale dependence of the gauge parameter \( \alpha \) (in other words, the dependence of the normalization point) is governed by the normalization condition as follows: the propagator at \( p^2 = \mu^2 \) must have the form
\[ G^{AB \mu \nu}_\mu(p) = \frac{\delta^{AB}}{p^2} \left( \left( g_{\mu \nu} - \frac{p_\mu p_\nu}{p^2} \right) + \alpha_0 \frac{p_\mu p_\nu}{p^2} \right). \quad (35) \]

From the equations for the characteristic curves and the expression (34) for the running coupling we obtain
\[ \alpha = \frac{13C \ln \left( \frac{\mu^2}{\Lambda^2} \right)^{(13/22)}}{3 \left( 1 + C \ln \left( \frac{\mu^2}{\Lambda^2} \right)^{(13/22)} \right)}, \quad (36) \]
where the constant \( C \) is determined from the normalization condition (35):
\[ C = \frac{3\alpha_0}{(13 - 3\alpha_0) \ln \left( \frac{p^2}{\Lambda^2} \right)^{13/22}}. \quad (37) \]

Thus the scale dependence of the gauge-fixing parameter is given by
\[ \alpha(\mu) = \frac{13\alpha_0 S}{13 - 3\alpha_0 (1 - S)}, \quad S = \left( \frac{\ln \left( \frac{\mu^2}{\Lambda^2} \right)}{\ln \left( \frac{p^2}{\Lambda^2} \right)} \right)^{13/22}. \quad (38) \]

The dependence of the propagator on the normalization point and the momentum is obtained by substituting the expressions for the running gauge-fixing parameter (38) and running coupling (34) into formula (33). After such substitution, we arrive at
\[ G^{\text{LOR}}(S) = \frac{13S}{13 - 3\alpha_0 (1 - S)}. \quad (39) \]
5.2 The case of maximal Abelian gauge

As contrasted from the case of Lorentz gauge, the gluon propagator in the maximal Abelian gauge cannot be represented in the form \((17)\); it can be parametrized by three functions as follows: \(G^{\text{off}}_L\), \(G^{\text{off}}_T\), and \(G^{(3)}\):

\[
G^{33}_{\mu\nu}(p) = \frac{1}{p^2} \left( \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) G^{(3)} + \hat{\beta} \frac{p_\mu p_\nu}{p^2} \right),
\]

\[
G^{ab}_{\mu\nu}(p) = \frac{\delta^{ab}}{p^2} \left( \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) G^{\text{off}}_T + \alpha \frac{p_\mu p_\nu}{p^2} G^{\text{off}}_L \right).
\]

These functions are not independent because there should be relations between them resulting from the Slavnov–Taylor identities.

The renormalization group equation for the one-particle irreducible Green’s function has the form

\[
\left( \mu^2 \frac{\partial}{\partial \mu^2} + \beta(g) \frac{\partial}{\partial g} + \delta_\alpha(g) \frac{\partial}{\partial \alpha} + \delta_\beta(g) \frac{\partial}{\partial \hat{\beta}} + 2\gamma \right) G_2(\mu, \alpha, \hat{\beta}, g) = 0,
\]

where \(G_2\) is one of the functions \(G^{\text{off}}_L\), \(G^{\text{off}}_T\), or \(G^{(3)}_T\) and the renormalization group functions are defined by the equations

\[
\beta(g) = \mu^2 \frac{\partial g}{\partial \mu^2}, \quad \gamma(g) = \mu^2 \frac{\partial \ln Z_A}{\partial \mu^2},
\]

\[
\delta_\alpha(g) = \mu^2 \frac{\partial \alpha}{\partial \mu^2}, \quad \delta_\beta(g) = \mu^2 \frac{\partial \hat{\beta}}{\partial \mu^2}.
\]

In the one-loop approximation, we obtain

\[
\beta(g) = -bg^3,
\]

\[
\gamma^{(3)}(g) = c^{(3)} g^2 = -\frac{11}{3} \frac{g^2}{16\pi^2},
\]

\[
\gamma^{\text{off}}(g) = c^{\text{off}} g^2 = -\frac{(17 - 3\alpha - 6\hat{\beta})}{12} \frac{g^2}{16\pi^2},
\]

\[
\delta_\alpha(g) = -\frac{(3\alpha^2 - 4\alpha + 9)}{3} \frac{g^2}{16\pi^2},
\]

\[
\delta_\beta(g) = \frac{22}{3} \frac{\hat{\beta}}{16\pi^2} \frac{g^2}{16\pi^2}.
\]

The set of simultaneous equations defining the characteristic curves for the equation \((42)\) can be represented in a symmetric form,

\[
\frac{d\lambda}{-1} = \frac{dg}{\beta(g)} = \frac{d\alpha}{\delta_\alpha} = \frac{d\hat{\beta}}{\delta_\beta} = \frac{dG_2}{-2\gamma G_2}.
\]

Now we solve these equations for the Green’s function of the “diagonal” gluon. Integra-
\[ I_1 = 2b\lambda + \frac{1}{g^2}, \]
\[ I_2 = g^2\hat{\beta}, \]
\[ I_3 = \arctan \left( \frac{3\alpha - 2}{\sqrt{23}} \right) - \frac{\sqrt{23}}{11} \ln g, \]
\[ I_4 = \frac{G_T^{(3)}}{\hat{\beta}}. \]

Thus the general solution of the equation (42) has the form

\[ G_T^{(3)} = \hat{\beta} \Psi(I_3, I_2, I_1), \]  

where \( \Psi(x, y, z) \) is an arbitrary function. The solution of the equation (42) for \( G_L^{(3)} \) is the same. A particular solution is obtained by a determination of the specific form of the function \( \Psi(x, y, z) \) from the boundary condition. It is convenient to specify the boundary conditions on the plane \( p^2 = \mu^2 (\lambda = 1) \) assuming that the perturbative expansion for \( G_T^{(3)} \) is valid at small \( g \). In the tree approximation, \( G_T^{(3)} = 1 \) and, therefore,

\[ \Psi(x, y, z) = \frac{1}{yz}. \]  

From this formula it follows that

\[ G_T^{(3)} = \frac{1}{1 + \frac{22}{3} \frac{g^2}{16\pi^2} \ln \frac{p^2}{\mu^2}}. \]  

Taking the scale dependence of \( g^2 \) into account yields

\[ G_T^{(3)} = \frac{\ln (\mu^2/\Lambda^2)}{\ln (p^2/\Lambda^2)}. \]
The scale dependence of the parameter $\beta$ can be determined from the equations for the characteristic curves and formula (34), the result is
\[
\hat{\beta} = \frac{11 C_\beta}{24 \pi^2} \ln \left( \frac{\mu^2}{\Lambda^2} \right). \tag{50}
\]

The parameter $C_\beta$ is determined from the normalization condition $\hat{\beta} = \hat{\beta}_0$ at $p^2 = \mu^2, L^+$
\[
\hat{\beta} = \hat{\beta}_0 \frac{\ln(p^2/\Lambda^2)}{\ln(\mu^2/\Lambda^2)}. \tag{51}
\]

The expression for the propagator for the "diagonal" gluon in the maximal Abelian gauge has a simpler form than the analogous expression in the Lorentz gauge. It can be accounted for by the remaining $U(1)$ symmetry in the maximal Abelian gauge, which implies the relation $Z_g = Z_\alpha^{-1/2}$ between the renormalization factors. Therefore, the functions $\beta(g)$ and $\gamma^{(3)}(g)$ are connected with each other by the formula $\beta(g) = g \gamma^{(3)}(g)$.

Now we solve the equation (42) for the transverse component of the Green’s function in the case of "off-diagonal" gluon. First integrals $I_1, I_2, I_3$ of the system (45) are the same as in the previous case. The fourth integral is given by
\[
I_4 = G_{Tg}^{\text{off}} g^{15/22} e^{\frac{3}{22} (3\alpha^2 - 4\alpha + 9)^{-\frac{1}{4}}}. \tag{52}
\]

Thus the solution of the equation (42), which is valid to both $G_{Tg}^{\text{off}}$ and $G_{Lg}^{\text{off}}$, takes the form
\[
G_{Tg}^{\text{off}} = g^{15/22} e^{\frac{3}{22} (3\alpha^2 - 4\alpha + 9)^{1/4}} \Phi(I_3, I_2, I_1), \tag{53}
\]

where $\Phi(x, y, z)$ is an arbitrary function, whose specific form is governed by the boundary conditions. For the sake of convenience, we set the boundary conditions on the plane $p^2 = \mu^2 (\lambda = 1)$. It is assumed that, at small values of $g$, the perturbation theory expansion for the function $G_{Tg}^{\text{off}}$ is valid. In the tree approximation (that is, to zeroth order in $g$), we obtain $G_{Tg}^{\text{off}} = 1$ and
\[
\Phi(x, y, z) = z^{-15/44} \exp\left(\frac{3}{22} yz\right) \left| \frac{3}{23} \cos \left( x - \frac{\sqrt{23}}{22} \ln z \right) \right|^{\frac{1}{4}} \tag{54}
\]

Therefore,
\[
G_{Tg}^{\text{off}} \left( \frac{p^2}{\mu^2}, g, \alpha, \hat{\beta} \right) = \left( \frac{p^2}{\mu^2} \right)^{g^2 \hat{\beta}/16\pi^2} \xi^{15/44} \left| \cos \xi + \frac{3\alpha - 2}{\sqrt{23}} \sin \xi \right|^{1/2}, \tag{55}
\]

where
\[
\xi = \frac{\sqrt{23}}{22} \ln \zeta, \quad \zeta = \left( 1 + \frac{11g^2}{24\pi^2} \ln \left( \frac{p^2}{\mu^2} \right) \right). \tag{56}
\]

The $\mu$ dependence of the parameters $\alpha$ and $\hat{\beta}$ is determined by the scale dependence of the running coupling (we restrict our attention to the one-loop approximation) and the values of the integration constants $I_2$ and $I_3$ in formulas (45). A specification of these
values is equivalent to specification of the normalization conditions, which, in the case under consideration, is as follows: the propagator at $p^2 = \mu^2$ has the form

$$G_{\mu\nu}^{ab} = -i\delta^{ab} \left( \frac{1}{p^2} \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) + \hat{\alpha}_0 \frac{p_\mu p_\nu}{p^4} \right).$$

(57)

In view of the relation (54), we obtain the equation

$$\zeta = \frac{\ln(p^2/\Lambda^2)}{\ln(\mu^2/\Lambda^2)}$$

(58)

giving the dependence of $\xi$ on $p^2$. Though it is a straightforward matter to derive the dependence of $\hat{\beta}$ on $p^2$, the $p^2$ dependence of $\alpha$ is derived in a more intricate way. To do this, we consider the factor

$$\left| \cos\xi + \frac{3\alpha - 2}{\sqrt{23}} \sin\xi \right|^{1/2}$$

and introduce the angle $\varphi = \arctan \frac{3\alpha - 2}{\sqrt{23}}$. Then we obtain

$$\left| \cos\xi + \frac{3\alpha - 2}{\sqrt{23}} \sin\xi \right| = \left| \frac{\cos(\xi - \varphi)}{\cos \varphi} \right|.$$  

(59)

From formula (45) it follows that

$$\varphi = I_3 + \frac{\sqrt{23}}{22} \ln g^2 \quad (I_3 \text{ is the integration constant independent of } \mu).$$

Taking the scale dependence of the coupling constant into account, we arrive at

$$\varphi(\mu^2) = I_3 - \frac{\sqrt{23}}{22} \ln \left( \frac{11}{24\pi^2} \ln(\mu^2/\Lambda^2) \right),$$

(60)

where the constant $I_3$ is determined from the normalization condition at $p^2 = \mu^2$:

$$\varphi(p^2) = \varphi_0 = I_3 - \frac{\sqrt{23}}{22} \ln \left( \frac{11}{24\pi^2} \ln(p^2/\Lambda^2) \right).$$

(61)

Thus we obtain

$$\varphi(\mu^2) = \varphi_0 + \frac{\sqrt{23}}{22} \ln \left( \frac{\ln(p^2/\Lambda^2)}{\ln(\mu^2/\Lambda^2)} \right)$$

(62)

and $\xi - \varphi = -\varphi_0$. From there formulas we derive the asymptotic behavior of the "off-diagonal" propagator:

$$G_{\text{off}} = \left( \frac{p^2}{\mu^2} \right)^{-\frac{3\alpha_0}{22 \ln(p^2/\Lambda^2)}} \left( \frac{\ln(p^2/\Lambda^2)}{\ln(\mu^2/\Lambda^2)} \right)^{-\frac{1}{2}} \left| \cos \left( \frac{3\alpha_0 - 2}{\sqrt{23}} \ln \left( \frac{\ln(p^2/\Lambda^2)}{\ln(\mu^2/\Lambda^2)} \right) - \varphi_0 \right) \cos \varphi_0 \right|^{-1/2}.$$

(63)

The parameter $\varphi_0$ in this formula is determined by the relation $\varphi_0 = \arctan \frac{3\alpha_0 - 2}{\sqrt{23}}$, where $\alpha_0$ is the gauge-fixing parameter at $p^2 = \mu^2$ (see (57)).

6 Discussion and conclusions

The obtained expression (63) is singular at the values of momentum given by

$$m(k) = \Lambda \left( \frac{\mu}{\Lambda} \right)^{\exp \left\{ \frac{3\alpha_0}{2\sqrt{23} \left( \frac{4}{\pi} + \varphi_0 + \pi k \right)} \right\}},$$

(64)
where \( k \) is an integer. An infinite number of singular points emerges both when \( k \to -\infty \) \((m(k) \to \Lambda)\) and when \( k \to \infty \) \((m(k) \to \infty)\). In the neighborhood of a singular point, the propagator behaves as \( G^{off}_{\mathcal{T}} \sim \frac{1}{\sqrt{p^2 - m^2(k)}} \). Neither such behavior nor the existence of the singularities of a propagator in the Euclidean domain has a simple physical interpretation. Such points are probably artifacts of the normalization condition used here, one-loop approximation, and the maximal Abelian gauge in the continuum limit. It should be noted that a mathematically rigorous (say, as in [11]) consideration of the question of whether this normalization condition is admissible can be the subject of a separate study. It should be emphasized that the maximal Abelian gauge in the continuum limit, that is, the condition

\[
(\partial_\mu + igA^3_\mu)(A^1_\mu + iA^2_\mu) = 0
\] (65)

provides a relation between the fields of the order \( A_\mu \sim 1 \) with the fields of the order \( A_\mu \sim \frac{1}{g} \), which may cause difficulties in computations in perturbation theory. The emergence of the terms \( \frac{1}{\alpha} \) in the interaction Lagrangian is only one of manifestations of this relation. Such terms emerge in the gauge-fixing part of the Lagrangian in a quite natural way.

For this reason, the maximal Abelian gauge is essentially nonperturbative even in the domain of the small values of coupling constant. This conclusion is consistent with the renormalization-group analysis of the propagator behavior at the one-loop level. All the above implies that the asymptotic behavior of the gluon propagator evaluated in this gauge on the lattice, where nonperturbative contributions are naturally taken into account, may substantially diverge from the results of the renormalization-group improved computations in the framework of perturbation theory performed in the present study.

The propagator of the "diagonal" gluon evaluated on a lattice in the domain of large momenta can be adequately parametrized by formula (49) with an appropriate choice of the parameters \( \Lambda, \mu \) and \( \beta_0 \).

The behavior of the "off-diagonal" propagator diverges substantially from our results. The reason is as follows:

1. There is no one by one correspondence between lattice MaG and perturbative MaG gauge;

2. Renormalization-group improved computations do not take into account nonperturbative contributions, which are naturally taken into account in the lattice calculations.

However, a comprehensive comparison of the renormalization-group improved perturbative behavior of the propagator with the behavior obtained numerically in a lattice gauge theory must include a consideration of the case when the gauge-fixing parameter \( \alpha \) equals zero. In this case, the quantization procedure in the framework of the path-integral approach involves introducing an auxiliary field. A treatment of the gluon propagator at \( \alpha = 0 \) may form the subject of a subsequent study.

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