Odderon in the Color Glass Condensate

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Abstract

We discuss the definition and the energy evolution of scattering amplitudes with \(C\)-odd ("odderon") quantum numbers within the effective theory for the Color Glass Condensate (CGC) endowed with the functional, JIMWLK, evolution equation. We explicitly construct gauge-invariant amplitudes describing multiple odderon exchanges in the scattering between the CGC and two types of projectiles: a color–singlet quark–antiquark pair (or ‘color dipole’) and a system of three quarks in a colorless state. We deduce the energy evolution of these amplitudes from the general JIMWLK equation, which for this purpose is recast in a more synthetic form, which is manifestly infrared finite. For the dipole odderon, we confirm and extend the non–linear evolution equations recently proposed by Kovchegov, Szymanowski and Wallon, which couple the evolution of the odderon to that of the pomeron, and predict the rapid suppression of the odderon exchanges in the saturation regime at high energy. For the 3–quark system, we focus on the linear regime at relatively low energy, where our general equations are shown to reduce to the Bartels–Kwiecinski–Praszalowicz equation. Our gauge–invariant amplitudes, and the associated evolution equations, stay explicitly outside the Möbius representation, which is the Hilbert space where the BFKL Hamiltonian exhibits holomorphic separability.

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1 Introduction

Since the advent of the Balitsky–Fadin–Kuraev–Lipatov (BFKL) equation [1,2] in the mid seventies, there has been significant progress in our comprehension of high–energy QCD, and several theoretical approaches have been proposed which aim at a resummation of the energy–enhanced radiative corrections to high–energy processes in perturbative QCD. The BFKL equation is a leading logarithmic approximation (LLA), which allows one to resum to all orders corrections of the form \((\alpha_s \ln s)^n\) to the scattering between two colorless objects via the exchange of two gluons in the \(t\)–channel. As a result of this resummation, the bare two–gluon exchange is replaced by the BFKL pomeron (the sum of an infinite series of ladder diagrams of ordinary perturbation theory), or, equivalently, by two reggeized gluons which interact with each other.

All the subsequent theoretical approaches proposed within perturbative QCD encompass the BFKL equation, and can be viewed as extensions of the latter towards increasing the complexity of the objects exchanged in the \(t\)–channel, and also towards enlarging the limits of the LLA.

The simplest object beyond the BFKL pomeron within perturbative QCD is the exchange of three interacting (reggeized) gluons in a symmetric color state. This object, which is negative (or “\(C\)–odd”) under charge conjugation \((C = -1)\), represents the lowest order perturbative contribution to the odderon, the \(C\)–odd exchange which dominates the difference between the hadronic cross sections for direct and crossed channel processes at very high energies [3].

The evolution of the three–gluon odderon exchange with increasing energy in the LLA is described by the BKP equation, established by Bartels [4] and Kwiecinski and Praszalowicz [5], which amounts to a pairwise iteration of the BFKL kernel (see also [6]). This equation can be immediately extended to describe the exchange of an arbitrary number \(n \geq 3\) of reggeized gluons with pairwise BFKL interactions [7,4,8,9,10]. The resulting formalism, also known as the generalized leading logarithmic approximation (GLLA), resums all radiative corrections that involve the maximally possible number of energy logarithms \(\ln s\) for a given number of exchanged gluons. At the moment, two exact solutions of the BKP equation for odderon evolution are available [11,12], and the subject continues to be under intensive debate [13,14] (see also the recent review paper [15] and the discussion below).

In the formalisms described so far, the number of gluons in the \(t\)–channel remains fixed in the course of the evolution. This is probably a good approximation in some intermediate kinematical region, but it fails to describe two interesting physical situations: First, it does not incorporate correctly the fluctuations in the number of gluons, as resulting from processes in which one (reggeized) gluon splits into two, or, more generally, a \(n\)–gluon state evolves into a \((n + m)\)–one, with \(m \geq 1\). Such processes are especially important...
in the dilute regime at relatively large transverse momenta (for a given energy), where gluon splitting is the main process through which higher–point correlations get built [16]. Second, the approximation in which the number of t–channel gluons is fixed cannot describe recombination processes in which (reggeized) gluons merge with each other, thus reducing the gluon density. Such processes are important in the high–energy regime where the gluon density becomes large enough (due to BFKL evolution and to the splitting processes alluded to above) to enhance recombination processes, which are then expected to lead to gluon saturation [17,18,19]. The inclusion of saturation is also necessary, for consistency, in studies of the unitarization of the scattering amplitudes, except for some exceptional kinematical configurations [20].

The simplest approach including gluon splitting in the framework of BFKL evolution is the color dipole picture developed by Mueller [20,21]. This picture is valid at large $N_c$, and describes pomeron multiplication via vertices at which one (BFKL) pomeron splits into two. A more ambitious program, which is not restricted to the large–$N_c$ approximation, is the extended generalized leading logarithmic approximation (EGLLA), initiated by Bartels [22], in which the gluon number changing vertices are explicitly computed in perturbative QCD (see Refs. [23,24,25,26,27,28,29] for further developments along this line and Ref. [15] for a review). By using such vertices, evolution equations allowing for gluon splitting have been written down in Refs. [24,27,29]. Also, the equivalence between the triple pomeron vertex in the dipole picture [21,30] and the one generated by EGLLA at large $N_c$ [23,25] has been verified in Refs. [24,28].

So far, the only formalism allowing for the systematic inclusion of gluon merging in the high–energy evolution is the Color Glass Condensate (CGC) [31], in which the reggeized gluons are replaced by classical color fields whose correlations get built in the course of the evolution. But the corresponding evolution is non–linear: the new gluons radiated at one step in the evolution (the analog of the ‘rungs’ in the BFKL ladders) are allowed to scatter off the classical color fields generated in the previous steps, and this is the mechanism leading to gluon merging. Because of the non–linear effects, the evolution couples $n$–point functions with various values of $n$, and can be most compactly summarized as a functional Fokker–Planck equation for the weight function describing the correlations: the Jalilian-Marian–Iancu–McLerran–Weigert–Leonidov–Kovner (JIMWLK) equation [32,33,34]. Alternatively, and equivalently [35], the evolution can be formulated as an hierarchy of equations for scattering amplitudes — the Balitsky equations [36] —, in which unitarity is manifest. Note however that gluon splittings are not included in the JIMWLK equation [16]; this is obvious from the fact that, in the dilute, or weak–field, limit, this equation reduces to an evolution in which the number of gluons in the $t$–channel stays constant [33,37]. An extension of the JIMWLK–Balitsky evolution which includes pomeron splitting has been proposed only very recently [16,38].
As it should be clear from this succinct presentation, the various formalisms proposed so far in perturbative QCD at high energies are quite different from each other, and the correspondences between them are not always transparent. We know for instance that all these approaches reproduce the Balitsky–Kovchegov (BK) equation \[36,39\], which is the simplest non–linear generalization of the BFKL equation, but only in the sense of a mean field approximation that has been recently challenged \[16,40,41,42\]. But the relation between the correlations (i.e., the \(n\)–point functions with \(n > 2\)) generated by the different approaches is much less understood. For instance, it has been shown only recently, by Kovchegov, Szymanowski and Wallon \[43\], that the perturbative odderon can be accommodated within Mueller’s dipole picture \[20\], and that the corresponding solution coincides with the Bartels–Lipatov–Vacca (BLV) solution \[12\] to the BKP equation.

In particular, in the regime where saturation effects can be neglected, one expects the CGC formalism and the more traditional approaches like GLLA to be equivalent with each other, but this has never been verified beyond the example of the 2–point function (i.e., of the BFKL equation). With this paper, we would like to make one more step towards elucidating this correspondence, by establishing the equivalence between the two approaches at the level of odderon exchanges (i.e., for a 3–point function). Specifically, we shall demonstrate that, in the weak–field limit, the JIMWLK evolution of the \(C\)–odd three–gluon exchanges reduces to the BKP equation, as expected.

But recovering the BKP equation from the CGC formalism is not the main purpose of the present analysis, but only a pretext for it. The CGC is the theoretical framework par excellence for a study of high–energy scattering and evolution in QCD near the unitarity limit, yet the odderon problem has never been addressed in this formalism before. Thus, a substantial fraction of the subsequent analysis will be devoted to the proper formulation of the odderon exchanges in the framework of the CGC, and to the derivation of the corresponding evolution equations from the general, JIMWLK, equation. This study of the odderon should be a good starting point towards understanding the multi–reggeon dynamics within the CGC formalism.

Our study will also emphasize some essential differences between the CGC formalism and the perturbative approach based on the BFKL Hamiltonian: The latter is adapted to the description of a single scattering via the exchange of a composite object — pomeron, odderon, or, in general, a system of \(n\) reggeized gluons — which evolves with increasing energy. It relies on “\(k_T\)–factorization” (see, e.g., \[44\]) to separate the dynamics in the transverse plane from that in the longitudinal direction, and express a scattering amplitude as the convolution of an universal \(Green's\ function\), which describes the exchanged object, with the process–dependent \(impact\ factors\), which connect this object to the external particles. From the above, one sees that the calculation is most nat-
urally carried on in *momentum space*.

By contrast, in the CGC formalism — which is specially tailored to describe unitarity corrections —, single and multiple scatterings are treated on the same footing, namely they are resummed in process–dependent, and gauge–invariant, *scattering amplitudes*, which are computed in the *eikonal approximation*, and thus are naturally constructed in *coordinate space*. There is no $k_T$–factorization any longer, nor universal Green’s functions: the longitudinal and transverse dynamics are tied up together in *Wilson lines*, which describe the eikonal scattering of the elementary particles which compose the projectile (the external object which scatters off the CGC, identified as the *target*).

These differences explain some of the subtleties that we shall meet when trying to compare results for the odderon in the two approaches. On one hand, the odderon is described by the universal Green’s function of three reggeized gluons, which obeys BKP equation in momentum space. On the other hand, the CGC scattering amplitudes depend upon the specific process at hand (they include the impact factor of the projectile) and obey non–linear evolution equations written in coordinate space. (In general, these are not closed equations, but just a part of Balitsky’s hierarchy [36].) Still, in the weak–field, or single–scattering, approximation, in which the evolution equations become linear, they must contain the same non–trivial information as the BKP equation, whatever is the process under consideration.

The authors of Ref. [43] have met with a similar difficulty when trying to compare the $C$–odd scattering amplitude of a dipole with the standard BKP odderon. In that case, they have been able to do so by using the respective solutions, which are explicitly known. Here, we shall follow a more general strategy, which applies to arbitrary processes, including those where the evolution equations are too complicated to be solved exactly. Namely, by inspection of two specific processes, we shall be able to identify the analog of the universal odderon Green’s function in the weak–field limit of the CGC formalism, and show that, when properly defined, this quantity obeys indeed the (coordinate version of the) BKP equation. As we shall momentarily explain, this CGC approach to the BKP equation not only establishes a correspondence between the two formalisms, but also reveals some new insights about the BKP equation itself.

The two specific processes that we shall consider are the CGC scattering with a *quark–antiquark color dipole* (a sub–process of the virtual photon—CGC scattering) and that with a *colorless 3–quark system* (a simple model for a baryon). For both cases we start by constructing the general, non–linear, amplitudes which describe multiple odderon exchanges (these turn out to be the imaginary parts of the respective $S$–matrix elements, themselves expressed in terms of Wilson lines), and then expand these amplitudes in the limit
where the CGC field is weak. After this expansion, both amplitudes reduce to (gauge–invariant) linear combinations involving a three–gluon Green’s function in a totally symmetric color state. Clearly, this Green’s function is a natural candidate for the BKP odderon in the CGC formalism. This interpretation is, however, hindered by the fact that the CGC Green’s functions are gauge–variant objects, for which the JIMWLK equation predicts infrared singularities (to be contrasted with the BKP equation, which is infrared safe). Although physically harmless — as they cancel in the gauge–invariant amplitudes —, these singularities complicate the correspondence with the BKP approach.

At this point comes one of the main new technical developments in this paper: We show that the JIMWLK Hamiltonian [33,34] can be rewritten in a new form, which is manifestly infrared finite (the original kernel in the transverse space is replaced by the dipole kernel [20], which decays much faster at large distances). When acting on gauge–invariant quantities, this new Hamiltonian is equivalent with (but simpler to use than) the original one, in the sense of generating the same evolution equations. But the new Hamiltonian generates infrared–finite equations also for the gauge–variant Green’s functions, and thus allows us to introduce the latter in a mathematically well–defined way. With this prescription, the equation satisfied by the CGC odderon Green’s function turns out to be the same as the Fourier transform to coordinate space of the BKP equation, as we shall check explicitly.

But this Fourier transform reserves some more surprises, as it could be anticipated from the fact that our equation in coordinate space is not exactly the same as the coordinate–space version of the BKP equation that is usually written in the literature\(^2\) (see, e.g., [15]). Rather, the two equations coincide with each other only if we require our CGC Green’s function, which in general is a totally symmetric function of three transverse coordinates, to vanish whenever two coordinates become identical. This property is sometimes referred to as “the Möbius representation” (see, e.g., [28]), and is interesting in that, when restricted to functions having this property, the BFKL Hamiltonian is conformally invariant [2] and exhibits holomorphic separability [8]. This mathematical simplification has led [9,10] to a powerful analogy between the BKP odderon problem (and, more generally, the problem of multi–reggeon exchanges in the limit of a large number of colors) and an integrable Heisenberg spin chain. In particular, the first exact solution to the BKP equation has been found, by Janik and Wosiek [11], by exploiting this analogy.

\(^2\) At the technical level, the difference originates in some ambiguities in the form of delta–functions which appear when Fourier transforming the momentum–space BKP equation to coordinate space, and which are generally interpreted in the sense of the Möbius representation [28]; that is, these delta–functions are simply ignored.
But although natural for the pomeron exchange (i.e., at the level of the 2–point function), where it entails no loss of generality, the restriction to the Möbius representation is not so natural for the higher \( n \)-point functions \((n \geq 3)\), as recently emphasized in Ref. [28]. For instance, the other known solution to the BKP equation, due to Bartels, Lipatov, and Vacca [12], which dominates at high energy and is perhaps of more relevance for the phenomenology (as it couples to a virtual photon), lies outside the Möbius representation.

Similarly, the use of the Möbius representation does not appear to be natural in the CGC formalism either. In fact, our both examples of gauge–invariant scattering amplitudes lie outside this representation: For the dipole case, there is no coupling to this functional space, as well known [12,43], whereas for the 3–quark case, this property is excluded by the initial conditions. We conclude that, at least for the problems that we shall discuss, the BKP equation must be solved in a Hilbert space more general than the Möbius representation.

But our analysis below will not be confined to the weak–field limit and its relation with the perturbative QCD approaches. As repeatedly emphasized, the CGC is a formalism for multiple scattering, in which non–linear amplitudes and the corresponding evolution equations are straightforward to construct. As an illustration, we shall derive the general evolution equations for the scattering amplitudes describing \( C \)-odd and, respectively, \( C \)-even exchanges in the dipole–CGC scattering (the non–linear generalizations of the odderon and, respectively, pomeron exchanges). In fact, these equations will be obtained by simply separating the real part and the imaginary part of the first equation in the Balitsky hierarchy [36]. Interestingly, the non–linear terms in these equations are found to couple the odderon and pomeron evolutions. In the mean field approximation in which the non–linear terms are assumed to factorize, the equation for the \( C \)-odd amplitude reduces to a non–linear equation originally proposed in Ref. [43]. Our analysis of this equation will confirm the conclusion [43] that the odderon exchanges are strongly suppressed by the unitarity corrections, and will allow us to deduce the mathematical law for this suppression. For the 3–quark system, we shall not write down the corresponding non–linear equation (since this appears to be too complicated to be illuminating). Rather, we shall rely on the relation between the corresponding \( C \)-odd amplitude and the respective one for the dipole to conclude that, in the weak–field regime, the dominant increase with the energy should be controlled again by the BLV solution [12], so like for the dipole case [43].

The plan of the paper is as follows. In Sect. 2 we give a general proof that the JIMWLK evolution of gauge–invariant observables is free of infrared problems, and we deduce an alternative form of the JIMWLK Hamiltonian which makes infrared finiteness manifest. In Sect. 3, we consider the weak–field limit of the JIMWLK evolution, and show that the use of the new Hamiltonian allows one to introduce well–defined CGC Green’s functions. In Sect. 4 we construct
the general amplitudes describing multiple \( C \)-odd exchanges for a color dipole and a 3-quark system. Then, in Sects. 5 and 6, we deduce the corresponding evolution equations, after having introduced first the odderon Green’s function in the CGC. Finally, in Sect. 7 we discuss the connection to the BKP equation.

2 The JIMWLK equation with the dipole kernel

In this section, we show that the JIMWLK equation can be considerably simplified when its action is restricted to gauge-invariant correlation functions, such as the scattering amplitudes. The resulting equation is still a functional differential equation, but with a different kernel in transverse space — the dipole kernel —, which has a more rapid fall-off at large distances, and therefore makes it easier to check that the evolution is free of infrared singularities.

2.1 The JIMWLK equation

The CGC formalism [31] is an effective theory for the small-\( x \) gluons in the light-cone wavefunction of an energetic hadron. In this formalism, the gluons with small longitudinal momenta, or small values of \( x \), are described as the classical color field radiated by ‘color sources’ (gluons and valence quarks) with higher values of \( x \), which are ‘frozen’ by Lorentz time dilation in some random configuration. Accordingly, the color fields at small-\( x \) are themselves random, with a distribution specified by the ‘weight function’ \( W_\tau[\alpha] \) (a functional probability density). Here, \( \tau \equiv \ln(1/x) \) is the rapidity, and \( \alpha \equiv \alpha^a(x^-, x) \) is the light-cone component of the color gauge field, and is the only non-trivial component in a suitable gauge (the ‘covariant gauge’; see below). Note that this field depends upon the light-cone longitudinal coordinate \(^3 x^- \equiv (t-z)/\sqrt{2} \), and upon the transverse coordinates \( x = (x, y) \), but not upon the (light-cone) time \( x^+ \equiv (t+z)/\sqrt{2} \), in agreement with the ‘freezing’ property mentioned above.

All the interesting physical quantities are expressed as operators built with \( \alpha \), say \( \mathcal{O}[\alpha] \), and the corresponding expectation values are obtained after averaging over the random field \( \alpha \):

\[
\langle \mathcal{O} \rangle_\tau \equiv \int \mathcal{D}\alpha \, \mathcal{O}[\alpha] \, W_\tau[\alpha]. \tag{2.1}
\]

Whereas, by itself, the weight function \( W_\tau[\alpha] \) is a non-perturbative object, its evolution with decreasing \( x \) (or increasing energy) can be computed in pertur-
bation theory, at least in the high energy regime where the intrinsic ‘saturation momentum’ \( Q_s(x) \) (which increases as a power of \( 1/x \) \([17,58]\)) is hard. The corresponding evolution has been computed in the non–linear generalization of the leading logarithmic approximation \([32,33]\), which allows one to extend the BFKL resummation \([1]\) in the high density region at saturation. In this resummation, the radiative corrections enhanced by the logarithm of the energy \( \ln s \sim \ln 1/x \), and the non–linear effects involving the classical field \( \alpha \), are all treated on the same footing, as effects of order one. The result of this calculation \([33]\) is a second–order, functional, differential equation for \( W_\tau[\alpha] \), which is known as the JIMWLK equation \([32,33,34]\) and reads:

\[
\frac{\partial}{\partial \tau} W_\tau[\alpha] = H W_\tau[\alpha] \equiv \frac{1}{2} \int_{xy} \frac{\delta}{\delta \alpha^a_x(x)} \eta^{ab}(x,y) \frac{\delta}{\delta \alpha^b_y(y)} W_\tau[\alpha], \quad (2.2)
\]

where the subscript \( xy \) on the integral sign denotes the integration over the transverse coordinates \( x \) and \( y \). The kernel \( \eta^{ab}(x,y) \) is a functional of \( \alpha \), upon which it depends via the Wilson lines \( \tilde{V}(x) \) and \( \tilde{V}^\dagger(x) \) built with \( \alpha \equiv \alpha^a T^a \) in the adjoint representation:

\[
\eta^{ab}(x,y) = \frac{1}{\pi} \int \frac{d^2 z}{(2\pi)^2} \mathcal{K}(x,y,z) \left( 1 - \tilde{V}^\dagger_x \tilde{V}_z \right)^{f_a} \left( 1 - \tilde{V}^\dagger_z \tilde{V}_y \right)^{f_b}, \quad (2.3)
\]

with the following transverse kernel:

\[
\mathcal{K}(x,y,z) \equiv \frac{(x - z) \cdot (y - z)}{(x - z)^2(z - y)^2}, \quad (2.4)
\]

and, e.g.,

\[
\tilde{V}^\dagger_x \equiv \tilde{V}^\dagger(x) = P \exp \left( ig \int dx^- \alpha^a(x^-,x) T^a \right), \quad (2.5)
\]

where \( P \) denotes path–ordering in \( x^- \), and the integration over \( x^- \) runs over the longitudinal extent of the hadron, which increases with \( \tau \). When decreasing \( x \), we include in the effective theory gluon modes with smaller longitudinal momenta, which by the uncertainty principle are localized at larger values of \( x^- \). The functional derivatives in Eq. (2.2) act on the color field created in the last step of the evolution (i.e., in the rapidity bin \( (\tau, \tau + d\tau) \)), which is therefore located at the largest value of \( x^- \). Thus, the action of the derivatives on Wilson lines like (2.5) reads as follows:

\[
\frac{\delta \tilde{V}^\dagger_x}{\delta \alpha^a_y(y)} = ig\delta^{(2)}(x-y) T^a \tilde{V}^\dagger_x, \quad \frac{\delta \tilde{V}_x}{\delta \alpha^a_y(y)} = -ig\delta^{(2)}(x-y) \tilde{V}_x T^a. \quad (2.6)
\]
By taking a $\tau$-derivative in Eq. (2.1) and using the JIMWLK equation (2.2), one can easily deduce the following evolution equation for a generic observable:

\[
\frac{\partial}{\partial \tau} \langle O \rangle_\tau = \left\langle \frac{1}{2} \int_{xy} \frac{\delta}{\delta \alpha^a_\tau(x)} \eta^{ab}(x, y) \frac{\delta}{\delta \alpha^b_\tau(y)} O \right\rangle_\tau.
\] (2.7)

For $O$ to represent a physical observable, this must be a gauge–invariant operator, or, more precisely, the expression of such an operator when evaluated in the covariant gauge in which $A^{\mu}_a(x) = \delta^{\mu+} \alpha_a(x^-, x)$. In such a case, one has seen on specific examples that the evolution described by Eq. (2.7) is infrared safe [33], and in what follows we shall give a general proof in that sense.

As a simple example, consider the scattering between the CGC and an external ‘color dipole’ : a quark–antiquark pair in a colorless state. The corresponding $S$–matrix operator can be computed in the eikonal approximation as:

\[
O = \frac{1}{N_c} \text{tr}(V_x^\dagger V_y) \equiv S(x, y; \alpha),
\] (2.8)

where $V_x^\dagger$ is a Wilson line in the fundamental representation (as obtained by replacing $T^a \rightarrow t^a$ in Eq. (2.5)), and represents the phase factor picked up by the quark while crossing the background field of the target. Similarly, $V_y$ is the corresponding phase factor for the antiquark. Plugging this operator into Eq. (2.7), one obtains [33] the following equation (with $\bar{\alpha}_s = \alpha_s N_c / \pi$):

\[
\frac{\partial}{\partial \tau} \langle \text{tr}(V_x^\dagger V_y) \rangle_\tau = \frac{\bar{\alpha}_s}{2\pi} \int d^2 z \mathcal{M}(x, y, z) \left\langle \frac{1}{N_c} \text{tr}(V_x^\dagger V_z) \text{tr}(V_z^\dagger V_y) - \text{tr}(V_x^\dagger V_y) \right\rangle_\tau,
\] (2.9)

after a rather lengthy calculation in which many terms which appear at intermediate steps cancel with each other. This is not a closed equation, but only the first equation in an infinite hierarchy originally derived by Balitsky [36]. The kernel appearing in this integral has been generated as:

\[
\mathcal{M}(x, y, z) \equiv \frac{(x - y)^2}{(x - z)^2(z - y)^2} = \mathcal{K}_{xxz} + \mathcal{K}_{yyz} - 2\mathcal{K}_{xyz},
\] (2.10)

and is recognized as the dipole kernel [20]. Note that the poles in this kernel at $z = x$ and $z = y$ are actually harmless because the operator within the brackets vanishes at these points. In fact, it is easy to check on Eq. (2.3) that such ‘ultraviolet’ (i.e., short–distance) poles cancel already in the general evolution equation (2.7), irrespective of the nature of the operator $O$. 

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The crucial new feature of the dipole kernel (2.10) as compared to the original kernel (2.4) in the JIMWLK equation is that this new kernel falls off much faster at large distances: \( M(x, y, z) \sim 1/z^4 \) when \( z \gg x, y \), which is enough to ensure the convergence of the integral in the r.h.s. of Eq. (2.9). That is, this equation and, similarly, all the higher equations in the Balitsky hierarchy [36], are infrared safe. As we shall explain in the next subsection, this is related to the fact that the corresponding operators are gauge invariant.

2.2 The dipole JIMWLK equation and the finiteness conditions

A brief inspection of the original JIMWLK equation, cf. Eqs. (2.2)–(2.4), reveals that, for a generic operator \( O \), there is a priori no guarantee that the corresponding evolution equation (2.7) should be infrared safe. Indeed, at large distances \( z \gg x, y \), the transverse kernel (2.4) decays only like \( K_{xyz} \sim 1/z^2 \), so Eq. (2.7) may develop logarithmic divergences (\( \sim \int d^2 z/z^2 \) for \( z \to \infty \)). In the calculation leading to (2.9), such divergences have actually canceled among various terms. In what follows, we shall argue that such a cancellation will always hold for the gauge–invariant operators. In the present context, where a gauge–fixing condition has been already chosen, what we mean by that is the invariance under the residual gauge transformations. An interesting consequence of this discussion will be that, for such operators, the corresponding evolution equation (2.7) can equivalently be rewritten in a simpler, and manifestly infrared finite form, which involves the dipole kernel (2.10):

\[
\frac{\partial}{\partial \tau} \langle O \rangle_{\tau} = \langle H_{dp} O \rangle \equiv -\frac{1}{16\pi^3} \int_{xyz} \frac{(x - y)^2}{(x - z)^2(z - y)^2} \times \left( \left( 1 + \hat{V}_x^\dagger \hat{V}_y - \hat{V}_x^\dagger \hat{V}_z - \hat{V}_z^\dagger \hat{V}_y \right)^{ab} \delta \frac{\delta}{\delta \alpha_1^b(x)} \delta \frac{\delta}{\delta \alpha_2^b(y)} O \right)_{\tau}.
\]

Our argument will be constructed as follows: First, we shall use Eqs. (2.3) and (2.7) to deduce the “finiteness conditions” that some operator \( O \) must satisfy in order for the corresponding evolution equation to be infrared safe. Then, we shall use these conditions to rewrite Eq. (2.7) in the form of Eq. (2.11). Finally, we shall demonstrate that the “finiteness conditions” are equivalent to invariance under the residual gauge transformations in the covariant gauge.

The conditions for infrared finiteness are easily written down once one realizes that the dangerous terms are those terms in Eq. (2.3) which do not involve either \( \hat{V}_z \) or \( \hat{V}_z^\dagger \). Indeed, after rewriting \( (1 - \hat{V}_x^\dagger \hat{V}_z)(1 - \hat{V}_z^\dagger \hat{V}_y) = 1 + \hat{V}_x^\dagger \hat{V}_y - \hat{V}_z^\dagger \hat{V}_z - \hat{V}_z^\dagger \hat{V}_y \), it becomes clear that the contribution of the first two terms, \( 1 + \hat{V}_x^\dagger \hat{V}_y \), to Eq. (2.7) is potentially divergent.
\[\int_{xyz} (x - z) \cdot (y - z) \frac{\delta}{(x - z)^2(y - z)^2} \delta \alpha^c_a(x) \cdot \left(1 + \tilde{V}_x^a \tilde{V}_y^a\right) \frac{\delta}{\delta \alpha^b_f(y)} \mathcal{O}\]

\[\rightarrow \int \frac{d^2z}{z^2} \int \frac{\delta}{\delta \alpha^c_a(x)(x - z)^2(y - z)^2} \cdot \left(1 + \tilde{V}_x^a \tilde{V}_y^a\right) \frac{\delta}{\delta \alpha^b_f(y)} \mathcal{O}, \quad (\text{for } z \to \infty),\]

whereas the contributions of the other terms should be finite, because correlations involving \(\tilde{V}_z\) or \(\tilde{V}_z^\dagger\) are expected to vanish when, e.g., \(|z - x| \to \infty\).

Clearly, for the divergence in Eq. (2.12) to go away, the coefficient of the divergent \(z\)-integral there must vanish, which in turn implies the two following finiteness conditions:

\[\int d^2x \frac{\delta}{\delta \alpha^c_a(x)} \mathcal{O} = 0, \quad (2.12)\]

\[\int d^2x (\tilde{V}_x)^{ab} \frac{\delta}{\delta \alpha^b_f(x)} \mathcal{O} = 0. \quad (2.13)\]

It is easy to check that the following 2\(n\)-point operators constructed from the Wilson lines in the fundamental representation

\[\mathcal{O}_n \equiv \text{tr} (M_1 M_2 \cdots M_n), \quad M_i \equiv \tilde{V}_x^a V_y^a, \quad (2.14)\]

(and arbitrary linear combinations and products of them \(\mathcal{O}_n, \mathcal{O}_n^2, \cdots\)) satisfy the finiteness conditions Eqs. (2.12) and (2.13). For example, consider the simplest case \(n = 1\), i.e., the dipole operator (2.8). The first condition (2.12) is trivially satisfied because

\[
\frac{\delta}{\delta \alpha^c_a(v)} \text{tr}(V_x^a V_y^a) = ig \left\{ \delta^{(2)}(v - x) - \delta^{(2)}(v - y) \right\} \text{tr}(V_x^a V_y^a) \delta^a_b
\]

which indeed vanishes after integration over \(v\), while the second condition (2.13) can be verified with the help of the formulae \((V)^{ab}_v = 2\text{tr}(t^a V_v t^b V_v^\dagger)\) and \(\text{tr}(t^a A)\text{tr}(t^a B) = \frac{1}{2} \text{tr}(AB) - \frac{1}{2N_c} \text{tr}(A)\text{tr}(B)\).

Let us now establish the dipole form of the JIMWLK equation, Eq. (2.11). We first decompose the original kernel (2.4) so that the dipole kernel is separated out (cf. Eq. (2.10)):

\[\frac{(x - z) \cdot (y - z)}{(x - z)^2(y - z)^2} = \frac{1}{2} \left[ -\frac{(x - y)^2}{(x - z)^2(y - z)^2} + 1 \left(\frac{1}{(x - z)^2} + \frac{1}{(y - z)^2}\right) \right].\]

Note that this is in fact the separation of infrared divergences: The last two terms will generate logarithmic divergences at large \(z\) when the JIMWLK
Hamiltonian is applied to a generic operator. However, it is easy to show that if the operator satisfies the finiteness conditions, the contributions coming from the last two terms $1/(x - z)^2$ and $1/(y - z)^2$ vanish. Thus, for such operators, we can replace $\mathcal{K}(x, y, z) \longrightarrow -(1/2) \mathcal{M}(x, y, z)$ in Eq. (2.3), and at the same time move the functional derivative $\delta/\delta \alpha^a_\tau(x)$ to the right of the Wilson lines. Indeed, the relevant commutator, namely,

$$\left[ \frac{\delta}{\delta \alpha^a_\tau(x)}, \left(1 + \bar{V}_x V_y - \bar{V}_x \tilde{V}_z - \tilde{V}_z V_y \right)^{ab} \right] = -ig\delta^{(2)}(x - y) \text{Tr}(\bar{V}_z V_y T^b)$$

vanishes when multiplied by the factor $(x - y)^2$ in the dipole kernel. This establishes Eq. (2.11).

Besides being conceptually more appealing (as infrared finiteness is now manifest), the rewriting of the JIMWLK equation as in Eq. (2.11) considerably simplifies the manipulations leading from Eq. (2.11) to final evolution equations like Eq. (2.9). The dipole kernel, that we expect in the final equations, is already present there, and many of the terms generated at intermediate steps when the calculations are based on the original equations, Eqs. (2.7)–(2.3), are simply absent when the calculations start with Eq. (2.11). For the more complicated operators that we shall encounter in the next sections, the usefulness of the dipole JIMWLK equation is indeed appreciable.

### 2.3 Physical meaning of the finiteness conditions

The operator (2.14) which has been seen to obey an infrared–finite evolution equation is the covariant–gauge expression of a gauge invariant operator. Indeed, one can rewrite this operator as the trace of a closed Wilson loop, for which gauge symmetry is manifest. To that aim, note that the end points at longitudinal infinity ($x^- = \pm \infty$) of two adjacent Wilson lines (say, $V_{x_i}^\dagger$ and $V_{y_i}$) can be connected by a Wilson line in the transverse direction, which is simply unity in the present gauge, in which $A_\mu^a = 0$. Therefore, one can connect the end points of all the Wilson lines which enter the trace in Eq. (2.14) can be connected in such a way to form a single, closed, Wilson loop (see Figure 1).

This observation suggests that there is an intimate relation between the infrared finiteness and gauge invariance, that we would like to clarify now. To this end, we first notice that the differential operators which enter the finiteness conditions, namely

$$\mathcal{G}_L[\omega_L] \equiv \omega_L \int d^2x \frac{\delta}{\delta \alpha^a_\tau(x)},$$

(2.16)
Fig. 1. A closed Wilson loop for $\text{tr}(V_x^I V_y^I V_z^I V_w^I)$ as an example of the operator (2.14). Solid lines are Wilson lines along the longitudinal directions which are manifest on the operator, and dashed lines are Wilson lines which connect two end points at longitudinal infinity $x^- = \pm \infty$, but are in fact unity in the covariant gauge.

$$G_R[\omega_R] \equiv -\omega^a_R \int d^2 \vec{x}(\tilde{V}_x)^{ab} \delta^{\delta\alpha^b}(\vec{x})$$

(2.17)

are, respectively, the generators of left and right, global, color rotations:

$$V_x^\dagger \rightarrow \Omega_L V_x^\dagger,$$

(2.18)

$$V_x^\dagger \rightarrow V_x^\dagger \Omega_R^\dagger,$$

(2.19)

where $\Omega_{L/R} = \exp(ig\omega_{L/R}^{a}t^a)$ is a constant $SU(N_c)$ matrix. Indeed, by using the formulae analogous to Eq. (2.6), one can explicitly check that the infinitesimal color rotations are given by

$$\delta_L V_x^\dagger = G_L[\omega_L]V_x^\dagger = ig\omega_L V_x^\dagger,$$

(2.20)

$$\delta_R V_x^\dagger = G_R[\omega_R]V_x^\dagger = -igV_x^\dagger \omega_R.$$

(2.21)

Thus, for any operator $\mathcal{O}$ built with Wilson lines, the finiteness conditions (2.12) and (2.13) are tantamount to the conditions of invariance under such global color rotations. It is trivial to check that the operator (2.14) is indeed invariant under the color rotations (2.18) and (2.19).

Now, with their one–sided action, the color rotations (2.18) and (2.19) look a priori different from the ordinary gauge transformations of the Wilson lines. Still, as we explain now, they are in fact the residual gauge transformations with respect to the ‘covariant gauge’ in which the CGC theory is usually
formulated (see [31] for details). We recall here that, in this gauge, the vector potential has only one non–zero component, the light–cone field $A^+_a \equiv \alpha_a$, which is independent of $x^+$. This structure of the field is preserved by a gauge transformation $A^\mu \rightarrow \Omega(A^\mu + \frac{i}{g} \partial^\mu)\Omega^\dagger$ in which the gauge function $\Omega$ depends only upon $x^-$. Thus a residual gauge transformation amounts to

$$A^+ \rightarrow \Omega(x^-) \left( A^+ + \frac{ig}{\alpha} \partial^+ \right) \Omega^\dagger(x^-), \quad \Omega(x^-) = e^{ig\omega^a(x^-)\epsilon^a}, \quad (2.22)$$

which induces the following transformation for the Wilson line $V^\dagger_x$:

$$V^\dagger_x \rightarrow \Omega(x^- = \infty) V^\dagger_x \Omega^\dagger(x^- = -\infty). \quad (2.23)$$

So far, the gauge function $\omega(x^-)$ is arbitrary. The ”global” color rotations introduced in Eqs. (2.18) and (2.19) are now obtained as the two (independent) special gauge transformations in which $\omega(x^-) \rightarrow 0$ (and thus $\Omega(x^-) \rightarrow 1$) at either $x^- \rightarrow -\infty$ (for the left rotation (2.18)) or $x^- \rightarrow +\infty$ (for the right rotation (2.19)). This establishes the interpretation of the finiteness conditions as the conditions for gauge symmetry $^4$.

Finally, let us show the infinitesimal transformation for the gauge field integrated over the longitudinal direction: $\alpha^a(x) = \int dx^- \alpha^a(x^-, x)$. As we will discuss shortly, this is a natural variable in the weak–field regime. Since $A^+ = \alpha(x^-, x)$ transforms as in Eq. (2.22), its infinitesimal change is given by $\delta \alpha(x^-, x) = \partial^+ \omega(x^-)$. Therefore, $\alpha^a(x)$ transforms as

$$\alpha^a(x) \longrightarrow \alpha^a(x) + \int_{-\infty}^{\infty} dx^- \partial^+ \omega^a(x^-) = \alpha^a(x) + \xi^a, \quad (2.24)$$

with $\xi^a \equiv \omega^a(x^- = \infty) - \omega^a(x^- = -\infty)$ being a pure number. This transformation law will be useful in checking the gauge invariance of various operators in the weak–field limit.

$^4$ Note also that the left and right generators form two independent SU($N_c$) algebra, as is expected for gauge transformations. More precisely, the differential operators $J^a_L(x) \equiv -\frac{1}{ig} \delta^a_{\delta x}$, and $J^a_R(x) \equiv \frac{1}{ig} (V_x)^{ab} \delta^a_{\delta y}$, which enter the definitions of the left and right generators, satisfy the following algebra $[\delta_{xy} \equiv \delta^{(2)}(x - y)]$

$$[J^a_L(x), J^b_L(y)] = if^{abc} J^c_L(x) \delta_{xy}, \quad [J^a_R(x), J^b_R(y)] = if^{abc} J^c_R(x) \delta_{xy},$$

and commute with each other $[J^a_L(x), J^b_R(y)] = 0$. 
3 Weak–field regime and Green’s functions in the CGC

In order to make contact with previous evolution equations proposed in perturbative QCD at high energy, like the BFKL [1] and BKP [4,5] equations, which neglect saturation effects and therefore apply only in the dilute regime at relatively high transverse momenta (well above the saturation scale), it is convenient to consider also the weak–field limit of the JIMWLK equation (here, with dipolar kernel). When $g\alpha \ll 1$, one can expand the Wilson lines in Eq. (2.3) to lowest nontrivial order, to obtain

$$ (1 - \hat{V}_z^\dagger \hat{V}_z)^{fa} \approx ig(\alpha^c(x) - \alpha^c(z))(T^c)_{fa}, \quad (3.1) $$

where

$$ \alpha^a(x) \equiv \int dx^- \alpha^a(x^-, x) \equiv \alpha_x^a \quad (3.2) $$

is the effective color field in the transverse plane, as obtained after integrating over the longitudinal profile of the hadron. Then, the kernel $\eta$ becomes simply quadratic in $\alpha_x^a$, and the evolution equation (2.11) reduces to

$$ \frac{\partial}{\partial \tau} \langle O \rangle = \frac{g^2}{16\pi^3} \int \frac{\langle x - y \rangle^2}{\langle x - z \rangle^2 \langle z - y \rangle^2} \times \left\langle \frac{\partial}{\partial \alpha_x^a} \frac{\partial}{\partial \alpha_z^b} f^{acf} f^{bdf} \delta \frac{\delta}{\delta \alpha_x^c(x)} \delta \frac{\delta}{\delta \alpha_z^d(y)} O \right\rangle. \quad (3.3) $$

For consistency with the above manipulations, the observable $O$ — which in general is some operator built with Wilson lines (see, e.g., Eq. (2.14)) — must be correspondingly expanded in powers of $g\alpha$. Note that the relevant order for this expansion depends upon the operator at hand, and also upon the channel that we consider for scattering (see Sect. 4). For instance, for the dipole operator shown in Eq. (2.8), the lowest nontrivial contribution is obtained after expanding the Wilson lines up to second order in $g\alpha$:

$$ V_z^\dagger[\alpha] \approx 1 + ig \int dx^- \alpha^a(x^-, x)t^a \quad (3.4) $$

$$ - \frac{g^2}{2} \int dx^- \int dy^- \alpha^a(x^-, x)\alpha^b(y^-, x)[\theta(x^- - y^-)t^a t^b + \theta(y^- - x^-)t^b t^a]. $$

Clearly, to this order, the ordering of the color matrices $\alpha^a(x^-, x)t^a$ in $x^-$ starts to play a role in the expansion of the Wilson lines. But this ordering is still irrelevant for the computation of the dipole $S$–matrix to lowest order, because...
of the symmetry of the color trace in Eq. (2.8) : $\text{tr}(t^a t^b) = \frac{1}{2} \delta^{ab} = \text{tr}(t^b t^a)$. Namely, one finds:

$$S(x, y; \alpha) \equiv \frac{1}{N_c} \text{tr}(V_x^\dagger V_y) \simeq 1 - \frac{g^2}{4 N_c} (\alpha^a_x - \alpha^a_y)^2,$$

(3.5)

where the final expression involves only the two–dimensional field $\alpha^a_x$, cf. Eq. (3.2). The latter property turns out to be shared by all the operators for weak (single) scattering that we shall discuss throughout this work. This feature, together with the structure of the simplified evolution Hamiltonian manifest on Eq. (3.3), enable us to consider the weak field approximation to the CGC effective theory as being restricted to the Hilbert space of the functions built with $\alpha^a_x$. Onto this space, the functional derivatives in Eq. (3.3) can be trivially replaced by $\delta/\delta \alpha^a_x$. Thus, in this approximation, the longitudinal structure of the color field becomes irrelevant.

Note that the weak–field Hamiltonian in Eq. (3.3) is quadratic both in $\alpha$ and in the functional derivative with respect to $\alpha$. Thus, when acting on the $n$–point function $\langle \alpha(x_1) \alpha(x_2) \cdots \alpha(x_n) \rangle_\tau$, this Hamiltonian does not change the number $n$ of fields. This has an important consequence for the weak–field evolution described by Eq. (3.3): During this evolution, the number of gluons in the $t$-channel remains fixed, familiar to the ‘multi–reggeons’ approaches based on BFKL evolution [7,4,5,8,10], to which we shall eventually compare the present formalism. In particular, this implies that the JIMWLK evolution in the weak–field regime is linear.

It is already known [32,33] that, in the weak field limit, the JIMWLK evolution of the dipole $S$–matrix reduces to the corresponding BFKL equation [1]. This can be easily checked on the first Balitsky equation, Eq. (2.9), by first rewriting this equation in terms of the dipole scattering amplitude,

$$-iT(x, y; \alpha) \equiv 1 - S(x, y; \alpha) = 1 - \frac{1}{N_c} \text{tr}(V_x^\dagger V_y) ,$$

(3.6)

and then linearizing the ensuing equation with respect to $T$, which is formally 5 appropriate in the weak scattering regime where $|T| \ll 1$. But for the following developments in this paper it is still instructive to give a rapid derivation of the BFKL equation, by using directly the weak field approximation in Eqs. (3.3) and (3.5). Specifically, in this approximation $-i\langle T(x, y) \rangle_\tau \simeq \langle N(x, y) \rangle_\tau$, with:

---

5 We ignore here the subtleties associated with fluctuations in the dilute regime which in general render such a linearization illegitimate even when $|\langle T \rangle_\tau| \ll 1$; see Ref. [16] for details.
\[ \langle N(x, y) \rangle_\tau \simeq \frac{g^2}{4N_c} ((\alpha_x^a - \alpha_y^a)^2)_\tau \]
\[ \equiv \frac{g^2}{4N_c} [f_\tau(x, x) + f_\tau(y, y) - 2f_\tau(x, y)], \quad (3.7) \]

with the following definition for the 2–point Green’s function of the color fields in the dilute regime:

\[ f_\tau(x, y) \equiv \langle \alpha^a(x)\alpha^a(y) \rangle_\tau = f_\tau(y, x). \quad (3.8) \]

Note that, although colorless (it carries no open color indices), the object in Eq. (3.8) is still not gauge invariant: the residual gauge transformation for \( \alpha_x^a \) consists in the constant shift \( \alpha_x^a \rightarrow \alpha_x^a + \xi^a \) [cf. Eq. (2.24)], but \( f_\tau(x, y) \) is not invariant under this operation. On the other hand, the linear combination yielding the scattering amplitude (3.7) is clearly invariant, as it should (since Eq. (3.7) has been obtained after a consistent expansion of the gauge–invariant operator (3.6)). Thus, \textit{a priori}, one is allowed to use the dipolar form of the evolution equation, Eq. (3.3), for the physical amplitude \( \langle (\alpha_x^a - \alpha_y^a)^2 \rangle_\tau \), but not for the Green’s function \( f_\tau(x, y) \). Still, since Eq. (3.3) is linear in \( O \), it is clear that the equation obeyed by \( \langle (\alpha_x^a - \alpha_y^a)^2 \rangle_\tau \) is correctly obtained by separately evolving each of the Green’s functions in the second line of Eq. (3.7), and then summing up the corresponding results. This argument shows that, in fact, it is legitimate to use the dipolar evolution equation (3.3) even for quantities which by themselves are \textit{not} gauge invariant, so like the Green’s function (3.8), provided these quantities are eventually used as building blocks in the construction of gauge–invariant observables. In such a case, the use of the dipolar Hamiltonian \( H_{dp} \) can be seen as a convenient prescription to regulate the infrared singularities which would appear at intermediate steps when using the original JIMWLK Hamiltonian in Eqs. (2.2)–(2.3).

Specifically, by using Eq. (3.3) for \( O = \alpha_x^a\alpha_y^a \), one easily finds

\[ \frac{\partial}{\partial \tau} f_\tau(x, y) = \frac{\bar{\alpha}_x}{2\pi} \int d^2 z \frac{(x - y)^2}{(x - z)^2(y - z)^2} \times \left( f_\tau(x, z) + f_\tau(y, z) - f_\tau(x, y) - f_\tau(z, z) \right). \quad (3.9) \]

This equation is well defined both in the infrared (because of the rapid decay of the dipole kernel at large values of \( z \)), and in the ultraviolet (the short–distance poles of the kernel at \( x = z \) and \( y = z \) are actually harmless because they have zero residue). By contrast, the equation which is obtained by acting on \( \alpha_x^a\alpha_y^a \) with the (weak–field version of the) original JIMWLK Hamiltonian\(^6\) contains terms which are potentially singular at large distances.

\(^6\) This equation can be found, e.g., as Eq. (3.10) in Ref. [45].
By using Eqs. (3.7) and (3.9), one finds the coordinate space (or ‘dipolar’) version of the BFKL equation for $\langle N(x, y) \rangle_\tau$, as expected:

$$\frac{\partial}{\partial \tau} \langle N(x, y) \rangle_\tau = \frac{\bar{\alpha}_s}{2\pi} \int d^2 z \frac{(x - y)^2}{(x - z)^2(y - z)^2} \times \left( \langle N(x, z) \rangle_\tau + \langle N(z, y) \rangle_\tau - \langle N(x, y) \rangle_\tau \right).$$

In this last equation, ultraviolet finiteness is ensured by the vanishing of the scattering amplitude at equal points, $\langle N(x, x) \rangle = 0$, a property consistent with Eq. (3.7) and which reflects ‘color transparency’.

In what follows, we shall take Eq. (3.9) (supplemented with an appropriate initial condition) as the definition of the 2–point Green’s function in the CGC formalism. This situation illustrates a general feature of the present approach, namely the fact that, because of infrared complications, the definition of Green’s functions meets with ambiguities which disappear only in the construction of gauge–invariant quantities. Of course, these Green’s functions are only intermediate objects, which are strictly speaking not needed: it is always possible to construct directly the equation obeyed by the gauge–invariant quantity of interest, which is then free of any ambiguity. The reason why it is nevertheless convenient to introduce such Green’s functions, it is because these are the CGC analogs of the multi–gluon exchanges considered in the more traditional approaches to high energy QCD [7,4,5,8,10], to which we would like to establish a connection in the forthcoming sections.

4 Odderon operators in the CGC

With this section, we start our discussion of the odderon within the CGC framework. To start with, we shall construct the operators describing multiple odderon exchanges in the scattering between the CGC and a relatively simple projectile, such as a color dipole, or three quarks in a colorless state. The dipole–CGC scattering can be viewed as a sub–process of the diffractive scattering of a virtual photon on some dense hadronic target (the CGC), a process in which odderon contributions are expected, e.g., in the production of $C$–even mesons like $\eta_c$ (see, e.g., [12,46]). As for the 3–quark system, this may be viewed as a crude ‘valence quark’ model of the baryon.
4.1 The dipole-CGC scattering

Consider first the simplest case, namely the high energy scattering of a $q\bar{q}$ dipole off the CGC, and let us briefly outline the construction of the corresponding $S$–matrix. As explained in Sect. 2.1, we need to first compute the $S$–matrix for a fixed configuration of the classical field $\alpha$, and then average over the latter. For the first step, we can use the eikonal approximation: $S(x, y; \alpha) = \langle \text{out}|\text{in} \rangle$, where the transverse positions of the quark ($x$) and the antiquark ($y$) are the same in the in–coming and the out–going states. One can write, schematically, ($i = 1, \ldots, N_c$ is the color index)

$$|\text{in}\rangle \sim \bar{\psi}^\text{in}_i(x)\psi^\text{in}_i(y)|0\rangle, \quad |\text{out}\rangle \sim \bar{\psi}^\text{out}_i(x)\psi^\text{out}_i(y)|0\rangle,$$

where the appropriate normalization is understood. The relation between the in–coming and the out–going fields is found by solving (the high–energy version of) the Dirac equation $(\partial_+ - ig\alpha^{a\mu})\psi = 0$ for a given gauge field configuration $\alpha$. This implies that $\psi^\text{out}_i = (V^\dagger_x)^{ij}\psi^\text{in}_j$ with the Wilson line in the fundamental representation. The $S$–matrix becomes:

$$S(x, y; \alpha) = \langle \text{out}|\text{in} \rangle = \frac{1}{N_c}(V^\dagger_x)^{ij}(V_y)^{kl}\delta^{kl}\delta^{ij} = \frac{1}{N_c}\text{tr}(V^\dagger_x V_y),$$

(we have restored the appropriate normalization), in agreement with Eq. (2.8). The physical $S$–matrix is finally obtained after averaging over the random classical color field, cf. Eq. (2.1), an operation which also introduces the dependence upon the energy (i.e., upon $\tau$):

$$S_\tau(x, y) = \frac{1}{N_c}\langle \text{tr}(V^\dagger_x V_y) \rangle_\tau. \quad (4.1)$$

Since non–linear in $\alpha$, the above formula describes in general multiple exchanges, which can be either even, or odd, under the operation of charge conjugation $C$. To single out $C$–even (‘pomerons’) or $C$–odd (‘odderons’) exchanges, one needs to project Eq. (4.1) onto incoming and outgoing states with appropriate $C$–parities. Since the charge conjugation for fermions is defined by $C\psi C^{-1} = -i(\bar{\psi}\gamma^0\gamma^2)T$, and $C\bar{\psi} C^{-1} = (i\gamma^0\gamma^2\psi)^T$, it is clear that the eigenstates of $C$ in the dipole sector are given by $(\bar{\psi}(x)\psi(y) \pm \psi(y)\bar{\psi}(x))|0\rangle$, where $+(-)$ sign yields the $C$–even(odd) state. These structures are natural because the charge conjugation is essentially the exchange of a quark and an antiquark.

Taking the $C$–odd dipole state as the in–coming state (this is the state selected by the virtual photon wavefunction), and the $C$–even dipole state as the out–
going state (this would be selected by a $\eta_c$ meson), one obtains the following $C$–odd contribution to the $S$-matrix:

$$S^{\text{odd}}(x, y) = \langle \text{out, even} | \text{in, odd} \rangle = \frac{1}{2N_c} \left\langle \text{tr}(V^\dagger_x V_y) - \text{tr}(V^\dagger_y V_x) \right\rangle. \quad (4.2)$$

This allows us to identify the operator for $C$–odd exchanges in the dipole–CGC scattering ("the dipole odderon operator") as

$$O(x, y) \equiv \frac{1}{2iN_c} \text{tr}(V^\dagger_x V_y - V^\dagger_y V_x) = -O(y, x), \quad (4.3)$$

where the factor of $i$ is introduced in order for this quantity to be real: indeed, since $V$ and $V^\dagger$ are unitary matrices, we have $[\text{tr}(V^\dagger V)]^* = \text{tr}(V V^\dagger)$.

One can directly check, by using the transformation property of the gauge fields under charge conjugation,

$$CA_{\mu}C^{-1} = -(A_{\mu})^T, \quad (4.4)$$

that the operator (4.3) is indeed $C$–odd: For a generic Wilson line $V$ constructed with $A_{\mu}$, Eq. (4.4) implies

$$CV C^{-1} = (V^\dagger)^T = V^*, \quad (4.5)$$

so that $C \text{tr}(V^\dagger_x V_y) C^{-1} = \text{tr}(V^\dagger_y V_x)$, or, finally, $C O(x, y) C^{-1} = -O(x, y)$.

Note that the $C$–odd contribution (4.2) is the imaginary part of the $S$–matrix element, or, equivalently, the real part of the scattering amplitude $T$, with $S = 1 + iT$:

$$\langle O(x, y) \rangle_T = \Im S_T(x, y), \quad (4.6)$$

which was to be expected. Correspondingly, the $C$–even, Pomeron exchange, amplitude, that we shall denote as $N(x, y)$, is identified with the real part of the $S$-matrix:

$$N(x, y) \equiv 1 - \frac{1}{2N_c} \text{tr}(V^\dagger_x V_y + V^\dagger_y V_x), \quad (4.7)$$

$$\langle N(x, y) \rangle_T = 1 - \Re S_T(x, y). \quad (4.8)$$

Operatorially, $S = 1 + iT = 1 - N + iO$. Note the obvious boundary conditions which follow from $S_T(x, x) = 1$ (or directly from the definitions (4.3), (4.7)):
\[ N(x, x) = O(x, x) = 0. \] (4.9)

Clearly, these conditions remain true after averaging over the random field \( \alpha \).

From perturbative QCD, we expect the lowest order contribution to the odderon exchange to be represented by three gluons tied together with the \( d^{abc} \) symbol, where \( d^{abc} = 2 \text{tr}(\{t^a, t^b\}t^c) \) is the totally symmetric tensor. The operator \( d^{abc} A_{\mu}^a(x) A_{\nu}^b(y) A_{\rho}^c(z) \) is indeed \( C \)-odd, as obvious from Eq. (4.4).

Let us check that a similar structure emerges also from the CGC operator (4.3) when this is evaluated in the weak-field limit. The lowest non-trivial contribution to Eq. (4.3) is obtained by expanding the Wilson lines there up to cubic order in the field \( \alpha \) in the exponent. (The terms quadratic in \( \alpha \) cancel in the difference of traces in Eq. (4.3).) By collecting the remaining terms, one obtains:

\[
O(x, y) \simeq -\frac{g^3}{24N_c} d^{abc} \left\{ 3(\alpha_a^a \alpha_b^b \alpha_c^c - \alpha_a^a \alpha_b^b \alpha_c^c) + (\alpha_a^a \alpha_b^b \alpha_c^c - \alpha_a^a \alpha_b^b \alpha_c^c) \right\}. \tag{4.10}
\]

As expected, this expression is cubic in \( \alpha^a \) with the color indices contracted symmetrically by the \( d \)-symbol. Note that, because of the symmetry properties of this symbol, the path-ordering of the Wilson lines in \( x^- \) has been irrelevant for computing \( O(x, y) \).

The linear combination of trilinear field operators in Eq. (4.10) is gauge invariant by construction. To render this more explicit, let us rewrite this operator as

\[
O(x, y) \simeq -\frac{g^3}{24N_c} d^{abc} (\alpha_a^a - \alpha_b^a)(\alpha_a^b - \alpha_b^b)(\alpha_a^c - \alpha_b^c). \tag{4.11}
\]

This is manifestly invariant under a residual gauge transformation, which consists in a constant shift of the field: \( \alpha^a \rightarrow \alpha^a + \xi^a \) (cf. Eq. (2.24)). In fact, an alternative way to construct the gauge-invariant linear combination appearing in Eq. (4.10) is to directly impose the (weak-field version of the) finiteness conditions (2.12) and (2.13) on the \( C \)-odd Green’s function \( d^{abc} (\alpha_a^a \alpha_b^b \alpha_c^c)_{\tau} \).

### 4.2 The 3-quark–CGC scattering

To describe the 3–quark colorless state, we shall use the following “baryonic” operator\(^7\) \( \epsilon^{ijk} \psi^i(x) \psi^j(y) \psi^k(z) \), where \( \epsilon^{ijk} \) is the complete antisymmetric symbol, and the color indices \( i, j, k \) can take the values 1, 2, or 3. Thus, the construction below applies only for \( N_c = 3 \). (The generalization to arbitrary \( N_c \)

---

\(^7\) A similar approach was adopted for the proton-proton scattering in Refs. [47,48].
is in principle possible, but the analysis becomes more complicated because
the baryonic operator is then built with $N_c$ quark fields.) By using the same
technique as for dipole–CGC scattering, one obtains the following $S$–matrix:

$$S_\tau(x, y, z) = \frac{1}{3!} \epsilon^{ijk} \epsilon^{lmn} \langle V_{il}^\dagger(x)V_{jm}^\dagger(y)V_{kn}^\dagger(z) \rangle_\tau,$$

(4.12)

where $x$, $y$, and $z$ are the transverse positions of the three quarks. This op-
operator is symmetric under any permutations of the three coordinates, and is
normalized as $S_\tau(x, x, x) = 1$. By using Eq. (4.4), it is easy to check that the
odderon contribution is given again by the imaginary part of the $S$-matrix :

$$\langle B(x, y, z) \rangle_\tau = \Re m S_\tau(x, y, z),$$

(4.13)

where the "3–quark odderon operator" $B(x, y, z)$ has been defined as

$$B(x, y, z) = \frac{1}{3!} \epsilon^{ijk} \epsilon^{lmn} V_{il}^\dagger(x)V_{jm}^\dagger(y)V_{kn}^\dagger(z) - c.c.$$  

(4.14)

This is totally symmetric too, and satisfies the boundary condition $B(x, x, x) = 0$, which is an immediate consequence of the normalization $S_\tau(x, x, x) = 1$.

The simplest way to see that the operator (4.14) is gauge invariant is to notice
that this can be rewritten in terms of the manifestly gauge invariant operators
shown in Eq. (2.14). Indeed, by using the identity

$$\frac{1}{3!} \epsilon^{ijk} \epsilon^{lmn} V_{il}(w)V_{jm}(w)V_{kn}(w) = \det V(w) = 1,$$

(4.15)

where $w$ is an arbitrary transverse coordinate, one can equivalently rewrite
the 3–quark odderon operator $B$ as

$$B(x, y, z) = \frac{1}{3!} \left[ \tr(V_x^\dagger V_y^\dagger V_z^\dagger V_{w}) - \tr(V_y^\dagger V_z^\dagger V_{w}^\dagger V_{y}^\dagger V_{z}^\dagger V_{w}) 
- \tr(V_y^\dagger V_z^\dagger V_{x}^\dagger V_{w}^\dagger V_{x}^\dagger V_{y}^\dagger V_{z}^\dagger V_{w}) 
+ \tr(V_x^\dagger V_y^\dagger V_{w}^\dagger V_{x}^\dagger V_{y}^\dagger V_{w}) + \tr(V_x^\dagger V_y^\dagger V_{z}^\dagger V_{x}^\dagger V_{y}^\dagger V_{z}^\dagger V_{w}) - c.c. \right].$$

(4.16)

Note that this expression involves not only dipolar operators, but also higher
multi-polar ones (quadrupoles and sextupoles). By construction, this expres-
sion is independent off $w$ when $N_c = 3$. Thus, it can be simplified by choosing

8 We have already used a similar relation when we have specified the normalization
of $S_\tau(x, y, z)$. 

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$w$ to be one of the quark coordinates, say $w = z$. Then, Eq. (4.16) reduces to:

$$B(x, y, z) = \frac{1}{3!2i} \left[ \text{tr}(V_x^\dagger V_z) \text{tr}(V_y^\dagger V_z) - \text{tr}(V_x^\dagger V_y^\dagger V_z V_y) - \text{c.c.} \right]. \quad (4.17)$$

which looks indeed considerably simpler, but where the symmetry in $x$, $y$ and $z$ is not manifest anymore (although, for $N_c = 3$, we know that this expression is totally symmetric, by construction).

In particular, when two of the coordinates are the same, the 3–quark odderon operator reduces to the dipole odderon operator, Eq. (4.3):

$$B(x, z, z) = O(x, z) = -B(x, x, z). \quad (N_c = 3) \quad (4.18)$$

This is physically reasonable, because the diquark state is equivalent to an antiquark as far as color degrees of freedom are concerned, and can be most easily checked by setting $y = z$ in the r.h.s. of Eq. (4.17). Note however that, if one rather sets $y = x$ in the same expression, then it is not immediately obvious that the ensuing expression for $B(x, x, z)$ is indeed equal to $-O(x, z)$ when $N_c = 3$, as it should. This is so because of the lack of manifest symmetry of Eq. (4.17), as alluded to above. Still, by diagonalizing the unitary matrix $V_x^\dagger V_z$, and after performing some simple algebraic manipulations (relying on the fact that the corresponding eigenvalues $\lambda_i$, $i = 1, 2, 3$, are pure phases and obey $\lambda_1 \lambda_2 \lambda_3 = 1$), it is possible to check that the equality $B(x, x, z) = -O(x, z) = -B(x, z, z)$ holds indeed.

In the weak field approximation, as obtained after expanding to lowest non–trivial order (i.e., to cubic order in $\alpha$) the Wilson lines in any of the previous expressions for $B(x, y, z)$, one finds again a gauge invariant linear combination of trilinear field operators with the color indices contracted with the $d$–symbol:

$$B(x, y, z) \simeq \frac{g^3}{144} d^{abc} \times \left\{ (\alpha^a_x - \alpha^a_z) + (\alpha^a_y - \alpha^a_z) \right\} \left\{ (\alpha^b_y - \alpha^b_x) + (\alpha^b_z - \alpha^b_x) \right\} \left\{ (\alpha^c_z - \alpha^c_y) + (\alpha^c_x - \alpha^c_y) \right\}. \quad (4.19)$$

In fact, in this weak–field regime, the 3–quark $C$–odd operator is fully determined by gauge symmetry together with the requirement of total symmetry with respect to the external coordinates: Indeed, it can be checked that Eq. (4.19) uniquely emerges from the $C$–odd Green’s function (5.1) after symmetrization and imposing the finiteness condition (2.12).
5 Odderon evolution in the dipole–CGC scattering

In this and the next sections, we shall apply the general JIMWLK equation (in its dipolar form, cf. Eq. (2.11)) to the operators describing odderon exchanges constructed in the previous section, in order to deduce the evolution equations for the respective, $C$–odd, scattering amplitudes.

We start with the simpler case of the dipole scattering, for which we shall discuss separately the weak–field limit (corresponding to a single scattering), and the general, non–linear, case (which includes multiple scattering). In fact, in this particular case, it is rather straightforward to write down directly the non–linear equations (see below), from which the equations for the weak–field limit can be then simply deduced by linearization. Still, the more detailed approach that we shall follow below is instructive as a preparation for the more tedious case of the 3–quark operator, to be discussed in Sect. 6.

5.1 Linear evolution and the odderon Green’s function

In the weak–field regime, where we can limit ourselves to a single odderon exchange, it is convenient to proceed as in Sect. 3 and introduce the (totally symmetric) odderon Green’s function

$$f_{\tau}(x, y, z) \equiv d^{abc}\langle \alpha_x^a \alpha_y^b \alpha_z^c \rangle_{\tau},$$

(5.1)

in terms of which the weak–field version of the dipole odderon operator (4.10) can be rewritten in a form similar to Eq. (3.7) for the pomeron:

$$O(x, y) \simeq -\frac{g^3}{24N_c} \left\{ 3 \left( f_{\tau}(x, y, y) - f_{\tau}(x, x, y) \right) + f_{\tau}(x, x, x) - f_{\tau}(y, y, y) \right\}.$$

(5.2)

The discussion of the pomeron Green’s function (3.8) in Sect. 3 applies to the odderon Green’s function (5.1) as well: The latter is not a gauge–invariant quantity, so its evolution under the original JIMWLK Hamiltonian would be afflicted by infrared singularities, which can however be regulated by using the (weak field version of the) dipolar Hamiltonian $H_{dp}$. This yields a mathematically well defined equation for $f_{\tau}(x, y, z)$, that we shall use as the definition of the odderon Green’s function.

Still as in the pomeron case, the equation obeyed by the scattering amplitude (5.2) — which is gauge–invariant — is not sensitive to the ambiguities which affect the definition of the Green’s function, and comes out the same whatever
form of the Hamiltonian is used in its derivation. In fact, the only reason for introducing the Green’s function (5.1) is the fact that it is for this function that we shall verify the BKP equation later on.

Specifically, by using Eq. (3.3) for \( \mathcal{O} = d^{abc} \alpha^a_x \alpha^b_y \alpha^c_z \), and after some lengthy algebra, one obtains

\[
\frac{\partial}{\partial \tau} f_\tau(x, y, z) = \frac{\alpha_s}{4\pi} \int d^2 w \frac{(x - y)^2}{(x - w)^2(y - w)^2} \left( f_\tau(x, w, z) + f_\tau(w, y, z) - f_\tau(x, y, z) - f_\tau(w, w, z) \right) + \{2 \text{ cyclic permutations}\},
\]

(5.3)

which is like applying the equation (3.9) for the 2–point Green’s function to each pair of points within \( f_\tau(x, y, z) \). It can be checked as before that this equation is well defined both in the infrared and in the ultraviolet. It is now straightforward to construct the equation obeyed by the linear combination in Eq. (5.2), and thus find that this is precisely the BFKL equation (3.10)

\[
\frac{\partial}{\partial \tau} \langle \mathcal{O}(x, y) \rangle_\tau = \frac{\alpha_s}{2\pi} \int d^2 z \frac{(x - y)^2}{(x - z)^2(y - z)^2} \times \left( \langle \mathcal{O}(x, z) \rangle_\tau + \langle \mathcal{O}(z, y) \rangle_\tau - \langle \mathcal{O}(x, y) \rangle_\tau \right)
\]

(5.4)

(we have also used the identity \( f_{eag} f_{dbf} d_{abc} = \frac{N_c}{2} d_{ede} \)), in agreement with an original observation by Kovchegov, Szymanowski and Wallon [43]. In view of the formal difference between the pomeron and the odderon operators, as given by Eqs. (3.7) and, respectively, (4.10), it may appear as a surprise that they obey both the same evolution equation. But this becomes more natural if one remembers that \( N \) and \( O \) are, respectively, the imaginary part and the real part of the scattering amplitude \( T \) (with \( S = 1 + iT \)), and that in the weak field regime \( T \) obeys a linear equation with real coefficients — the linearized version of the first Balitsky equation, Eq. (2.9) — which is therefore separately satisfied by its real and imaginary parts.

But since the initial conditions corresponding to \( N \) and \( O \) are different (in particular, they have different \( C \)-parities), so are also the respective solutions, and their behaviors at high energy. In Appendix A, we compute within the CGC formalism the \( C \)-odd initial conditions for some simple targets: a bare quark and a \( q\bar{q} \) dipole. Namely, if the target is a single quark with transverse position \( x_0 \), one obtains

\[
\langle \mathcal{O}(x, y) \rangle_{\tau=0} = \frac{\alpha_s^3}{12} \frac{(N_c^2 - 4)(N_c^2 - 1)}{N_c^3} \ln^3 \frac{|x - x_0|}{|y - x_0|},
\]

(5.5)
whereas for the more interesting case of a dipolar target (with the quark being at \( x_0 \) and the antiquark at \( y_0 \)), one rather finds

\[
\langle O(x, y) \rangle_{\tau=0} = \frac{\alpha_s^3 (N_c^2 - 4)(N_c^2 - 1)}{12 N_c^3} \ln^3 \left| \frac{|x-x_0||y-y_0|}{|x-y_0||y-x_0|} \right|. \tag{5.6}
\]

These expressions are in agreement with the corresponding results in Ref. [43] up to an overall numerical factor. As expected, these initial conditions are antisymmetric under the exchange of \( x \) and \( y \), and thus satisfy the boundary condition (4.9). It is easily checked that this boundary condition is preserved by the evolution according to Eq. (5.4), as necessary for this equation to be well defined.

The high–energy behavior of the odderon solution to Eq. (5.4) has been analyzed too in Ref. [43], where it has been shown that the projection of the general BFKL solution onto \( C \)–odd initial conditions selects \( (C \)–odd) BFKL eigenfunctions whose maximal intercept is equal to one. It turns out that these are the same eigenfunctions that were previously identified, by Bartels, Lipatov, and Vacca [12], as exact solutions to the momentum–space BKP equation [4,5]. Thus, in contrast to the pomeron solution to the BFKL equation, which at high energy rises exponentially with \( Y \left( N(Y) \sim e^{\alpha_P Y} \right) \), the corresponding odderon solution rises only slowly, as a power of \( Y \sim \ln s \).

But, of course, these types of high–energy behavior (for either \( N \) or \( O \)), which are mathematical consequences of the BFKL equation, are physically acceptable only so far as this equation is a correct approximation, that is, within the limited range of energies where the unitarity corrections are indeed negligible. For higher energies, the evolution is governed by more complicated, non–linear, equations, to which we now turn.

5.2 Non–Linear evolution

For dipole–CGC scattering, the general evolution equations obeyed by the average amplitudes \( \langle N(x, y) \rangle_\tau \) and \( \langle O(x, y) \rangle_\tau \) in the strong field regime can be easily inferred from the first Balitsky equation (2.9): Since the operators \( N(x, y) \) and \( O(x, y) \) are, respectively, the real part and the imaginary part of the dipole \( S \)–matrix \( S(x, y) = (1/N_c)\text{tr}(V_x^\dagger V_y) \), cf. Eqs. (4.3) and (4.7), it is clear that the respective equations can be simply obtained by separating the real part and the imaginary part in Eq. (2.9). One thus obtain:
\[
\frac{\partial}{\partial \tau} \langle O(x, y) \rangle_{\tau} = \frac{\alpha_s}{2\pi} \int d^2 z \frac{(x - y)^2}{(x - z)^2(z - y)^2} \times \left[ \langle O(x, z) \rangle_{\tau} + \langle O(z, y) \rangle_{\tau} - \langle O(x, y) \rangle_{\tau} \right],
\]

(5.7)

\[
\frac{\partial}{\partial \tau} \langle N(x, y) \rangle_{\tau} = \frac{\alpha_s}{2\pi} \int d^2 z \frac{(x - y)^2}{(x - z)^2(z - y)^2} \times \left[ \langle N(x, z) \rangle_{\tau} + \langle N(z, y) \rangle_{\tau} - \langle N(x, y) \rangle_{\tau} \right],
\]

(5.8)

As is generally the case for the Balitsky equations, the equations above do not close by themselves, but rather belong to an infinite hierarchy. Interestingly, the non-linear terms in these equations couple the evolution of \(C\)-odd and \(C\)-even operators. For instance, the last term, quadratic in \(O\), in the r.h.s. of Eq. (5.8) for \(\langle N \rangle_{\tau}\) describes the merging of two odderon into one pomeron. This process has not been recognized in previous studies of the Balitsky hierarchy, but the vertex connecting one pomeron to two odderon has been already computed in lowest order perturbation theory [26], and it would be interesting to compare such previous results with the corresponding vertex in Eq. (5.8) (which is essentially the dipole kernel).

But the odderon–pomeron coupling which turns out to have the most dramatic consequences is the one encoded in the last terms in Eq. (5.7) for \(\langle O \rangle_{\tau}\) : As we shall shortly argue, this coupling leads to a rather rapid suppression of the \(C\)-odd contributions to scattering in the high energy regime where unitarity corrections start to be important (i.e., where \(\langle N \rangle_{\tau} \sim O(1)\)). To construct the argument without having to resort to an infinite hierarchy of equations, we shall restrict ourselves to the mean field approximation in which the non-linear terms in Eqs. (5.7)–(5.8) are assumed to factorize. In the strong field regime, which will be our main focus below, we expect this approximation to be qualitatively correct [40].

In this mean field approximation, Eqs. (5.7)–(5.8) reduce to a closed system of coupled, non–linear, equations for \(\langle N \rangle_{\tau}\) and \(\langle O \rangle_{\tau}\) :

\[
\frac{\partial}{\partial \tau} \langle O(x, y) \rangle_{\tau} = \frac{\alpha_s}{2\pi} \int d^2 z \frac{(x - y)^2}{(x - z)^2(z - y)^2} \times \left[ \langle O(x, z) \rangle_{\tau} + \langle O(z, y) \rangle_{\tau} - \langle O(x, y) \rangle_{\tau} \right],
\]

(5.9)
\[
\frac{\partial}{\partial \tau} \langle N(x, y) \rangle = \frac{\bar{\alpha}_s}{2\pi} \int d^2 z \frac{(x - y)^2}{(x - z)^2(z - y)^2} \times \left[ \langle N(x, z) \rangle + \langle N(z, y) \rangle - \langle N(x, y) \rangle - \langle N(x, z) \rangle \langle N(z, y) \rangle + \langle O(x, z) \rangle \langle O(z, y) \rangle \right].
\] (5.10)

The first of these equations has been already proposed in Ref. [43], as a plausible non–linear generalization of Eq. (5.4). As for Eq. (5.10), this is the Kovchegov equation [39] supplemented by a new term describing the merging of two odderons.

The Kovchegov equation has been extensively studied over the last few years, both analytically and numerically, and although the exact solution is not known, its general properties are by now well understood [49,50,51,52,53,54,55,56,57,58,59]. In its most synthetic description, due to Munier and Peschansky [59] (see also [50]), this solution can be viewed as a front connecting the saturation regime where \( N = 1 \) — this is reached for dipole sizes \( r = |x - y| \) much larger than the saturation length \( 1/Q_s(\tau) \) — to the unstable regime at \( r \ll 1/Q_s(\tau) \), where the amplitude is small \( (N \ll 1) \), but it rises rapidly with \( \tau \), according to BFKL equation. When increasing \( \tau \), the front propagates towards smaller values of \( r \) (or larger transverse momenta), and its instantaneous position defines the saturation momentum \( Q_s(\tau) \). The latter is found to rise exponentially with \( \tau \): \( Q^2_s(\tau) = Q^2_0 e^{c_\alpha s \tau} \), with \( c \) a numerical constant determined by the BFKL dynamics [17,56,57,58].

As we shall argue below, this mean–field picture of the pomeron exchanges is not significantly modified by the odderon contribution to Eq. (5.10). In particular, the values \( N = 1 \) and \( O = 0 \) remain as (stable and, respectively, unstable) fixed points of the evolution described by Eqs. (5.9)–(5.10), but to them one should add a new fixed point, namely \( O = 0 \), which is asymptotically approached at high energy.

It is first easy to check that \( N = 1 \) and \( O = 0 \) are indeed fixed points at high energy. To also see that this is the only combination of fixed points in this limit, notice from Eq. (5.9) that, when increasing \( N \), the non–linear terms in this equation act towards suppressing the odderon [43]. It is thus consistent to assume that, at high energy, the odderon contribution represents only a small perturbation to the Kovchegov equation, so that the pomeron saturates in the standard way: \( N(r, \tau) \approx 1 \) for \( r \gg 1/Q_s(\tau) \). Also, the dominant energy behavior of the saturation momentum, i.e., the value of the saturation exponent \( c \), should not change, since this is fully determined by the linear (BFKL) part of the equation for \( N \). Using this assumption, it is possible to study the approach of \( O \) towards zero, and thus check the consistency of our hypothesis.

\footnote{Until the end of this section, we shall mostly use the simplified notations \( N \) and \( O \) for the respective expectation values.}
Namely, for sufficiently high energy, such that \( r \gg 1/Q_s(\tau) \), the integral in the r.h.s. of Eq. (5.9) is dominated by relatively large dipoles, for which \( N \approx 1 \). In this regime, the non-linear terms in this equation precisely cancel the first two linear terms there, and the equation simplifies to

\[
\frac{\partial}{\partial \tau} \langle O(\mathbf{x}, \mathbf{y}) \rangle_\tau \simeq -\bar{\alpha}_s \int_{Q_s^{-2}(\tau)}^r \frac{dz^2}{z^2} \langle O(\mathbf{x}, \mathbf{y}) \rangle_\tau = -\bar{\alpha}_s \ln[Q_s^2(\tau)r^2] \langle O(\mathbf{x}, \mathbf{y}) \rangle_\tau, (5.11)
\]

which together with \( \ln[Q_s^2(\tau)r^2] = c\bar{\alpha}_s(\tau - \tau_0) \) immediately implies:

\[
\langle O(\mathbf{x}, \mathbf{y}) \rangle_\tau \simeq \exp\left\{ -\frac{c}{2} \bar{\alpha}_s^2(\tau - \tau_0)^2 \right\} \langle O(\mathbf{x}, \mathbf{y}) \rangle_{\tau_0} \text{ for } r \gg 1/Q_s(\tau). (5.12)
\]

As anticipated, this is a rapidly decreasing function of \( \tau \), which is actually the same as the function describing the approach of the real part of the \( S \)-matrix towards the ‘black-disk’ limit \( S = 0 \) (the Levin-Tuchin law) [49,37]. We expect the fluctuations neglected in the mean field approximation leading to Eqs. (5.9)–(5.10) to modify (actually, decrease) the value of the overall coefficient in the exponent of Eq. (5.12), but preserve the above qualitative picture [40].

6 Odderon evolution in the scattering of the 3-quark system

The new feature which makes the 3-quark system conceptually interesting is the fact that the corresponding scattering amplitude depends upon three independent transverse coordinates. Therefore, already the lowest-order odderon amplitude, as given in Eq. (4.19), involves configurations in which the three exchanged gluons are attached to different quark legs, and which are thus probing the complete functional dependence of the odderon Green’s function defined in Eq. (5.1). As we shall further argue in Sect. 7, this in turns implies that the 3-quark—CGC scattering is a good theoretical laboratory to study the general solution to the BKP equation.

Our discussion in this section will be restricted to the weak-field version of the 3-quark odderon operator, Eq. (4.19), which describes scattering via the exchange of a single odderon. This is indeed sufficient to discuss the correspondence with the BKP equation in the next section. Within the CGC formalism, there is no difficulty of principle (other than the tediousness of the corresponding algebraic manipulations) which would prevent us from deriving the evolution equations satisfied by the general, non-linear, 3-quark operators in Eqs. (4.14) or (4.16). However, these general equations are complicated
and not very illuminating: Through their non–linear terms, they couple the 3–quark operator to other color structures. This is especially manifest if one uses the form (4.16) of the 3–quark amplitude: this involves operators of various multi-polar orders, which in the Balitsky hierarchy are coupled to other operators of even higher multi-polar moments.

By using the expression of $B(x, y, z)$ as a linear combination of odderon Green’s functions, as manifest in Eq. (4.19), together with the equation (5.3) obeyed by the latter, one can deduce after a straightforward but lengthy calculation the following linear evolution equation for $\langle B_{xyz}\rangle_\tau \equiv \langle B(x, y, z)\rangle_\tau$:

$$\frac{\partial}{\partial \tau} \langle B_{xyz}\rangle_\tau = \frac{3\alpha_s}{4\pi^2} \int d^2 w \frac{(x - y)^2}{(x - w)^2(y - w)^2} \times \left( \langle B_{xwz}\rangle_\tau + \langle B_{wyz}\rangle_\tau - \langle B_{xyz}\rangle_\tau 
- \langle B_{wwz}\rangle_\tau - \langle B_{xxw}\rangle_\tau - \langle B_{yyw}\rangle_\tau - \langle B_{xyw}\rangle_\tau \right) + (2 \text{ cyclic permutations}).$$ (6.1)

(Of course, the same equation could be obtained also directly from Eq. (3.3) with $O = B_{xyz}$, but the corresponding manipulations would be even more tedious; the introduction of the Green’s function (5.1) at intermediate steps has also the advantage to better organize the calculation.) Note that Eq. (6.1) is a closed equation for $\langle B_{xyz}\rangle_\tau$, which was expected in view of gauge invariance: the only gauge invariant operators available are $B_{xyz}$ and $O_{xy} = B_{xyy}$ (cf. Eq. (4.18)).

Consider the structure of Eq. (6.1) in some detail: The linear combination of $B$’s in the integrand vanishes at the points $w = x$ and $w = y$ where lie the poles of the dipole kernel, so the latter are again innocuous. Also, one can easily check that the above equation is consistent with the relation (4.18) between the dipole and the 3–quark odderon amplitudes: if one sets $z = y$, Eq. (6.1) reduces indeed to Eq. (5.4) with $N_c = 3$. Lastly, Eq. (6.1) manifestly preserves the symmetry of the scattering amplitude under the permutation of its three coordinate variables.

The initial condition can be calculated within the CGC formalism, in a similar way as for dipole–CGC scattering (cf. Appendix A). For example, for a single quark at transverse position $x_0$, one finds

$$\langle B_{xyz}\rangle_{\tau=0} = \frac{5}{34} \alpha_s^3 \ln \frac{|x - x_0||y - x_0|}{|z - x_0|^2} \ln \frac{|y - x_0||z - x_0|}{|x - x_0|^2} \ln \frac{|z - x_0||x - x_0|}{|y - x_0|^2}. \tag{6.2}$$

As expected, this is symmetric under the permutation of the three external
coordinates, and satisfies the boundary condition \( \langle B_{xxx} \rangle_\tau = 0 \).

In the next section, we shall argue that the high energy behavior of the solution \( \langle B_{xyz} \rangle_\tau \) to Eq. (6.1) is controlled by the BLV solution [12] to the BKP equation, and thus has a maximal intercept equal to one.

7 Comparison with previous approaches

Within the traditional perturbative QCD approach to small–\( x \) evolution, the odderon exchange is viewed as the exchange of a composite object made of three reggeized gluons which evolves with energy according to the Bartels–Kwiecinski–Praszalowicz (BKP) equation [4,5]. From the point of view of the CGC formalism, this exchange is a single scattering, and should be compared to the weak–field limit of the corresponding CGC equations, as derived in the previous sections. In what follows, we shall demonstrate that the BKP odderon corresponds to the CGC Green’s function introduced in Eq. (5.1): The BKP equation, which is traditionally written in momentum space, is essentially the Fourier transform of Eq. (5.1) for \( f_\tau (x, y, z) \). Based on our previous discussion of gauge–invariant scattering amplitudes, we shall then argue that the natural Hilbert space for discussing odderon exchanges is in fact larger than the one which is generally used in the literature, and within which the BFKL Hamiltonian shows holomorphic separability [8,10].

7.1 Relation to the BKP equation

The BKP equation is traditionally written in momentum space as

\[
\frac{\partial}{\partial \tau} F_\tau (k_1, k_2, k_3) = \frac{1}{2} \sum_{i=1}^{3} \int d^2k'_1 d^2k'_2 d^2k'_3 \; \delta^{(2)} (k'_1 + k'_2 + k'_3 - q) \; \delta^{(2)} (k_i - k'_i) \times H_{\text{BFKL}} (k_{i-1}, k_{i+1}; k'_{i-1}, k'_{i+1}) F_\tau (k'_1, k'_2, k'_3),
\]

where \( F_\tau (k_1, k_2, k_3) \) is the Green’s function for the exchange of three reggeized gluons (with transverse momenta \( k_1, k_2, \) and \( k_3 \), respectively) in a \( C \)-odd, color singlet, state. Furthermore, \( k_4 \equiv k_1 \) (and similarly \( k'_4 \equiv k'_1 \)), and \( q \) is the momentum transfer. The factor \( \frac{1}{2} \) accounts for the fact that two of the gluons are in the color octet state. Finally, \( H_{\text{BFKL}} \) is the non-forward BFKL kernel including the virtual terms:

\[
H_{\text{BFKL}} (k_1, k_2; k'_1, k'_2) = \frac{k_1^2 k'_2^2 + k_2^2 k'_1^2 - (k_1 - k'_1)^2 (k_1 + k_2)^2}{k_1^2 k'_2^2 (k_1 - k'_1)^2}.
\]
\[-\pi \delta^{(2)}(\mathbf{k}'_1 - \mathbf{k}_1) \left( \ln \frac{k_1'\pi}{\epsilon^2} + \ln \frac{k_2\pi}{\epsilon^2} \right), \quad (7.2)\]

where \(\epsilon\) is an infrared cutoff which does not affect physical results.

The function \(F_\tau(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\) is often referred to as a ‘scattering amplitude’, but in fact the physical amplitudes are obtained only after convoluting this with appropriate impact factors. A generic odderon amplitude reads:

\[
\mathcal{O}(q) = \int d^2 k_1 d^2 k_2 d^2 k_3 \delta^{(2)}(k_1 + k_2 + k_3 - q) \\
\times \Phi_{\text{proj}}(k_1, k_2, k_3) F_\tau(k_1, k_2, k_3), \quad (7.3)
\]

where gauge symmetry requires the projectile impact factor \(\Phi_{\text{proj}}(k_1, k_2, k_3)\) to vanish when \(k_i = 0\) for some \(i\) (indeed, a zero–momentum gluon ‘sees’ the projectile as a whole, and the latter is globally colorless). Note that, when writing the scattering amplitude as in Eq. (7.3), the impact factor of the target is implicitly included in the definition of \(F_\tau\). This is in line with the general approach in this paper, where the target is a generic ‘color glass condensate’, and the CGC Green’s functions like those in Eqs. (3.8) or (5.1) include the relevant information about the target impact factor.

In the case where the projectile is a proton, the odderon impact factor is expected in the form (see, e.g., [48])

\[
\Phi_{\text{proton}}(k_1, k_2, k_3) = \int d^2 x d^2 y d^2 z |\Psi_{\text{proton}}(x, y, z)|^2 (2 e^{ik_1 x} - e^{ik_1 y} - e^{ik_1 z}) \\
\times (2 e^{ik_2 y} - e^{ik_2 z} - e^{ik_2 x})(2 e^{ik_3 z} - e^{ik_3 x} - e^{ik_3 y}), \quad (7.4)
\]

where \(|\Psi_{\text{proton}}(x, y, z)|^2\) is the proton light–cone wavefunction, with \(x, y, z\) denoting the coordinates of the three valence quarks relative to their barycenter. (The condition \(x + y + z = 0\) is implicit in the definition of \(|\Psi_{\text{proton}}|^2\).) The exponential terms within the parentheses correspond to all the possible attachments of the three exchanged gluons to the quark lines in the proton. Note that \(\Phi_{\text{proton}}(k_1 = 0, k_2, k_3) = 0\), etc., as expected.

Furthermore, if the projectile is a virtual photon, the odderon couples to the dipole component of the photon Fock space, so that:

\[
\Phi_{\gamma^*}(k_1, k_2, k_3) = \int dz d^2 r |\Psi_{\gamma^*}(z, r)|^2 \\
\times (e^{ik_1 z} - e^{-ik_1 z})(e^{ik_2 z} - e^{-ik_2 z})(e^{ik_3 z} - e^{-ik_3 z}), \quad (7.5)
\]

where \(r\) is the dipole size, and \(\Psi_{\gamma^*}\) is the light–cone wavefunction describing the dissociation of the virtual photon, and can be computed in perturbation
theory with respect to $\alpha_{\text{EM}}$ [60].

To make contact between this more traditional approach and our previous results in this paper, it is convenient to introduce first the momentum space version of the odderon Green’s function introduced in Eq. (5.1). We thus define:

$$f_\tau(k_1, k_2, k_3) \equiv \int \frac{d^2x d^2y d^2z}{(2\pi)^6} e^{-ik_1x-ik_2y-ik_3z} f_\tau(x, y, z),$$  \hspace{1cm} (7.6)

in terms of which the scattering amplitudes $\langle O(x, y) \rangle_\tau$ and $\langle B(x, y, z) \rangle_\tau$ can be rewritten as

$$\langle O(x, y) \rangle_\tau = \int d^2k_1 d^2k_2 d^2k_3 f_\tau(k_1, k_2, k_3) \times (e^{ik_1x} - e^{ik_1y})(e^{ik_2x} - e^{ik_2y})(e^{ik_3x} - e^{ik_3y}),$$  \hspace{1cm} (7.7)

and, respectively,

$$\langle B(x, y, z) \rangle_\tau = \int d^2k_1 d^2k_2 d^2k_3 f_\tau(k_1, k_2, k_3)(2e^{ik_1x} - e^{ik_1y} - e^{ik_1z}) \times (2e^{ik_2x} - e^{ik_2y} - e^{ik_2z})(2e^{ik_3x} - e^{ik_3y} - e^{ik_3z}).$$  \hspace{1cm} (7.8)

Note the similarity between the exponential factors in these equations and those in Eqs. (7.4) and (7.5) for the impact factors. This reflects the fact that the couplings between the exchanged gluons and the quarks (or antiquarks) in the projectile have been explicitly included in our definition of the scattering amplitudes. By inspection of the previous equations, it becomes clear that the CGC Green’s function $f_\tau(k_1, k_2, k_3)$ should correspond to the function $F_\tau(k_1, k_2, k_3)$ of the traditional BKP approach. To be able to compare these quantities, one also needs the equation satisfied by $f_\tau(k_1, k_2, k_3)$, which is obtained after taking a Fourier transform in Eq. (5.3). A lengthy but straightforward calculation shows that the resulting equation is the same as the BKP equation (7.1) up to terms proportional to delta–functions $\delta^{(2)}(k_i) \ (i = 1, 2, 3)$, which are however irrelevant for the calculation of the scattering amplitudes, since they do not contribute to the convolutions in Eqs. (7.3) or (7.7)–(7.8). This demonstrates the equivalence between the two formalisms, in so far as the single odderon exchanges are considered.
7.2 Comments on the Hilbert space for odderon solutions

Following the remarkable discovery by Lipatov \[8\] that the BFKL Hamiltonian exhibits holomorphic separability when restricted to functions which belong to the Möbius space — by which we mean the functions \( f_\tau(x, y, z, \ldots) \) which vanish when any two coordinates coincide with each other \( (f_\tau(x, x, z, \ldots) = 0, \text{etc.}) \) —, much effort has been devoted towards finding solutions to the BKP equation (and, more generally, to its generalization which describes the exchange of \( n \) reggeized gluons) within this particular Hilbert space of functions. This situation is mathematically appealing since the restriction of the BKP equation to a given holomorphic sector describes a dynamical system which has a sufficient number of hidden conserved charges (three in the case of the odderon) to be completely integrable\[10\] \[9,10\], and which in fact can be identified as the XXX Heisenberg model of spin \( s = 0 \) \[10\].

However, as recently reiterated in Ref. \[28\], a careful inspection reveals that the full BKP Hamiltonian contains extra delta functions like \( \delta^{(2)}(x - y) \) in addition to the separable Hamiltonian given in \[8,9,10\]. One can discard these delta functions and restore holomorphic separability by working in the Möbius space. But this is only natural in the case of the BFKL pomeron, i.e., for the 2–point Green’s function \( f_\tau(x, y) \), since in that case one can always redefine

\[
\tilde{f}_\tau(x, y) \equiv f_\tau(x, y) - \frac{1}{2} f_\tau(x, x) - \frac{1}{2} f_\tau(y, y),
\]

which ensures that \( \tilde{f}_\tau(x, x) = 0 \), without affecting the calculation of scattering amplitudes\[11\]: The subtracted terms in Eq. (7.9), being independent of one of the coordinates, give vanishing contributions when convoluted with a colorless impact factor. However, for the odderon problem \( (n = 3) \), the restriction to the Möbius space is a highly nontrivial issue — it cannot be simply achieved via subtractions which preserve the physical amplitudes —, and therefore has implications on the physical relevance of the various solutions to the BKP equation.

And, indeed, among the two exact solutions to the (odderon) BKP equations which do not vanish rapidly at high energies \[11,12\], only one of them — the Janik–Wosiek (JW) solution \[11\], which has an intercept slightly lower than one — lies indeed in the Möbius representation, e.g.,

\[10\] For the generalization of the BKP equation to a system of \( n \) reggeized gluons, with \( n > 3 \), the same property holds only in the large–\( N_c \) limit in which one can restrict the BFKL–like pairwise interactions to neighboring gluons \[8,9,10\].

\[11\] In fact, a brief comparison with Eq. (3.7) reveals that the subtracted 2–point Green’s function can be identified with the pomeron scattering amplitude.
and has been constructed by exploiting conformal symmetry and integrability. But, clearly, this solution does not couple to a dipole in the present, leading–logarithmic, approximation. Besides, although in principle it can couple to a 3–quark system, the corresponding amplitude constructed according to Eq. (4.19) (where we identify $d^{abc} \langle \alpha_x \alpha_y \alpha_z \rangle_\tau \equiv f^{JW}(x, y, z)$) has the rather curious property to vanish at equal points ($\langle B(x, z, z) \rangle_\tau = 0$, etc.), for which there is no compelling physical justification. For instance, the initial condition (6.2) does not have this property. Thus, the solution to Eq. (6.1) corresponding to this particular initial condition will not belong to the Möbius space.

Similarly, the other exact solution known for the odderon BK P equation, due to Bartels, Lipatov, and Vacca (BLV) [12], does not belong to the Möbius space either. For this solution, the dominant intercept is exactly one, so the corresponding amplitude has only a weak energy dependence (weaker than any power of the energy). As explicitly verified in Ref. [43], using the BLV solution within Eq. (7.7) for the dipole odderon amplitude $\langle O(x, y) \rangle_\tau$ is equivalent to constructing the general $C$–odd solution to the BFKL equation (5.4). Thus, the BLV solution appears as the physical Green’s function describing the odderon exchange between a virtual photon and some generic target (in the dilute regime where saturation effects in the target are unimportant).

Furthermore, by inserting the BLV solution into Eq. (7.8), one obtains a particular solution to Eq. (6.1) for the amplitude $\langle B(x, y, z) \rangle_\tau$ describing odderon exchanges between a 3–quark system and a dilute target\textsuperscript{12}. The following argument suggests that this particular solution should in fact yield the dominant behavior of $\langle B \rangle_\tau$ at high energy; that is, without actually solving Eq. (6.1), one can conclude that the intercept for $\langle B \rangle_\tau$ is exactly one, and is determined by the BLV solution. The argument relies on the relation (4.18) between the odderon amplitudes for the 3–quark system and that for the dipole. Let $1 + \omega$ denote the intercept for $\langle B \rangle_\tau$, that is,

$$
\langle B(x, y, z) \rangle_\tau = h(x, y, z; \tau) e^{\omega \tau},
$$

(7.11)

where $h$ is only slowly varying with $\tau$ (slower than any exponential). Setting $y = z$, one finds

$$
\langle O(x, z) \rangle_\tau = h(x, z, z; \tau) e^{\omega \tau}.
$$

(7.12)

Since, on the other hand, $\langle O(x, z) \rangle_\tau$ must be a $C$–odd solution to the BFKL equation (5.4), the analysis in Ref. [43] demonstrates that the largest possible

\textsuperscript{12}To fully determine the physical amplitude, one must still convolute the BLV Green’s function with the target impact factor.
value of $\omega$ is $\omega = 0$. This argument misses the component of the BKP solution which vanishes at equal points. However, the largest intercept of this kind, corresponding to the JW solution, is known to be less than one \[11\].

8 Conclusions

In this paper we have given the first discussion of the odderon problem within the framework of the effective theory for the color glass condensate. The genuinely CGC part of the analysis has been rather straightforward: The operators describing $C$–odd exchanges between the CGC and two simple projectiles — a $q\bar{q}$ color dipole and a 3–quark system — have been constructed in the eikonal approximation, which includes multiple scattering to all orders via Wilson lines. By acting with the JIMWLK Hamiltonian on these operators, we have then deduced the evolution equations satisfied by the corresponding scattering amplitudes. This is a standard, but generally quite lengthy, procedure, that we have considerably simplified by introducing the dipolar form of the JIMWLK Hamiltonian. In particular, for a dipole projectile, we have recovered the equations previously obtained in Ref. \[43\], that we have generalized here beyond the mean field approximation.

What turned out to be more subtle, however, was the correspondence with the traditional perturbative QCD approach (in particular, with the BKP equation) in the limit where the scattering is weak. Although our equations become linear in this limit, they still apply to gauge–invariant scattering amplitudes, and not directly to Green’s functions. Besides, they are \textit{a priori} written in coordinate space. But the use of the dipolar version of the JIMWLK Hamiltonian has allowed us to write down a well–defined equation for the odderon Green’s function, which after Fourier transformation to momentum space turned out to be the same as the BKP equation.

Our analysis has emphasized the subtlety involved in the Fourier transformation of the BKP equation, and the importance of the structure of the external probe for selecting physical solutions to this equation. While the latter point was already stressed in Refs. \[12,43\] in the context of the dipole scattering, the ability of our formalism to deal with a 3–quark system (which has three independent coordinates) has made this point even clearer, thus shedding light on the Hilbert space to be used in relation with the BKP equation. From the viewpoint of Eq. (6.1), the solutions which belong to the Möbius representation (i.e., which vanish at equal points: $\langle B(x, z, z) \rangle_\tau = 0$, etc.) are very special ones, and are unlikely to be realized during the evolution from physical initial conditions. Rather, the solutions to Eq. (6.1), and therefore also to the BKP equation, which are singled out by our gauge–invariant 3–quark amplitude and the respective initial conditions do \textit{not} vanish at equal points.
This observation, together with the relation Eq. (4.18), has led us to conclude that the highest odderon intercept for the 3-quark–CGC scattering is exactly one, so like for the dipole–CGC scattering [43], and is described again by the Bartels–Lipatov–Vacca solution [12] to the BKP equation.

As mentioned in the Introduction, our weak field analysis of the odderon problem \((n = 3, \, C = -1)\) is intended as a first step in a systematic study of multireggeon \((n \geq 3)\) exchanges in the CGC formalism, with the aim of clarifying the relation between this formalism and more traditional approaches, like GLLA. Although successful in establishing the correspondence with the BKP equation, our previous analysis has also revealed a few subtleties which may lead to difficulties when trying to extend this approach to processes with more than three reggeons. We have seen indeed that the Green’s functions in the CGC formalism make sense only as building blocks (in the sense of linear combinations) for scattering amplitudes in the weak field limit. Thus, in order to have a meaningful definition for a \(n\)-point Green’s function (corresponding to a \(n\)-reggeon exchange), one must first identify appropriate gauge–invariant amplitudes whose weak–field expansion starts at order \(n\) in \(\alpha_s(x)\). The potential difficulty with this approach, however, is that there is \textit{a priori} no guarantee that the nonlocality in \(x^-\) inherent in the Wilson lines will disappear in the expansion leading to the Green’s functions. Recall, for instance, our previous construction of the weak–field odderon operators, Eqs. (4.10) and (4.19), from the corresponding non–linear operators, Eqs. (4.3) and (4.14), respectively: In that case, the nonlocality in \(x^-\) has disappeared from the final results only ‘accidentally’, because of the presence of the totally symmetric tensor \(d^{abc}\). An alternative procedure which looks promising would be to construct the gauge–invariant linear combination directly in the weak–field limit, starting with a non–invariant Green’s function and imposing on it the finiteness condition (2.12). It remains as an interesting open question whether any of the methods mentioned above can be used to construct arbitrary \(n\)–reggeon exchanges. We leave this and related issues for future work.

**Acknowledgments**

We are grateful to Carlo Ewerz for useful discussions and for comments on an early version of the draft. Y.H. thanks Dima Kharzeev, Anna Stasto and Kirill Tuchin for useful conversations. Two of the authors (K.I. and L.M.) thank Yukio Nemoto for the discussion about the baryonic Wilson line operators. We would like to thank Basarab Nicolescu for constantly encouraging us to address the odderon problem in the framework of the color glass condensate. Y. H. is supported by Special Postdoctoral Research Program of RIKEN, and K.I. is supported by the program, JSPS Postdoctoral Fellowships for Research Abroad. This manuscript has been authorized under Contract No. DE-AC02-98CH10886 with the U. S. Department of Energy.
A C-odd initial conditions in the dipole-CGC scattering

The gauge field $\alpha^a_\mathbf{x}$ is created by a color source $\rho^a$ in the target CGC. It is given by (in the covariant gauge)

$$\alpha^a_\mathbf{x} = \frac{1}{4\pi} \int d^2 \mathbf{z} \ln \frac{1}{(\mathbf{x} - \mathbf{z})^2 \mu^2} \rho^a(\mathbf{z}), \quad (A.1)$$

where $\mu$ is an infrared cutoff which will disappear from the final results. For a single quark at transverse position $\mathbf{x}_0$, the color source is given by

$$\rho^a(\mathbf{z}) = Q^a \delta^{(2)}(\mathbf{z} - \mathbf{x}_0), \quad (A.2)$$

and for a dipole made of a quark at $\mathbf{x}_0$ and an antiquark at $\mathbf{y}_0$,

$$\rho^a(\mathbf{z}) = Q^a \left[ \delta^{(2)}(\mathbf{z} - \mathbf{x}_0) - \delta^{(2)}(\mathbf{z} - \mathbf{y}_0) \right], \quad (A.3)$$

where $Q^a$ is the color charge $Q^a = g \int \psi^c t^a \psi$. We obtain the initial conditions for the evolution equation by substituting the gauge fields created by these "unevolved" targets into Eq. (4.10) and (4.11), and taking the average over the "random" gauge field. For the unevolved targets the average over the random configuration simply reduces to the color average. More precisely, in order to evaluate the average $\langle Q^a Q^b Q^c \rangle$, we replace the $c$-number charge $Q^a$ by a color matrix $g t^a$ and take the trace $\frac{1}{N_c} \mathrm{tr}(g t^a g t^b g t^c)$. Thus, it can be evaluated as follows:

$$d^{abc} \langle Q^a Q^b Q^c \rangle \rightarrow d^{abc} \frac{1}{N_c} \mathrm{tr}(g t^a g t^b g t^c) = \frac{g^3}{N_c} d^{abc} \frac{1}{2} \mathrm{tr}(\{t^a, t^b\} t^c) = \frac{g^3}{4N_c} d^{abc} d^{abc} = \frac{g^3}{4N_c^2} (N_c^2 - 4)(N_c^2 - 1). \quad (A.4)$$

This yields the results (5.5) and (5.6). The technique adopted here for the average was first proposed by Iancu and Mueller for the onium-onium scattering to show the equivalence between the color dipole picture and the CGC formalism [45]. In fact, this procedure goes beyond the original McLerran-Venugopalan (MV) model. This is because the MV model is formulated with the Gaussian random source and the average of odd number of sources such as $\langle \rho^a \rho^b \rho^c \rangle$ is simply vanishing. In order to describe the odderon, it is necessary to extend the MV model so that there is nontrivial correlation among three sources. On the other hand, such nontrivial correlation is correctly encoded in the CGC framework, as we claim in the present paper. Indeed, the dipole JIMWLK equation (3.3) in the weak-field limit suggests a similar equation
for the weight function $W_{\tau}[\alpha]$, but the Gaussian weight function is not the exact solution to this evolution equation. It arises only in the mean-field like approximation discussed in Ref. [37].

References


