The $SU(2|3)$ Spin Chain Sigma Model

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Abstract: The one-loop planar dilatation operator of $\mathcal{N} = 4$ supersymmetric Yang-Mills is isomorphic to the hamiltonian of an integrable $PSU(2, 2|4)$ spin chain. We construct the non-linear sigma model describing the continuum limit of the $SU(2|3)$ subsector of the $\mathcal{N} = 4$ chain. We explicitly identify the spin chain sigma model with the one for a superstring moving in $AdS_5 \times S^5$ with large angular momentum along the five-sphere.

1 Introduction

The AdS/CFT correspondence is a fascinating proposal relating the weak coupling regime of a gauge theory to a strong coupling regime in string theory, and vice versa. A proof of the correspondence would require complete access to the strong coupling regime of each theory and, in particular, a detailed understanding of the quantization of the string action in an $AdS_5 \times S^5$ background, which is yet an unsolved problem. There is however a maximally supersymmetric plane-wave limit of the $AdS_5 \times S^5$ background for the IIB string, that can be quantized in the light-cone gauge [1]. On the gauge theory side this limit corresponds to focusing on local operators with a large R-symmetry charge J , of the form $\text{Tr}(Z^J \dots)$, where Z is one of the $\mathcal{N} = 4$ complex scalars [2]. On the gravity side these operators are described in terms of small closed strings with their center of mass moving with large angular momentum J along a circle in S^5 [3].

More general string configurations with several large angular momenta along S^5 were afterwards proposed to correspond to operators composed of the three $\mathcal{N} = 4$ complex scalars [4]. The energy of these semiclassical strings turned out to admit an analytic expansion on the effective parameter λ/J^2 , with λ the 't Hooft coupling constant of the gauge theory. This expansion suggested a direct comparison with the anomalous dimensions of large Yang-Mills operators. The calculation of anomalous dimensions for operators with a large number of constituent fields was rendered accessible by the deep observation that the one-loop planar dilatation operator of $\mathcal{N} = 4$ supersymmetric Yang-Mills [5] is isomorphic to the hamiltonian of an integrable $PSU(2, 2|4)$ spin chain [6, 7]. Higher loop integrability remains an open problem, although strong arguments in favor exist for some sectors of $\mathcal{N} = 4$ [5, 8, 9]. With the help of the powerful Bethe ansatz

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technique, a perfect agreement between the anomalous dimensions of gauge operators and energies of dual semiclassical strings was shown up to two-loops [10]. It should however be stressed a disagreement starting at three-loops [9], which is currently one of the most intriguing questions on the AdS/CFT correspondence.

A promising path towards a more complete understanding of the correspondence between operators with large quantum numbers and semiclassical string solutions is to directly compare the action describing the continuum limit of the spin chain in the coherent state basis with the dual string non-linear sigma model. In [11, 12], the continuum limit of the $SU(2)$ Heisenberg spin chain, associated to $\mathcal{N}=4$ operators composed of two complex scalar fields, was shown to reproduce the action of strings moving with large angular momentum along an S^3 section of S^5 . In this note we will consider $\mathcal{N}=4$ operators containing both scalar and fermion fields and we will derive the continuum limit of the associated spin chain [13, 14, 15]. We will then show that the associated spin chain sigma model is able to reproduce the dual superstring action including fermionic terms (see also [16]). In particular, we will exhibit the spin chain counterpart of an interesting phenomenon present on the string theory side: the coupling of fermions to the RR five-form.

2 The $SU(2|3)$ spin chain sigma model

We will consider operators composed of the three $\mathcal{N}=4$ complex scalars and two combinations of the sixteen gaugino components transforming as $(\mathbf{2}, \mathbf{1})$ under the Lorentz group, and left invariant under the $SU(3)$ subgroup of the R-symmetry group under which the scalars transform in the fundamental representation. This set of operators maps to a $SU(2|3)$ spin chain based on the $\mathbf{3|2}$ representation. Its bosonic sector corresponds to a ferromagnetic $SU(3)$ Heisenberg spin chain. The one-loop hamiltonian of the $SU(2|3)$ chain is [8]

$$H = \frac{\lambda}{8\pi^2} \sum_{l=1}^L (1 - SP_{l,l+1}) , \quad (1)$$

where $SP_{l,l+1}$ denotes the super-permutation operator between two neighboring sites, and L is the number of sites of the chain. The $SU(2|3)$ sector of $\mathcal{N}=4$ Yang-Mills is closed to all loops under renormalization and its dilatation operator is known up to three loops [8]. However, in the following we will restrict our analysis to one-loop.

2.1 The $PSU(1|1)$ subsector

As a warming up step, we will first consider the simpler case of $PSU(1|1) \subset SU(2|3)$. This subsector is associated to operators of the schematic form $\text{Tr}(Z^J \lambda^{J'})$, with $L = J + J'$ denoting the length of the associated spin chain. At each site of the chain sits a two-dimensional vector space with one bosonic and one fermionic generator, $|b\rangle$ and $|f\rangle$, respectively. In order to study the continuum limit, we will introduce a set of coherent states at each site of the chain by applying an arbitrary $PSU(1|1)$ rotation to $|b\rangle$,

$$|n\rangle = e^{i\xi S_y} e^{i\chi S_x} |b\rangle = (1 - \frac{1}{2} \zeta^* \zeta) |b\rangle + \zeta |f\rangle , \quad (2)$$

where S_x and S_y are the two odd generators of $PSU(1|1)$, ξ and χ are two Grassmann variables, and we have introduced the complex combination $\zeta = -\xi + i\chi$. The coherent states (2) verify $\langle n|n\rangle = 1$. They form an overcomplete basis, with the resolution of the

identity

$$\int d\zeta d\zeta^* |n\rangle\langle n| = \mathbb{1} . \quad (3)$$

A path integral description of the partition function, as an integral over the over-complete set of coherent states, shows the equivalence between the spin chain and the following discrete sigma model

$$S = - \int dt \left[i \langle \mathbf{n} | \frac{d}{dt} | \mathbf{n} \rangle + \langle \mathbf{n} | H | \mathbf{n} \rangle \right] , \quad (4)$$

where we have introduced the product of coherent states along the chain $|\mathbf{n}\rangle = |n_1 \cdots n_L\rangle$.

In order to derive the effective action describing the continuum limit of the spin chain, it is generically necessary to take into account quantum effects of short wavelength configurations [12]. However, when we are only interested in capturing the physics at first order in $\frac{\lambda}{L^2}$, it is enough to substitute in (4) finite differences between variables at neighboring sites by derivatives. After an straightforward calculation we obtain

$$S = - \frac{L}{2\pi} \int d\sigma dt \left[i \zeta^* \partial_t \zeta + \frac{\lambda}{2L^2} \partial_\sigma \zeta^* \partial_\sigma \zeta \right] , \quad (5)$$

which describes a non-relativistic fermion, with mass $m = \frac{L^2}{\lambda}$. This action reproduces, at first order in λ/L^2 , the spectrum of fermionic fluctuations in the pp -wave limit [2].

2.2 The $SU(1|3)$ subsector

We will analyze now the $SU(1|3)$ subsector, associated to operators composed of the three complex scalars and one fermion. The derivation of the continuum limit of the associated $SU(1|3)$ spin chain follows the same steps as above. We will introduce a set of spin coherent states at each site of the chain by applying an arbitrary rotation to a bosonic eigenstate of the Cartan generators. Since such state is left invariant by an $SU(1|2)$ subgroup and just multiplied by a phase via an additional $U(1)$, the set of coherent states will be isomorphic to the quotient $SU(1|3)/SU(1|2) \times U(1)$. Hence to describe them we need four real variables and two Grassmann ones. An easy way to construct the $SU(1|3)$ coherent states is to consider first those for a $PSU(1|1)$ subsector and then apply to them an arbitrary $SU(3)$ rotation. The resulting states will have the same form (2) as before, but with $|b\rangle$ denoting now a $SU(3)$ coherent state in the fundamental representation. The four real variables necessary to describe $|b\rangle$ can be chosen as [13]

$$|b\rangle = \cos \theta \cos \psi e^{i\varphi} |1\rangle + \cos \theta \sin \psi e^{-i\varphi} |2\rangle + \sin \theta e^{i\phi} |3\rangle , \quad (6)$$

where $\theta, \psi \in [0, \pi/2]$ are associated to $SU(3)$ rotations that mix 13 and 12 respectively, and $\phi + \varphi, \phi - \varphi \in [0, 2\pi]$ to the Cartan generators. The coherent states satisfy again $\langle n | n \rangle = 1$ and expand the identity in an expression analogous to (3).

The continuum limit of the chain is obtained as before by evaluating (4) over coherent state configurations that vary smoothly along the chain. The bosonic part of the action, describing the continuum limit of the ferromagnetic $SU(3)$ Heisenberg spin chain is [13, 14]

$$S_B = \frac{L}{2\pi} \int d\sigma dt \left[C_0 - \frac{\lambda}{2L^2} e^2 \right] , \quad (7)$$

where C_0 is the time component of the 1-form

$$C = -i \langle b | d | b \rangle = \sin^2 \theta d\phi + \cos^2 \theta \cos(2\psi) d\varphi , \quad (8)$$

and we have defined

$$e^2 = \theta'^2 + \cos^2 \theta \left(\psi'^2 + \sin^2(2\psi) \varphi'^2 \right) + \sin^2 \theta \cos^2 \theta (\phi' - \cos(2\psi) \varphi')^2, \quad (9)$$

which represents energy density of the $SU(3)$ sigma model, $H = \frac{\lambda}{4\pi L} \int d\sigma e^2$. The fermionic piece of the action is given by

$$S_F = -\frac{L}{2\pi} \int d\sigma dt \left[i\zeta^* D_t \zeta + \frac{\lambda}{2L^2} (D_\sigma \zeta^* D_\sigma \zeta - e^2 \zeta^* \zeta) \right], \quad (10)$$

with $D\zeta = d\zeta - iC\zeta$. One more condition should be added in order that field configurations of the sigma model $S = S_B + S_F$ represent Yang-Mills operators: the trace required to define gauge invariant operators, implies that we need to impose translational invariance on the spin chain configurations, *i.e.* $P_\sigma = 2\pi\mathbf{Z}$.

Definitions of the coherent states that differ by a phase should be considered equivalent. Therefore in order to parameterize the set of coherent states, we have to make a choice of global phase. Under a change in the definition $|n\rangle \rightarrow e^{i\alpha}|n\rangle$, we have $\zeta \rightarrow e^{i\alpha}\zeta$ and $C \rightarrow C + d\alpha$. Consistently, this transformation leaves invariant S_F and only changes the bosonic term S_B by an irrelevant total derivative. However, it is important to notice that a phase choice can not be fixed globally over the set of $SU(3)$ coherent states. In particular, the one implicit in (6) becomes singular at $\theta = \frac{\pi}{2}$ and $\psi = 0, \frac{\pi}{2}$. The appearance of the covariant derivatives in (10) is a consequence of the non-triviality of the line bundle $\{e^{i\alpha}|b\rangle\}$, contrary to the simpler $PSU(1|1)$ case.

2.3 The $SU(2|3)$ sigma model

We will parameterize the coherent states of the complete $SU(2|3)$ chain as follows,

$$|n\rangle = (1 - \frac{1}{2}\zeta_1^* \zeta_1 - \frac{1}{2}\zeta_2^* \zeta_2 + \frac{3}{4}\zeta_1^* \zeta_1 \zeta_2^* \zeta_2)(|b\rangle + \zeta_1|f_1\rangle + \zeta_2|f_2\rangle), \quad (11)$$

with $|b\rangle$ again the $SU(3)$ coherent state (6), $|f_1\rangle$ and $|f_2\rangle$ fermionic states belonging to the fundamental representation of $SU(2|3)$, and ζ_1 and ζ_2 two complex Grassmann variables. The term factored out in (11) insures $\langle n|n\rangle = 1$.

The action describing the continuum limit of the chain is obtained following the steps by now familiar. The bosonic part of this action gives again the $SU(3)$ non-linear sigma model (7). For the fermionic part we obtain

$$S_F = -\frac{L}{2\pi} \int d\sigma dt \left\{ i\zeta_1^* D_t \zeta_1 + i\zeta_2^* D_t \zeta_2 - iz^* D_t z + \frac{\lambda}{2L^2} [D_\sigma \zeta_1^* D_\sigma \zeta_1 + D_\sigma \zeta_2^* D_\sigma \zeta_2 - D_\sigma z^* D_\sigma z - e^2 (\zeta_1^* \zeta_1 + \zeta_2^* \zeta_2 - 2z^* z)] \right\}, \quad (12)$$

where $z = \zeta_1 \zeta_2$, and the covariant derivatives are $D\zeta_j = d\zeta_j - iC\zeta_j$ (with $j = 1, 2$) and $Dz = dz - 2iCz$. This action describes two non-relativistic fermions with quartic interactions. At the quadratic level it coincides with (10) for each of the fields ζ_j .

3 The superstring action

In this section we will compare the spin chain sigma model with the type IIB superstring action in $AdS_5 \times S^5$. We will start by briefly analyzing this problem for the bosonic

part of the string action. Since on the gauge theory side we are considering only bosonic operators composed of scalar fields, we must restrict to strings moving on S^5 . Large operators composed of L fields, as those involved in the continuum limit of the chain, map to strings rotating with large total angular momentum $J = L$. We will set the gauge $t = \kappa\tau$, where t and τ are respectively the target and worldsheet times.

Let us call α the angular variable on S^5 whose associated conserved quantity is the angular momentum L . The limit of large L corresponds to strings rotating close to the speed of light in the α direction [4]. This implies that their energy is $\frac{E}{L} = \frac{\sqrt{\lambda}}{L}\kappa \sim 1$ and thus κ must be large ¹. In order to analyze the string action, it is most convenient to distinguish between fast and slow components of the string movement [11, 12]. The fast movement can be absorbed by shifting $\alpha \rightarrow \alpha + t$. After this redefinition, it is easy to see that the string propagation verifies $\dot{X}^M = \mathcal{O}(\frac{1}{\kappa^2})$ in all S^5 directions [11]. Expanding now the string sigma model in powers of $1/\kappa$, we obtain a non-relativistic action describing the slow movement of the string. The $SU(3)$ spin chain sigma model (7) is recovered from the leading term in this expansion [13, 14].

The fermionic part of the type IIB superstring action in $AdS_5 \times S^5$ is very involved, already at the classical level, due to the coupling of the fermions to the RR-field. However it has been worked out explicitly for the fermionic quadratic terms in [17]. Choosing the κ -symmetry so that both Majorana-Weyl spinors in ten dimensions are equal, the Green-Schwarz superstring action expanded to quadratic order near a particular bosonic string solution (with a flat induced metric) is

$$L = -2i\bar{\vartheta}\left(\rho^a D_a + \frac{i}{2}\epsilon^{ab}\rho_a\Gamma_*\rho_b\right)\vartheta, \quad D_a = \partial_a + \frac{1}{4}\omega_a^{AB}\Gamma_{AB}, \quad (13)$$

where ρ_a and ω_a^{AB} (with $a = 0, 1$) are the pullback onto the string worldsheet of the $AdS_5 \times S^5$ gamma matrices and spin connection. The second term in (13) is a mass term, and has its origin in the coupling to the RR 5-form [17]. We have defined $\Gamma_* = i\Gamma_0\Gamma_6\Gamma_7\Gamma_8\Gamma_9$, with $\Gamma_*^2 = \mathbb{1}$. Γ_A are the flat space ten-dimensional gamma matrices, where the indices $M = (0, 6, 7, 8, 9)$ correspond to AdS_5 and $M = (1, 2, 3, 4, 5)$ to S^5 .

Our aim is to compare (13) with the spin chain action for the fermionic fluctuations (10). Hence we choose as bosonic background for (13) a string rotating on S^5 . Actually, in order to simplify the calculation while keeping at a general level, we will restrict to strings rotating in an S^3 section of S^5 . They map to large $\mathcal{N} = 4$ Yang-Mills operators composed of two complex scalars, and correspond to setting $\theta = 0$ in (7).

The fermionic action (10) describes two-dimensional fermions with canonically normalized kinetic terms, coupled to a gauge field which carries the information about the bosonic background. Contrary, in the fermionic string action (13) the background is felt both on the matrices ρ_a and on the covariant derivatives D_a . For the comparison of both actions, the first step is to trivialize the matrices ρ_a and concentrate the effect of the background only on D_a . This can be achieved by applying a suitable rotation S to the spinor ϑ , as is described in detail in [15]. After this, (13) becomes

$$L = -2i\kappa^2\bar{\Psi}\left[\Gamma_0(\partial_t + i\Pi C_t) - \frac{1}{\kappa}\Gamma_3(\partial_\sigma + i\Pi C_\sigma) + \left(1 - \frac{e^2}{2\kappa^2}\right)\Gamma_0\Gamma_3\Gamma_2\bar{\Pi}\right]\Psi, \quad (14)$$

where C_a coincide with the components of the 1-form (8), e^2 is given by (9) and $\Pi = -i\Gamma_0\Gamma_1\Gamma_2\Gamma_3$ and $\bar{\Pi} = \Gamma_6\Gamma_7\Gamma_8\Gamma_9$. The new spinor field is defined by $\Psi = \sqrt{\frac{e}{\kappa}}S\vartheta$.

¹The large angular momentum limit that must be considered to compare with the spin chain results requires $L \gg \sqrt{\lambda}$.

In deriving (14), the bosonic background was again divided in fast and slow components and its effect on the fermionic excitations was taken into account only to order $1/\kappa$. For consistency, in the time evolution of the fermions we should also distinguish between fast and slow oscillations. The fast oscillations are due to the leading contribution, of order κ^0 , to the last term of (14). After they are subtracted, we obtain from the ten-dimensional spinor Ψ four fermions with the same action (13) [15]. This four-fold degeneracy can also be understood from the gauge theory point of view. There are four complex combinations of the sixteen gaugino components that are left invariant by the $SU(2)$ subgroup of the R-symmetry group under which two of the $\mathcal{N}=4$ complex scalars transform in the fundamental representation.

Hence the continuum limit of the gauge theory spin chain is able to reproduce both bosonic and fermionic parts of the superstring action. Notice in particular that the term proportional to e^2 in (10) has its origin, from the string theory point of view, in the coupling of the fermions to the background RR 5-form, namely, the last term in both (13) and (14). It will be very interesting to pursue further this analysis and compare the fermionic quartic terms in (12) with the superstring action.

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