Causal structures and causal boundaries

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Abstract. We give an up-to-date perspective with a general overview of the theory of causal properties, the derived causal structures, their classification and applications, and the definition and construction of causal boundaries and of causal symmetries, mostly for Lorentzian manifolds but also in more abstract settings.

PACS numbers: 04.20.Gz, 04.20.Cv, 02.40.Ky

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Causality is the relation between causes and their effects, or between regularly correlated phenomena. Causes are things that bring about results, actions or conditions. No doubt, therefore, causality has been a major theme of concern for all branches of philosophy and science for centuries. Perhaps the first philosophical statement about causality is due to the presocratic atomists Leukippus and Democritus: “nothing happens without the influence of a cause; everything occurs causally and by need”. This intuitive view was a matter of continuous controversy, eventually cleverly criticized by Hume, later followed by Kant, on the grounds that probably our belief that an event follows from a cause may be simply a prejudice due to an association of ideas founded on a large number of experiences where similar things happened in the same order. In summary, an incomplete induction.

The task of science and particularly Physics, however, is establishing or unveiling relations between phenomena, with a main goal: predictability of repetitions or of
new phenomena. Nevertheless, when we state that, for instance, an increase on the pressure of a gas reduces its volume, we could equally state that a decrease of the volume increases the pressure. The ambiguity of the couple cause-effect is even greater when we reverse the sense of time. And this is one of the key points. The whole idea of causality must be founded upon the basic a priori that there is an orientation of time, a time-arrow. This defines the future (and the past), at least on our neighbourhood and momentarily.

Consequently, all branches of Physics, old and new, classical or modern, have a causality theory lying underneath. This is specially the case after the unification process initiated by Maxwell with Electromagnetism and partly culminated by Einstein with the theories of Relativity, where time and space are intimately inter-related. Soon after, Minkowski was able to unite these two concepts in one single entity: spacetime, a generalization of the classical three-dimensional Euclidean space by adding an axis of time. A decisive difference with Euclidean space arises, though: time has a different status, it mingles with space but keeping its identity. It has, so to speak, a “different sign”. This immediately leads to the most basic and fundamental causal object, the cone which embraces the time-axis, leaving all space axes outside. And upon this basic fact, by naming the two halves of the cone (future and past), we may erect a whole theory of causality and causal structure for spacetimes.

Four-dimensional spacetimes (see definition 2.1) are the basic arena in General and Special Relativity and their avatars, and almost any other theory trying to incorporate the gravitational field or the finite speed of propagation of signals will have a (possibly $n$-dimensional with $n \geq 4$) spacetime at its base. The maximum speed of propagation is represented in this picture by the angle of the cone at each point. And this obviously determines the points that may affect, or be influenced by, other points. In Physics, spacetimes represent the Universe, or that part of it, we live in or want to model. Thus, the notion of spacetime is pregnant with “causal” concepts, that is, with an inherent causality theory. Causality theory has played a very important role in the development of General Relativity. It is fundamental for all global formalisms, for the theory of radiation, asymptotics, initial value formulation, mathematical developments, singularity theorems, and many more. There are books dealing primarily with causality theory ([6, 73]) and many standard books about General Relativity contain at least a chapter devoted to causality [64, 112, 172]. Recently, modern approaches to “quantum gravity” have borrowed concepts widely used in causality theory such as the causal boundary (AdS/CFT correspondence [75]) or abstract causal spaces (quantum causal sets [13]).

Roughly speaking the causal structure of a spacetime is determined at three stages: primarily, by the mentioned cones—called null cones—, at each event. This is the algebraic level. Second, by the the connectivity properties concerning nearby points, which is essentially determined by the null cone and the local differential structure through geodesic arcs. This is the local stage. And third, by the connectivity at large, that is, between any possible pair of points, be them close or not. This connection must be achieved by sequences of locally causally related events, usually by causal curves —representing the paths of physical small objects. This is the global level. These three stages taken together determine the causal structure of the spacetime. In short, the causal structure involves the sort of relations, properties, and constructions arising between events, or defined on tensor objects, which depend essentially on the existence of the null cones, that is to say, on the existence of a sole time in front of all spatial dimensions. Unfortunately, a precise definition of “causal structure” is in
general lacking in the literature, as it is more or less taken for granted in the formalisms employed by each author. One of the aims of this review is to provide an appropriate, useful and rigorous definition of “causal structure” of sufficient generality.

From a mathematical viewpoint, a spacetime is a Lorentzian manifold: an n-dimensional semi-Riemannian manifold \((V, g)\) where at each point \(x \in V\) the metric tensor \(g|_x\), which gives a local notion of distances and time intervals, has Lorentzian signature \([117]\), the axis with the different sign \((n > 2)\) indicating “time” (cf. definition 2.1 for more details). Even though a great deal of research has been performed in four-dimensional Lorentzian manifolds for obvious reasons, almost all the results do not depend on the manifold dimension and we will always work in arbitrary dimension \(n\). This is also adapted to more recent advances such as String Theory, Supergravity, et cetera. The field equations or physical conditions that the metric tensor \(g\) must satisfy in order to lead to an acceptable representation of an actual spacetime are in principle out of the scope of this review. There is an entire branch of Mathematics called Lorentzian geometry whose subject is the study of Lorentzian manifolds and it encompasses topics (such as causality theory) which traditionally have been studied by relativists. While the study of proper Riemannian manifolds (the metric tensor is positive definite at each point of the manifold so there is no time and no causality) presents a status in which important questions such as the presence and study of singularities, geodesic connectivity, existence of minimizing geodesics, splittings, or the completion of the manifold are ruled by powerful theorems, the matter is radically different in Lorentzian geometry where equivalent or analogous results require a case-by-case discussion with no general rule, and sometimes the answers to the “same” mathematical problems are entirely different. This makes of Lorentzian geometry a more difficult topic where there are still open questions regarded as bearing the same degree of importance as those already solved in proper Riemannian geometry.

Perhaps the main advantage of proper Riemannian geometry in this regard is the existence of a well-defined notion of distance between points, a feature absent in Lorentzian geometry where only a pseudo-distance can be defined \([6]\). As a result many new possibilities arise in Lorentzian geometry, see e.g. \([6, 64, 117, 172]\). Among these new possibilities, as we will largely discuss in this review, those dealing with the causal structure and the causal completion of spacetimes are specially interesting mathematically and physically.

From both these perspectives, a generic causal structure collects all information about Lorentzian manifolds not related to the particular geometrical form, or causal characteristics, of any particular spacetime. That is to say, the properties which totally depend on the existence of any Lorentzian metric \(g\) on the manifold, and are absent if \(g\) is removed. A particular causal structure will then be a class of Lorentzian manifolds carrying equivalent causal properties. Since any good causal property must be conformally invariant (because the null cones remain intact under conformal transformations) many authors have assumed implicitly that the causal structure is fully determined by the conformal class of metrics: all metrics proportional to each other with a positive proportionality factor. However, recent studies (see subsections 4.2, 4.3 and \([47]\)) indicate that this view may be too restrictive, and that non-conformally related metrics can belong to the same causality class in a well-defined way. Of course, these more general causal classes, while not “internally conformal”, are conformally invariant! This opens a wide new world concerning causality which is still to be explored in detail. Related to this is the question of the classification of “exact solutions”, i.e. particular spacetimes satisfying the field equations of a given
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theory and thus of physical interest, in terms of their causal structure. For instance, nowadays there is a large number of known exact solutions to Einstein’s field equations in four dimensions (the standard reference is [161]). Some of them have been analyzed from a global point of view [64, 128, 172, 6], but this study of global causal properties has not been performed in a majority of cases.

Though most of the theory of causal structures has been carried out in the framework of Lorentzian manifolds, the definition of generic causal structures showed the possibility of a different line of thought making abstraction of the existence of the metric. This is possible because well-defined binary relations can be set between points of a Lorentzian manifold according to their connectivity by causal curves. As these binary relations fulfill very precise properties, one can devise abstract sets, called causal spaces, with no metric —nor even differentiable structure— but possessing binary relations whose properties resemble those present in spacetimes [89, 25, 174]. These more abstract sets are also considered in this review and an account of the most popular causal spaces is given in section 3. Probably, most if not all of the results presented in this review for spacetimes extend in a way or another to abstract causal spaces. Sometimes this has been made explicit in the review, but not always. As a matter of fact, for example, the new results of sections 4.2 and 4.3 can be extended to abstract causal spaces and in this sense the word “causal structure” also acquires a precise meaning for them.

The important issue of the correspondence between the mathematical model employed to describe the spacetime (be it a Lorentzian manifold or an abstract causal space) and the real things that we see or the experimental measures we may perform has also been explored in the literature [34], and we give a brief account in subsection 3.3.

An important branch of causality theory with many ramifications is the theory concerning causal boundaries. These are essentially sets of “ideal points” which, when added appropriately to a particular spacetime, make it complete in a definite sense. The idea is to attach a boundary to the spacetime under study keeping the causal properties and bringing infinity to finite values of judiciously chosen coordinate systems. Also singularities are to be described as boundary points. Causal completions stemmed from the Lorentzian analog of the conformal compactification procedure which can be performed on proper Riemannian manifolds. In the Lorentzian case we can still carry out conformal compactifications, or other sensible completions, but the boundary points representing infinity can be usually be classified into further subclasses [122, 123], and there is no unequivocal procedure to achieve the causal completions. Unfortunately, the conformal compactification is not always practical nor satisfactory, because either it is too difficult to achieve in simple cases, or one is interested in completing the spacetime without utilizing external elements, but employing only elements of the spacetime itself. This is precisely the idea behind the Geroch, Kronheimer and Penrose construction, (cf. subsection 6.2), the most sophisticated and successful approach to causal boundaries historically. There is however not a single proposal to accomplish this and by now a wide range of techniques and procedures to construct a “causal boundary” for a spacetime are available, each of them with its own advantages and disadvantages. This is an important line of research, with many relevant applications. Just to mention a few: (i) the definition of asymptotically simple spacetimes and asymptotic flatness, see e.g. [64, 128, 122, 123, 162, 172], has revealed a wealth of interesting properties of the conformal boundary permitting explicit computations of the gravitational power
radiated to infinity and the construction of conserved quantities [124, 39, 162]; (ii) the study of the global characteristics, and global causal properties of spacetimes. For instance, it is very easy to tell apart globally hyperbolic spacetimes (see definition 4.1) from the rest by studying properties of the boundary; and (iii) more recently, causal boundaries have received new interest due to the presence of the causal boundary concept in the Maldacena conjecture [101, 75]. We review all new and classical matters concerning causal boundaries in section 6.

Having reached this point, it should be clear that studies related to the causal structure and the causal boundary in any of their varieties has been a high priority among relativists and differential geometers. One can also convince oneself by taking a glimpse at the long list of references—and the references therein! Why then a topical review on causal structures and causal boundaries? The answer is, in our opinion, twofold: there seemed to be convenient an up-to-date revision and compilation of the ideas sketched in previous paragraphs, pointing out those already out-fashioned or obsolete, and those with a promising future. And this should be complemented with a compilation of the new achievements, and the new possibilities opened in the field, which have been taking place in recent years. Examples of these are the developments of (realizations of) semigroups and monoids [72, 71, 115], the potentialities of causal symmetries and causal preserving vector fields [48, 49], new results about the splitting of Lorentzian manifolds [11, 43], new advances [106, 105, 107, 78, 58, 59, 47] and renewed interest [101, 75] in the construction of causal boundaries, and others. It is also helpful to set links between ideas presented in different places and under various motivations and a review like this could come into aid towards this goal. The review is intended to provide an introduction with enough basic information to make the article interesting and informative for non-specialist scientists. It can also serve as a reference source for those researchers interested in deepening their knowledge on any of the topics presented herein. Moreover, we have really tried to bring to physicists attention some useful mathematical tools available in journals and books mainly addressed to mathematicians, and conversely, to recall mathematicians of many efforts and results which may be unknown to them.

In writing this review we were confronted with the decision as to what topics should be covered keeping the length to a reasonable size. The guiding principle was to achieve a good compromise between historical relevance, impact on other research, applicability, and future promises held by published result. Perhaps one of the topics with a greater influence is Penrose’s idea of attaching a boundary to a given spacetime and this is why a major part of the review is devoted to the implications and generalizations of this fruitful idea. Other relevant topics treated are the precise definition of causal structure, the axiomatic study of causality theory, and the classification of spacetimes according to their global causal properties as there are a number of recent investigations which have profited from these classic topics in causality. A full account of the matters covered together with the outline of the review is given in subsection 1.1.

Bearing in mind the above ideas, we set no time limits on the papers or books surveyed and only their compliance with the chosen topics was taken into account. In general we do not mean to be exhaustive in our exposition of each paper, rather, we give overviews in such a way that the reader may get a general idea and extract enough information to decide if the reference is worth looking at. Thus, proofs of the results are in general omitted. Examples are not included either with exceptions.

The work of a large number of authors is described in the review. As is an
unfortunate custom among us, scientists, each author often follows his/her own notation and conventions. Hence it was a bit challenging to write a readable text while keeping faithful to the contents of the references. For the sake of clarity, we have maintained a consistent standard notation throughout, and we have adapted the original notation in the references to ours in order to provide a unified clear treatment of the subjects.

1.1. Plan of the paper with a brief description of contents

In this review there are sections which may be instructive for young researchers, Ph.D. students, and non-specialist scientists interested in the subject, as they contain introductions and enough basic information to make them informative for a general readership. Other parts of the review give an account of recent results which due to their novelty might arouse the scientific curiosity of all interested readers. Generally speaking, a non-specialist may be interested in reading sections 2, 4 and, if willing to learn the basics about causal boundaries, also parts of section 6 (for instance subsections 6.1, 6.2 and §6.5.1.) A mathematically oriented researcher may be interested in the more abstract constructions of section 3, in the theory of semigroups and monoids and the causal symmetries of section 7, or in the topological results of section 5. A theoretical physicist would probably like to skip section 3, but then he/she is advised to read subsection 3.3. On the other hand, sections 2 (except subsection 2.1), 3 and subsection 4.1 may be skipped by experts in the field, as we have tried to use a standard notation. If a reader wishes to pay attention only to the causal boundaries for spacetimes, then he/she can go directly to section 6 and consult, if necessary, other parts of the review to which we refer there. Finally, the review contains up-to-date information, recent advances and a handful of new applications which are not well known, not even for some experts. These new lines of research can be found in subsections 2.1, §4.2.1, 4.3, 5.2, §6.3.4, §6.3.6, §6.4.5, §6.5.2, §6.5.3 and the whole section 7. Next, we give a brief outline of the contents of each section in order to help readers to choose according to their own interests.

Section 2 is a basic summary of Lorentzian causality in which all the standard concepts such as null cone, causal curve and basic sets used in the causal analysis are introduced, and we set the notation followed in the review. Despite this being a basically standard section, the relatively novel concept of future (and past) tensor, which is relevant for the entire theory of causality and permits to carry over the classical null cone structure to all tensor bundles is presented here in subsection 2.1.

Section 3 contains an account of four axiomatic approaches to sets possessing causal relations, trying to reproduce at least part of the basic causal building blocks present in a Lorentzian manifold. These are the Kronheimer-Penrose causal spaces, Carter’s etiological spaces, the Ehlers-Pirani-Schild axioms for a physical spacetime, later improved by Woodhouse, and the causal set model for a quantum spacetime. The relationship between these approaches, their common features and differences are analyzed. Apart from evident historical reasons, we have decided to put section 2 before section 3 because this is very instructive, and also because almost all the abstract concepts introduced in section 3, which do not need of any smooth underlying structure, were in fact inspired by the Lorentzian geometry ideas.

Section 4 accounts for different procedures to classify Lorentzian manifolds and abstract causal/etiological spaces. The standard hierarchy of causality conditions is reviewed and briefly commented. An improved recent scheme to classify Lorentzian
manifolds based on setting causal mappings between them is here reproduced. In this framework, the ultimate abstract definition of causal structure for a differentiable manifold is brought to attention, and its implications investigated. A correct version of a classically sought theorem, which was unproved until recently in its precise form, stating the local equivalence of all Lorentzian manifolds from the causal point of view is given here. The definition and potentialities of causal chains of Lorentzian structures on a given manifold are also recalled.

The interplay between topology and causality is the subject of section 5. Classical topics such as possible topologies on spaces of causal curves or spaces of Lorentzian metrics are explained here in a succinct manner. Remarkable recent results on splittings of globally hyperbolic Lorentzian manifolds are included in this section.

Section 6 is the longest of the review and is devoted to an exhaustive account of the major attempts of finding “the boundary” of a Lorentzian manifold. This section could very well deserve a full topical review on its own. The contents include all classical approaches explained in a concise form, but new important and promising advances have also been described, such as the intriguing new ideas by Marolf and Ross, the proof by Harris of the universality of the classical future boundary using categories, or the new definition of causal boundary and causal completions using causal mappings. A basic introduction to the Penrose diagrams with a comparison to their newly defined generalization called causal diagrams is also provided.

In section 7 we discuss new concepts concerning the role of continuous transformations preserving the causal properties of spacetimes. This is a new line of research which incorporates some recent mathematical advances concerning realizations of semigroups and general cone structures. The transformations are studied from a finite viewpoint first (causal symmetries) and also from the infinitesimal perspective by defining their generators (causally-preserving vector fields). The set of all such transformations is no longer a group, but a monoid which is in turn a subset of a more general algebraic structure known as semigroup. Actions of semigroups on manifolds have been studied in mathematics and a brief account of this work is given in order to bring it to the knowledge of mathematical physicists.

Finally section 8 suggests new possible avenues of research, possible applications, and some interesting open problems which would be desirable to solve.

1.2. Conventions and Notation

An $n$-dimensional differentiable manifold $V$ (sometimes $M$) will be the basic arena in this review (except in sections 3 and some other related places). Thus, $V$ is endowed with a differentiable structure [6, 64, 117, 172], that is to say, given any two local charts $V_1, V_2 \in \mathcal{D}$ of a given atlas $\mathcal{D}$ with $V_1 \cap V_2 \neq \emptyset$ the induced diffeomorphism between the corresponding open sets of $\mathbb{R}^n$ is of class $C^k$, $k \in \mathbb{N}$ ($k$ times differentiable with continuity). Two atlases $\mathcal{D}_1, \mathcal{D}_2$ are said to be $C^k$-compatible if $\mathcal{D}_1 \cup \mathcal{D}_2$ provides another $C^k$ differentiable structure for $V$. We will further assume that $V$ is Hausdorff, oriented and connected.

One can define at any point $x \in V$ the tangent space $T_x(V)$, the cotangent space $T^*_x(V)$ and by means of the tensor product “$\otimes$” of these vector spaces the space of $r$-contravariant $s$-covariant tensors $T^r_s|_x(V)$ [6, 64, 117, 172]. They give rise to the bundles $T(V)$, $T^*(V)$ and $T^r_s(V)$ respectively. Boldface letters will be used to denote elements of any of the mentioned spaces (and also for the sections of the bundles), their distinction being usually obvious contextually. Indices on $V$ are occasionally
used and represented by lowercase Latin letters. The push-forward and pull-back of a map \( \Phi \) are denoted by \( \Phi^\ast \) and \( \Phi^\prime \) respectively.

In this review we will be mostly concerned with a particular type of pseudo-Riemannian (also called semi-Riemannian) manifolds [117] which are manifolds where a metric tensor \( g \) is defined: \( g \) is a non-degenerate symmetric (at least) \( C^1 \) section of the bundle \( T_2^0(V) \), so that in local coordinates \( \det(g_{ab}(x)) \neq 0 \), \( g_{ab}(x) = g_{ba}(x) \), \( \forall x \in V \) and the signature of \( g \) is constant on \( V \). In any semi-Riemannian manifold there is a canonical isomorphism between \( T_+^0(V) \) and \( T_+^0(V) \) which is induced by \( g \) and thus indices on tensors can be “raised and lowered” adequately. When the signature of \( g \) has a definite sign the manifold is called proper Riemannian or simply Riemannian. However, generic causal properties and the intuitive concepts of ‘time flow’, ‘future and past’, etcetera, can only be defined if the signature is Lorentzian, so that one of the dimensions (time) has a different status with respect to the rest (space). This allows for two possible (equivalent) choices, \((+,-,\ldots,-)\) or \((-,+,\ldots,+),\) depending on whether a vector pointing on the time direction is chosen to have positive or negative length. Such pseudo-Riemannian manifolds are called Lorentzian manifolds [6, 64, 117]. Our convention will be the first one, a choice determined by the particular goals we have in mind (causal properties), so that any \( \vec{v} \in T_+^0(V) \) will be called timelike, null or spacelike if \( g_{|v}(\vec{v},\vec{v}) \) is respectively greater than, equal to or less than zero. Non-spacelike vectors are commonly treated together and then also simply called causal vectors. An important point is that such a Lorentzian metric \( g \) cannot be defined on manifolds with arbitrary topology (this is a significative difference with the case of a proper Riemannian metric which can be defined on every differentiable manifold). We will come back to this important point in section 5. The set of all Lorentzian metrics on a given differentiable manifold \( V \) will be denoted by \( \text{Lor}(V) \).

The interior, exterior, closure and boundary of a set \( \zeta \) are denoted by \( \text{int}\zeta \), \( \text{ext}\zeta \), \( \overline{\zeta} \) and \( \partial\zeta \), respectively, and the inclusion, union and intersection of sets are written as \( \subset \), \( \cup \) and \( \cap \). We will also need some basic terminology concerning binary relations in set theory. A binary relation \( R \) on a set \( X \) is a subset of the Cartesian product \( X \times X \).

For any \((x, y) \in R \) we will write \( xRy \). A relation \( R' \) is larger than \( R \) if \( R \subseteq R' \). The restriction of \( R \) to a subset \( Y \subset X \) is the new relation \( R \cap (Y \times Y) \) denoted by \( R_Y \). The inverse relation \( R^{-1} \) of \( R \) is another binary relation defined as

\[
R^{-1} = \{ (x, y) \in X \times X : (y, x) \in R \}. 
\]

In particular, the relation is called symmetric if \( R = R^{-1} \), while it is said antisymmetric if \( xRy \) and \( yRx \) imply necessarily that \( x = y \). Other relevant cases are reflexive relations \((xRx)\), anti-reflexive relations \((x, x) \not\in R \), and transitive relations \((xRy, yRz \Rightarrow xRz)\). A relation which is reflexive and transitive is called a preorder, and antisymmetric preorders are called partial orders, while symmetric preorders are also called equivalence relations. A relation \( R \) orders linearly the set \( Y \) if for any pair \( x, y \in Y \) either \( yRx \) or \( xRy \). The reflexive relation \( R \) will be called horisomatic if for any finite sequence \( \{x_i\}_{1 \leq i \leq m} \) such that \( x_iR_{i+1} \forall i \neq m \) then

(i) \( x_1Rx_m \Rightarrow x_hRx_k, \forall h, k \text{ with } 1 \leq h \leq k \leq m \).

(ii) \( x_mRx_1 \Rightarrow x_h = x_k, \forall h, k \text{ with } 1 \leq h \leq k \leq m \).
2. Essentials of causality in Lorentzian Geometry

We start by reviewing the basic concepts and causal properties of Lorentzian manifolds. As we will see in section 3 some of these key concepts can be generalized to sets which are not Lorentzian manifolds (in some cases they are not even differentiable manifolds), usually keeping the same notation and terminology. We will give brief definitions of most of the concepts involved in order to fix the nomenclature followed in this review.

A Lorentzian manifold admitting a global nonvanishing timelike vector field $\vec{\xi}$ is said to be time orientable. Such manifolds constitute the main underlying structure for most parts of Physics through the following definition.

Definition 2.1 (Spacetime) Any oriented, connected Hausdorff $C^\infty$ Lorentzian manifold with a time orientation and a $C^1$ metric $g$ is called a spacetime. The points of a spacetime are called events.

The conditions imposed in this definition may vary slightly in the literature (compare for instance the definitions of [6, 64, 128]). In particular time orientability is not always required. Nevertheless, the existence of a consistent time orientation is crucial for the global causal structure and therefore we are forced to assume it here. Moreover any non-time-orientable Lorentzian manifold has a double-cover which can be time oriented, hence we can always perform our study in this double covering. The condition of paracompactness is added in too many references, but this is redundant because, as it was shown by Spivak [159], a pseudo-Riemannian manifold is necessarily paracompact.

An important point concerning definition 2.1 is that the $C^1$ condition for the metric tensor cannot be improved to a larger $C^k$ with $k > 1$ if one wishes to describe situations with different matter regions (such as the interior and exterior of stars, or shock waves) where these regions must be properly matched. The discontinuities of the matter content variables —its energy density, for instance— arise usually, via appropriate field equations (e.g. Einstein’s field equations [64, 161] or any of their relatives), through discontinuities on the second derivatives of the metric tensor, which must thus be allowed. Unfortunately, this poses enormous problems concerning causality structure, specially because the basic purely local causal properties of the spacetime, which are fundamental for the construction of the whole theory of causality, might not hold in general $C^1$ metrics. A longer description of this problem can be found in [153, 158, 31], and will be briefly considered later in subsections 2.2, 3.2 and 5.1. In order to avoid this annoying problem though —despite of being completely fundamental!—, we will implicitly assume for most of this review that $g$ is at least of class $C^2$.

In what remains of this section we follow the logical steps which allow to build a sensible notion of causality: first we consider the direct algebraic implications that the existence of the Lorentzian metric $g$ has at every single point $x \in V$ (subsection 2.1); then we go a step further and construct the local causal structure of the spacetime, that is, at appropriate local neighbourhoods of any point (subsection 2.2)); finally, we explore the causal relations between non-locally related points (subsection 2.3), which require the study of causal connectivity properties, that is, using causal curves, and leads to the definition of the fundamental sets used in causality theory.
2.1. The null cone. Causal tensors

Our first step towards the study of general properties of Lorentzian manifolds concerns the classification that the Lorentzian metric \( g \) induces on the tangent bundle. As is obvious, the condition of time-orientability incorporated in definition 2.1 implies that any causal vector \( \vec{v} \in T_x(V) \) can be classified as either future-directed or past-directed according to whether \( g(\vec{\xi}, \vec{v}) > 0 \) or \( < 0 \) (recall that two causal vectors can be orthogonal with respect to \( g \) only if they are null and proportional; besides, without loss of generality we have implicitly assumed that the future is defined by the arrow of \( \vec{\xi} \).) This provides \( T_x(V) \) with a two-sheeted cone, the null cone, which is the most basic causal object. Future-directed timelike (respectively null) vectors lie inside the “upper” part of the null cone (resp. on the upper cone itself), and similarly the past-directed causal vectors on the lower part of the cone.

Surprisingly, the extension of this classification to higher rank tensors has only been formulated very recently in [9]. Probably, the main difficulty was that there is no simple relation between the length \( g(t, t) = t_a...b t^a...b \) of a tensor \( t \in T_x(V) \) and its “causal character”. Nevertheless, one can use an alternative equivalent definition of causal vector which can be translated to all tensors, namely, a vector \( \vec{v} \in T_x(V) \) is future directed if and only if \( g(\vec{v}, \vec{u}) \geq 0 \) for all future-directed vectors \( \vec{u} \in T_x(V) \).

Hence, future tensors can be defined as follows [9].

**Definition 2.2 (Causal tensors)** A tensor \( t \in T_x(V) \) is said to be future (respectively past) if \( t(\vec{u}_1, \ldots, \vec{u}_s) \geq 0 \) (resp. \( \leq 0 \)) for all future-directed vectors \( \vec{u}_1, \ldots, \vec{u}_s \in T_x(V) \). A causal tensor is a tensor which is either future or past.

Observe that \( t \) is a future tensor if and only if \( -t \) is a past tensor. This definition extends straightforwardly to tensor fields and the bundles \( T^r_s(V) \). It can be easily seen that the set of all causal tensors has an algebraic structure of a graded algebra of cones generalizing the null cone.

Several useful characterizations of causal tensors, as well as applications and their basic properties, were given in [9] and subsequently improved and enlarged in [47, 48, 49, 46]. Definition 2.2 can be equivalently stated [9] by (i) just demanding that \( t(\vec{k}_1, \ldots, \vec{k}_r) \geq 0 \) for all future-directed null vectors \( \vec{k}_1, \ldots, \vec{k}_r \) or (ii) requiring that \( t(\vec{u}_1, \ldots, \vec{u}_r) > 0 \) for all future-directed timelike vectors \( \vec{u}_1, \ldots, \vec{u}_r \). Simple criteria to ascertain when a given tensor is causal are given in [154, 9, 47, 49].

Of course, the condition of future tensor was known long ago in General Relativity for the case of symmetric rank-2 covariant tensors, but with another name and purpose: the future property is what was usually called the “dominant energy condition” for the energy-momentum tensor. This was a condition to be demanded upon energy-momentum tensors likely to describe a physically acceptable matter content. Thus, the property of a future tensor is sometimes referred to as the “dominant property”, see [154, 9, 46] and references therein.

A more important property of causal tensors (specially in the case of rank-2 tensors) is their relation to maps that preserve the null cone. This was proved in [9] and shows the deep connection of causal tensors with the elementary causal structure of the Lorentzian manifold. This connection and the properties and applications derivable thereof were exploited in [47] to generalize the notion of causal structure, and in [48, 49] to look for finite and infinitesimal transformations which preserve the causal structure, or part of it; see also [46] for a self-contained full exposition.

We will come back to these novel matters later on in sections 4 and 7, specially in subsections 4.2, 4.3, 7.1 and 7.2.
2.2. Local causality

The second step in the erection of the causal program is the extension to local neighbourhoods. To that end we need the classical ideas on local Riemannian normal coordinates and normal neighbourhoods, and the definition of causal curves.

Recall that a piecewise $C^k$ curve with $k \in \mathbb{N}, k \geq 1$, is a set of $C^k$ maps (called arcs) $\lambda_j : I_j \rightarrow V, j \in \mathbb{N}$ where $\{I_j\}$ is a countable set of open intervals of the real line such that the set $\bigcup_j \lambda_j(I_j)$ is a continuous curve in $V$.

**Definition 2.3 (Causal and timelike curves)** A piecewise $C^k$ curve $\gamma \subset V$ is said to be future-directed timelike (resp. null, causal) if its tangent vector at all points $x$ where it is well defined is a timelike (resp. null, causal) future-directed vector of $T_x(V)$, and furthermore the causal orientations of all the arcs $\lambda_j(I_j)$ are consistent.

Note that the tangent vector to a piecewise $C^k$ curve is well defined at every point except possibly at the intersections $\lambda_j(I_j) \cap \lambda_{j+1}(I_j)$, which are called corners. At these corners there may be two different tangent vectors, and the consistency condition included in definition 2.3 requires simply that these two vectors point into the same causal orientation (the future, say) at every corner. Obviously, causal curves cannot change their time orientation within any of its arcs due to the differentiability of the $\lambda_j$, and thus a piecewise $C^k$ causal curve is future directed if all the “pairs of tangent vectors” at their corners are future pointing.

For global causality one needs to consider limits of sequences of differentiable curves, which of course do not need to be piecewise differentiable curves themselves. This will be briefly considered in subsection 2.3.

The exponential map (see [6, 64, 117, 153] for details) from an open neighbourhood $\mathcal{O}$ of $\vec{0} \in T_x(V)$ into a neighbourhood of $x \in V$ maps a given $\vec{v} \in \mathcal{O}$ into the point that reaches the geodesic starting at $x$ with tangent vector $\vec{v}$ a unit of affine length away from $x$ provided this is defined. Given that geodesics depend continuously on the initial conditions $x$ and $\vec{v}$, by choosing adequate neighbourhoods this exponential map is a homeomorphism, and actually a diffeomorphism if $g$ is $C^2$. This exponential map allows to define the classical Riemannian normal coordinates [6, 35, 153] at a neighbourhood of $x$. Any such neighbourhood is called a normal neighbourhood of $x$ and they can be chosen to be convex [173, 40]. The change to normal coordinates must be at least of class $C^1$ to keep a differentiable atlas, and consequently the geodesics must depend differentially on the initial conditions. As mentioned before, this will not happen in physical situations requiring the matching of two different regions with different matter contents across a common boundary. It is well-known that under such circumstances there exists a local coordinate system, called admissible by Lichnerowicz [97], in which the metric is $C^1$ piecewise $C^2$, see e.g. [97, 108]. Thus, in these situations there is no guarantee that the normal coordinate neighbourhoods are differentiable at the matching hypersurface. This is crucial for causality and therefore to all theories and matters which rely on it; see e.g. subsection 6.1 in [153] for a detailed list of troubles arising in relation with singularity theorems due to this problem. In general, there are many statements concerning causality theory which have only been proven under the restrictive assumption of a $C^2$ metric, some notable exceptions can be consulted in [25, 29, 64, 158, 31]. An example of these difficulties will be commented at the end of §3.2.1, and some results obtained in [158, 31] in subsection 5.1. Having mentioned these fundamental difficulties—which are usually ignored or dismissed without mention—, and in order to avoid complicated subtleties, we will
consider, when needed, and for the rest of the review unless otherwise stated, that the metric is $C^2$, keeping in mind that this problem should be eventually considered and solved.

One of the most important results in causality theory concerns the local causal properties of spacetime and states that the causality within normal neighbourhoods of any $x \in V$ is analogous to that of flat spacetime. The exact meaning here of the word “analogous” is very important: it has usually been interpreted as synonymous of “equivalent”. As we will largely discuss in subsection 4.2, this equivalence can actually be proven rigorously whenever an appropriate definition of causal equivalence is provided, and this proper definition cannot be the usual and simple local conformal relation between spacetimes. This latter part was explicitly showed by Kronheimer and Penrose in [89], while the solution to the problem, selecting the right definition for causal equivalence, was only very recently given in [47]. We postpone the discussion of these matters to section 4, and here we only want to give the basic facts supporting the “analogy” between the causality in local neighbourhoods and that of flat spacetime.

To be precise, let us define the future light cone (resp. its interior) of $x \in V$ as the image of the future null cone (resp. its interior) in $O \subseteq T_x(V)$ by the exponential map. Observe that the light cone is only defined on a normal neighbourhood of $x$. Then, one can prove a fundamental proposition (see [6, 64, 87, 153]). The following is its strongest version we are aware of in terms of differentiability [153] (see also [25]).

**Proposition 2.1** Any continuous piecewise $C^1$ future-directed causal curve starting at $x \in V$ and entirely contained in a normal neighbourhood of $x$ lies completely on the future light cone of $x$ if and only if it is a null geodesic from $x$, and is completely contained in the interior of the future light cone of $x$ after the point at which it fails to be a null geodesic.

Observe that a non-differentiable curve composed by arcs of null geodesics is not a null geodesic. Thus, any future-directed curve constituted by an arc of null geodesic from $x$ to $y$ followed by another null geodesic at the corner $y$ lies on the future light cone up to $y$ and enters into its interior from $y$ on. Notice also that any future-directed causal curve which is not a null geodesic at $x$ immediately enters and remains in the interior of the light cone with vertex at $x$, in particular all timelike curves from $x$ are completely contained in this interior.

### 2.3. Global causality

The third step towards the study of general causal properties of Lorentzian manifolds is to show how this structure can be used to define certain global objects or properties on the manifold. We are specially interested in the possible relations arising between “far apart” points of the manifold because they can be generalized to more abstract sets other than differentiable Lorentzian manifolds.

The nice simple local causal structure shown in Proposition 2.1 does not hold globally in general, unless the spacetime is extremely well-behaved (such as for instance in flat spacetime), as we will see. Thus, all non-simple results concerning causality arise only as global aspects of the spacetimes, and one needs to investigate and try to control the properties of Lorentzian manifolds globally. This is somehow frustrating, and paradoxical, if we have in mind physical theories such as General Relativity, which are local in essence, as the differential equations determining the metric tensor field are obviously local. In such physical theories, in order to prove very important results,
such as the singularity theorems, or the properties of horizons, black holes, and so on, the causality theory is absolutely necessary. But this is a global set of properties, while the theory, as already said, is only local. The traditional solution to solve this dichotomy has been the use of extensions, so that if a local solution was found one tries to extend it beyond its domain of validity as many times as needed until the resulting spacetime is inextensible \[64, 153\]. The problem with this strategy is that extensions are not unique, nor they can be determined by physical motivations or mathematical properties, see \[153\] for a detailed discussion. In other words, extensions are arbitrary. And therefore, the global solutions needed to ascertain the causal properties of a spacetime rely on a very weak foundation.

Nevertheless, if one assumes that the global Lorentzian manifold is known, or given by any means, the causality analysis can be pursued without problems. This is an interesting task on its own, as we can learn the different possible causal structures compatible with spacetimes, and the possible difficulties, or surprises, that may arise.

The basic objects permitting the global analysis are causal curves, as they connect points in \( V \) which do not have to lie on the same normal neighbourhood.

2.3.1. Continuous causal curves on Lorentzian manifolds. The simplest causal curves are the piecewise \( C^k \) future- or past-directed timelike and causal curves introduced in definition 2.3. Curves will usually be denoted in this review with the letter \( \gamma \) —or \( \gamma(t) \) if we want to make explicit the parametrization of the curve. The piecewise \( C^k \) condition imposed in definition 2.3 is needed in order to be able to define a tangent vector at the points of each arc \( \lambda_j(I_j) \) of the curve. However, it is possible and indeed necessary to generalize the previous definition in order to include curves which are only continuous see \[6, 64, 87, 117, 128\].

**Definition 2.4 (Continuous causal curves)** A continuous curve \( \gamma \subset V \) is said to be causal and future-directed if for every point \( x \in \gamma \) there exists a convex normal neighbourhood \( \mathcal{N}_x \) of \( x \) such that for every pair \( y, z \in \gamma \cap \mathcal{N}_x \) \( y = \gamma(t_1), z = \gamma(t_2) \) with \( t_2 > t_1 \) there is a \( C^1 \) future-directed causal arc contained in \( \mathcal{N}_x \) from \( y \) to \( z \).

It is easily shown that continuous causal curves are in fact Lipschitz and thus differentiable almost everywhere \[128, 87\].

Other standard concepts dealing with causal curves are presented next.

**Definition 2.5 (Inextensible curves)** A point \( x \) is said to be a future (resp. past) endpoint of a continuous future (resp. past) directed causal curve \( \gamma(t) \) if for every neighbourhood \( \mathcal{U}_x \) of \( x \) there exist a value \( t_0 \) such that \( \gamma(t) \subset \mathcal{U}_x \) for all \( t > t_0 \) (resp. \( t < t_0 \)). Causal curves with no future (past) endpoints will be called future (past) endless. A curve \( \gamma \) is inextensible if there is no curve \( \gamma' \) containing \( \gamma \) as a proper subset.

Clearly inextensible causal curves have no endpoints. The set of future-directed causal curves with a past endpoint \( a \) and a future endpoint \( b \) will be denoted by \( C(a, b) \). This set is a topological space under the \( C^0 \) topology defined next.

**Definition 2.6 (\( C^0 \) topology)** The collection of sets of curves \( \mathcal{O}(\mathcal{U}) = \{ \gamma \in C(a, b) : \gamma \subset \mathcal{U} \} \) where \( \mathcal{U} \subset V \) is an open set, constitute a basis for a topology in \( C(a, b) \) called the \( C^0 \) topology.

It is possible to define convergence of curve sequences contained in \( C(a, b) \) using this topology. Other different notions of convergence on \( C(a, b) \) (or more general sets of curves) can be defined and studied although we will not pursue this matter further in this review (see \[6, 64, 87, 128\]).
2.3.2. Basic sets used in causality theory. We are now prepared to define the fundamental binary relations between the points of a Lorentzian manifold according to whether they can be joined by timelike, null or causal curves (or none of them).

**Definition 2.7 (Causal relations)** Let \( p, q \in V \):

- \( p \) is chronologically related with \( q \), written \( p \ll q \), if \( C(p, q) \) contains timelike curves;
- \( p \) is causally related with \( q \), written \( p < q \), if \( C(p, q) \) is not empty;
- the relation \( p \rightarrow q \) means that \( p < q \) but not \( p \ll q \).

These relations are standard in causality theory and they can be found in many textbooks, e. g. [6, 64, 87, 117, 128, 172]. We summarize next their basic properties as they will be needed later.

**Proposition 2.2** For a Lorentzian manifold \( V \) the binary relations ``\( \ll \)'' and ``\(< \)'' fulfill the following basic properties

(i) \( < \) is reflexive.
(ii) \( < \) and \( \ll \) are transitive.
(iii) \( p < q \) and \( q \ll z \Rightarrow p \ll z \). \( p \ll q \) and \( q < z \Rightarrow p < z \).

As we will see in the next section these properties can be abstracted to more general sets which do not need even to be topological spaces. Using them the sets \( I^+(p) \) (chronological future (+) and past (−) of \( p \)), \( J^+(p) \) (causal future of and past \( p \)) and \( E^\pm(p) \) (future and past horismos of \( p \)) are defined as (from now on we only give the definitions for future objects assuming the obvious generalization for their past counterparts)

\[
I^+(p) = \{ x \in V : p \ll x \}, \quad J^+(p) = \{ x \in V : p < x \}, \quad E^+(p) = J^+(p) - I^+(p),
\]

from which we can construct \( I^+(\mathcal{U}) \), \( J^+(\mathcal{U}) \) and \( E^+(\mathcal{U}) \) for an arbitrary set \( \mathcal{U} \subset V \)

\[
I^+(\mathcal{U}) = \bigcup_{p \in \mathcal{U}} I^+(p), \quad J^+(\mathcal{U}) = \bigcup_{p \in \mathcal{U}} J^+(p), \quad E^+(\mathcal{U}) = \bigcup_{p \in \mathcal{U}} E^+(p).
\]

There are some variants of these definitions in which an auxiliary set \( \mathcal{W} \) is employed

\[
I^+(p, \mathcal{W}) \equiv \{ x \in V : p \text{ and } x \text{ can be joined by a timelike curve contained in } \mathcal{W} \},
\]

and similar definitions for \( J^+(p, \mathcal{W}) \), \( E^+(p, \mathcal{W}) \) and \( I^+(\mathcal{U}, \mathcal{W}) \), \( J^+(\mathcal{U}, \mathcal{W}) \), \( E^+(\mathcal{U}, \mathcal{W}) \)

These sets have well known topological properties which again can be found in e. g. [64, 128, 6, 117, 172, 87, 153]:

(i) \( I^+(\mathcal{U}) \) is always open.
(ii) \( I^+(\overline{\mathcal{U}}) = I^+(\mathcal{U}) \).
(iii) \( \overline{I^+(\mathcal{U})} = \{ x \in V : I^+(x) \subseteq I^+(\mathcal{U}) \} = J^+(\overline{\mathcal{U}}) \).
(iv) \( \partial J^+(\mathcal{U}) = \partial I^+(\mathcal{U}) = \{ x \in V : x \notin I^+(\mathcal{U}) \text{ and } I^+(x) \subseteq I^+(\mathcal{U}) \} \).
(v) \( \text{int}(J^+(\mathcal{U})) = I^+(\mathcal{U}) \).

Another type of subset, playing an important role in one of the most important constructions of causal boundary, is that of future (or past) set which can be defined using the chronological sets. Recall that a set \( F \) is achronal if \( F \cap I^+(F) = \emptyset \) or in other words no pair of points in \( F \) can be joined by a timelike future-directed curve.
Definition 2.8 (Future sets, achronal boundary) A set $F \subset V$ is called a future set if $I^+(F) \subseteq F$. The boundary $\partial F$ of a future set $F$ is called an achronal boundary.

Note that the terminology “achronal boundary” is standard but may be misleading as there are boundaries of non-future sets which are achronal [128], and this is why some authors have changed or omitted this terminology [87, 117, 153]. An interesting property of achronal boundaries is the next (see e.g. [64, 87, 117, 128, 153] for a proof.)

Proposition 2.3 Any achronal boundary is a closed achronal Lipschitz hypersurface.

The definitions stated before are focused on how points of a Lorentzian manifold influence each other. However, we are sometimes interested in those points of $V$ influenced solely by a given region of the Lorentzian manifold. This is taken care of in the following definition of future Cauchy development, which is again standard. The future Cauchy development $D^+(U)$ of a set $U \subset V$ is the set of points of $V$ which can be influenced exclusively by points of $U$. More precisely

Definition 2.9 (Cauchy development) $x \in D^+(U)$ if and only if every past-endless past-directed causal curve containing $x$ intersects $U$.

Future and past Cauchy horizons $H^\pm(U)$ of $U$ are defined as the future and past boundaries of the Cauchy developments of $U$. They are formed by the points in the closure $\overline{D^+(U)}$ which cannot be joined by a future-directed timelike curve with any other point of $D^+(U)$:

$$H^+(U) = \overline{D^+(U)} - I^-(D^+(U)).$$

The total domain of dependence and the total Cauchy horizon of $U$ are defined respectively as $D(U) = D^+(U) \cup D^-(U)$, $H(U) = H^+(U) \cup H^-(U)$. When $U$ is closed and achronal, some general properties of these sets are [6, 64, 87, 117, 128, 153]

(i) $\text{int} D^+(U) = I^+(U) \cap \overline{D^+(U)}$.
(ii) $H^+(U)$ is an achronal set.
(iii) $\overline{D^+(U)} = \{x \in V : \text{every endless timelike past directed curve from } x \text{ meets } U\}$.
(iv) $I^+[H^+(U)] = I^+(U) - \overline{D^+(U)}$.
(v) $\partial D^+(U) = H^+(U) \cup U$.

3. Causality in abstract settings

In the previous section we have reviewed the main concepts used in the causal analysis of Lorentzian manifolds. In these manifolds, causality stems from the peculiar properties that a metric of Lorentzian signature provides, which ultimately allows to classify vectors, tensors, fields, sets, and curves as future, past or neither of the two. This causal character for curves led to set certain relations between points of the manifold. In this section we follow the opposite way and ask ourselves which are the properties of manifolds or sets where binary relations resembling those obtained between points of a Lorentzian manifold (cf. proposition 2.2) are present. This approach could be termed as axiomatic, as we are trying to isolate what is genuine of causality regardless of the existence of a Lorentzian metric. This axiomatic viewpoint
was not studied systematically in the literature until the 1960’s. Perhaps the two most important and illuminating investigations dealing with this subject can be found in the work by Kronheimer and Penrose [89], where the properties presented in proposition 2.2 are axiomatized, and in Carter’s thorough analysis [25] where a further generalization is achieved. These two papers are the main subject of this section and their contents will be discussed in some detail.

3.1. The Kronheimer-Penrose causal spaces

The paper by Kronheimer and Penrose [89] is the first general study of causal spaces. Roughly speaking these spaces are sets in which binary relations $<, <,$ and $\rightarrow$ with properties similar to those presented in definition 2.7 have been set. The precise definition as is given in Kronheimer-Penrose’s paper is presented next.

**Definition 3.1 (Causal space)** The quadruple $(X, <, <, \rightarrow)$ is called a causal space if $X$ is a set and $<, <,$ and $\rightarrow$ are three binary relations on $X$ satisfying for each $x, y, z \in X$ the following conditions

(i) $x < x$.
(ii) if $x < y$ and $y < z$, then $x < z$.
(iii) if $x < y$ and $y < x$ then $x = y$.
(iv) not $x < x$.
(v) if $x < y$ then $x < y$.
(vi) if $x < y$ and $y < z$ then $x < z$.
(vii) if $x < y$ and $y < z$ then $x < z$.
(viii) $x \rightarrow y$ if and only if $x < y$ and not $x < y$.

The relations $<, <,$ and $\rightarrow$ are called respectively causality, chronology and horismos.

As we can see, this definition is fully inspired in the Lorentzian causal structure outlined in section 2. However, a very important remark is that points (iii) and (iv) have been added, and they do not necessarily hold in Lorentzian manifolds. This is connected to the existence of closed timelike or causal loops, which are perfectly permitted in generic Lorentzian manifolds, though they may be seen as undesirable, unphysical or even paradoxical. Kronheimer and Penrose want to avoid this “ill-behaved” causal spaces from the start. It must be borne in mind, however, that there are spacetimes violating (iii) and/or (iv). Having said this, any Lorentzian manifold satisfying (iii) and (iv) (these are called chronological and causal spacetimes, see definition 4.1 below) is a causal space as we deduce from proposition 2.2. However, a causal space is a rather more general structure as $X$ does not need to be even a topological space, although if a topology is present in $X$ its interplay with some relevant sets constructed from the causal and chronological relations was also considered in [89] as we will discuss in section 5.

Immediate properties arising from definition 3.1 are $\rightarrow \subseteq <, \subseteq \subseteq <, <$ is a partial order, $\subseteq$ is anti-reflexive and transitive and $\rightarrow$ is horismotic. Another direct consequence is that for any three points $x, y, z$ in a causal space $X$ such that $x < y < z$ and $x \rightarrow z$ then $x \rightarrow y \rightarrow z$.

From the relations of definition 3.1 we may construct their inverses getting a new quadruple which turns out to be a new causal space called dual causal space. In the
case of a Lorentzian manifold it corresponds to the interchange of \(<, \ll\) and \(\rightarrow\) by \(\gg, \gg\) and \(\leftarrow\) being these relations defined using past-directed causal curves instead of future-directed ones. Any subset \(Y\) of a causal space can be transformed into a causal space by just considering the binary relations \(<_Y, \ll_Y\) and \(\rightarrow_Y\). In this case \(Y\) is a causal subspace of \(X\).

The existence of these relations in a set \(X\) is enough to define the generalizations of the chronological future, causal future and future horismos for any element \(p \in X\)

\[
I^+(p) \equiv \{x \in X : p \ll x\}, \quad J^+(p) \equiv \{x \in X : p < x\}, \quad E^+(p) \equiv J^+(p) \setminus I^+(p).
\]

From them the chronological, causal and future horismos of any subset of the causal space \(X\) are defined straightforwardly. Using the standard definition of chain for any given binary relation one can then define chronological chains and causal chains which are in a sense the generalization of timelike and causal curves, respectively.

Other interesting objects are

\[
[p, q] = \{z \in X : p < z < q\}, \quad <x, y> = \{z \in X : x \ll z \ll y\}
\]

which can be seen equivalent to \(J^+(p) \cap J^-(q)\) and to \(I^+(x) \setminus I^-(y)\). The relation \(x||y\) means that neither \(x < y\) nor \(y < x\) and a set \(S \subseteq X\) such that \(x||y\) for all \(x, y \in S\) is said to be acausal (notice that in [89] such a set is called “spacelike”, but this would enter in conflict with the usual definition of spacelike subsets in spacetimes, as there are spacelike sets which are not acausal.)

As we see a great deal of the basic notions and sets introduced in causality theory are present in abstract causal spaces. Kronheimer and Penrose study further different aspects of causal spaces some of which are detailed next.

**Definition 3.2 (Regular causal spaces)** A causal space is called regular if for any four different points \(x_1, x_2, y_1, y_2\) such that \(x_i \rightarrow y_j, i, j = 1, 2\) then \(x_1||x_2\) if and only if \(y_1||y_2\).

Causal spaces can be further classified according to the properties of the chronology \(\ll\).

**Definition 3.3** The causal space \(X\) is said to be

(i) future reflecting if \(I^-(x) \subset I^-(y)\) whenever \(I^+(x) \supset I^+(y)\). \(X\) is reflecting if it is future and past reflecting.

(ii) weakly distinguishing if both \(I^+(x) = I^+(y), I^-(x) = I^-(y)\) entail \(x = y\).

Future distinguishing if \(I^+(x) = I^+(y)\) implies \(x = y\), and similarly for past distinguishing.

(iii) full if the following two conditions and their dual versions hold:

- \(\forall x \in X, \exists p \in X\) such that \(p \ll x\);
- \(\forall p_1, p_2 \ll x \exists q\) such that \(p_1 \ll q, p_2 \ll q\) and \(q \ll x\).

Note that these three properties refer exclusively to the relation \(\ll\), and thus we can talk about reflection, distinguishing and fullness of any anti-reflexive and transitive binary relation \(\ll\). Point (iii) is always satisfied in a spacetime due to the fundamental proposition 2.1 and the openness of the sets \(I^+\). Regarding (i) and (ii), we will actually meet them again in definition 4.1 as they are basics steps in the standard causal hierarchic classification of Lorentzian manifolds.

A causal space \(X\) can be endowed with a natural topology in view of the properties of chronological futures and past of manifolds.
Definition 3.4 (Alexandrov topology) The Alexandrov topology for a causal space $X$ is the coarsest topology in which the sets $I^+(x)$ and $I^-(x)$ are open for every element $x \in X$.

Observe that the sets $I^+(x) \cap I^-(y)$ are open for all $x, y \in X$. We will provide further details about this and other topologies in section 5.3.

The next important topic addressed in [89] is the study of the necessary restrictions to be imposed on a set equipped with at least one binary relation with the properties of either $<$, $\ll$ or $\rightarrow$ such that the remaining necessary binary relations can be added in order to get a causal space. The notation is fixed when we have a set $X$ and two binary relations (say $\rightarrow$ and $<$) specified on it as follows:

Definition 3.5 For the triad $(X, \rightarrow, <)$ it is said that $\rightarrow$ is horismos compatible (resp. regularly compatible) with the causality $<$ if there exists a relation $\ll$ such that the quadruple $(X, <, \ll, \rightarrow)$ is a (regular) causal space. The set 

$$\{\rightarrow \subseteq X \times X : \exists \ll \text{ such that } (X, \rightarrow, <, \ll) \text{ is a (regular) causal space}\}$$

is denoted by \{(reg.hor|cau $<$)\}. The definitions of \{(reg.hor|chr $<$)\}, \{(reg.chr|cau $<$)\} et cetera, are similar.

The construction of causal spaces from spaces with a single relation (a chronology, a causality or a horismos) can then be attacked separately [89].

Definition 3.6 (Construction from the horismos) Let $\rightarrow$ be a horismotic relation on a set $X$. Then, two further relations $\ll^\mathbb{U}$ and $\ll^\mathbb{U}$ can be defined on $X$ as follows

(i) $x \ll^\mathbb{U} y \iff$ there exists a finite sequence $(u_i)_{1 \leq i \leq n}$ satisfying $x = u_1 \rightarrow u_2 \rightarrow \ldots \rightarrow u_n = y$,

(ii) $x \ll^\mathbb{U} y \iff x \ll^\mathbb{U} y$ and not $x \rightarrow y$.

It is not difficult to check that $(X, <, \ll^\mathbb{U}, \rightarrow)$ is a causal space. Therefore $\ll^\mathbb{U} \in \{\text{cau}|\text{hor} \rightarrow\}$ and $\ll^\mathbb{U} \in \{\text{chr}|\text{hor} \rightarrow\}$. Moreover for any $\ll \in \{\text{cau}|\text{hor} \rightarrow\}$ the relations $x \ll^\mathbb{U} y$ imply that in fact $x < y$ and the same for $\ll$ and $\ll^\mathbb{U}$, hence $\ll^\mathbb{U}$ and $\ll^\mathbb{U}$ can be defined in an alternative and more transparent way

$$\ll^\mathbb{U} = \cap\{\text{cau}|\text{hor} \rightarrow\}, \quad \ll^\mathbb{U} = \cap\{\text{chr}|\text{hor} \rightarrow\}.$$  \(3.1\)

Definition 3.7 A causal space $(X, <, <, \rightarrow)$ is a $\mathbb{U}$-space if the equalities $\ll\ll$ and $\ll\ll$ hold.

The construction from the chronology relation is as follows

Definition 3.8 (Construction from the chronology) Consider a set $X$ endowed with an anti-reflexive and transitive binary relation $\ll$. Two further relations $\ll^\mathbb{B}$ and $\ll^\mathbb{B}$ on $X$ can then be defined as follows

$x \ll^\mathbb{B} y \iff I^+(x) \supset I^+(y)$ and $I^-(x) \subset I^-(y)$

$x \ll^\mathbb{B} y \iff x \ll^\mathbb{B} y$ and not $x \ll y$. 

Causal structures and causal boundaries
For any $< \in \{ \text{cau} | \text{chr} \ll \}$ we have that $x < y \Rightarrow x \ll y$. Conversely, if $x \ll y$ there always exists $< \in \{ \text{cau} | \text{chr} \ll \}$ such that $x < y$. Hence

$$\ll = \cup \{ \text{cau} | \text{chr} \ll \}, \quad \rightarrow = \cup \{ \text{hor} | \text{chr} \ll \}$$

In this case the quadruple $(X, \ll, \ll, \rightarrow)$ does not always satisfy (iii) of definition 3.1. In fact, $(X, \ll, \ll, \rightarrow)$ is a causal space if and only if $<$ is weakly distinguishing.

**Definition 3.9** A causal space $(X, <, \ll, \rightarrow)$ is called a $\mathfrak{B}$-space if $\ll = \ll$ and $\rightarrow = \rightarrow$.

From the above considerations we deduce that any $\mathfrak{B}$-space must be future distinguishing. Furthermore a sufficient condition for the causal space $(X, <, \ll, \rightarrow)$ to be a $\mathfrak{B}$-space is that $x < y$ whenever $x \ll y$.

Finally the construction of a causal space from a space with a single causal relation $<$ can be achieved as follows.

**Definition 3.10 (Construction from the causality)** Let $<$ be a partial order on a set $X$. Define two new binary relations $\rightarrow \subseteq X$ by

(i) $x \rightarrow y \iff x < y$ and $< \text{ linearly orders every } [u, v] \subset [x, y]$.

(ii) $x \ll y \iff x < y$ and not $x \rightarrow y$.

Then, $(X, <, \ll, \rightarrow)$ is a causal space.

In this case, the counterpart of (3.1) and (3.2) is

$$\rightarrow \subseteq X, \quad \ll = \cap \{ (\text{reg}) | \text{chr} \ll \}$$

$\mathfrak{C}$-spaces are defined similarly to $\mathfrak{U}$-spaces and $\mathfrak{B}$-spaces. $\mathfrak{C}$-spaces do not need to be regular. A sufficient condition for a regular causal space $(X, <, \ll, \rightarrow)$ to be a $\mathfrak{C}$-space is the inclusion $\rightarrow \subseteq \rightarrow$, which is equivalent to $\ll \subseteq \ll$.

### 3.1.1. Connectivity properties

Kronheimer and Penrose also considered the possible representation that the intuitive ideas of “null geodesic” or “null arc” could have in their abstract setting. The main definition is

**Definition 3.11 (Girders, hypergirders and proper beams)** Let $G = (g_i)_{1 \leq i \leq N}$ be a finite chain of $N \geq 3$ points ordered by the relation $<$ (i.e. $g_i < g_{i+1}, \forall i < N$). $G$ is called a girders if $g_i \rightarrow g_{i+2}$. A nonempty subset $H \subset X$ is called a hypergirder if $\forall x, y \in H$ there exists a girders $G \subset H$ containing $x$ and $y$. A maximal hypergirder is called a proper beam.

The definition of girders implies that in fact $g_i \rightarrow g_{i+1}$ so the elements of $G$ can be arranged according to the following diagram

$$\cdots \rightarrow g_i \rightarrow g_{i+1} \cdots$$

As we see, a hypergirder is the analogous to a null geodesic arc on Lorentzian manifolds, while proper beams are the generalization of inextensible null geodesics. Actually, using of Zorn’s lemma one can prove that any hypergirder is a subset of a proper beam.

Two points $x$ and $y$ are said proximate if they belong to some girders, or in other words, if there exists a girders $x \rightarrow z \rightarrow y$. Then, the following result holds true for regular causal spaces.
Theorem 3.1 Each pair of proximate points belong to a single proper beam if and only if the underlying causal space is regular.

To include certain pathological cases proper beams are generalized to beams. These are sets which are either proper beams or just consist of two points \(\{a, b\}\) ordered by the horismos and not contained in any hypergirder. This generalization allows us to formulate the following result.

Theorem 3.2 In any causal space any non-trivial set linearly ordered by the horismos is a subset of some beam.

To end this subsection giving a flavour of the power of this abstract construction and the relation to standard or intuitive theorems and results on Lorentzian manifolds, we present a theorem involving beams which generalizes the well known decomposition of \(\partial J^+(S)\) for spacetimes in null generators, their past endpoints and its edge (compare with the spacetime versions in e. g. \([6, 64, 128]\)).

Theorem 3.3 Let \(A\) be any acausal subset of a regular causal space \(X\) and construct

\[
A_0 = \{x \in A : x < z \text{ for some } z \in E^+(A)\}, \quad S = E^+(A) - (A - A_0).
\]

The set \(B \cap S\) is called a generator of \(S\) if \(B\) is a beam intersecting \(S\) more than once. In this case

(i) \(S\) is the union of its generators.
(ii) Each generator has a first point in \(A_0\).
(iii) The set of last points of the generators, if not empty, is acausal.
(iv) Any point which is common to two different generators is either the first point or the last point of both.

A generalization of the Kronheimer-Penrose construction with applications to quantum physics can be found in \([165, 166]\).

3.2. Carter's study of causal spaces

Carter's paper [25] takes a stride forward and gives a more accurate analysis of the causal binary relations. Almost all the definitions and results in [25] are established for a space-time manifold but the author cares to point out which of these results can or cannot be generalized to arbitrary sets. This is done by the introduction of etiological spaces‡, see subsection 3.2.1. A basic difference with [89] is that axioms (iii) and (iv) in definition 3.1 are not assumed from the beginning, which is more adapted to what actually happens in General Relativity and general spacetimes. This allows for a thorough analysis of the questions of causality violation ("vice") and causal good behaviour ("virtue") in general causal spaces—we will further treat these matters in section 4. Carter’s paper is very detailed and contains a large number of concepts which makes it, together with [89], one of the most complete references dealing with abstract causal theory. Of course, here we cannot cover all the material presented in [25] but we hope to convey the most important ideas.

To start with, the basic definition we adopted in definition 2.1 for spacetime is more restrictive than that used in [25], where a spacetime is taken to be just

‡ Carter proposed the name etiology for the branch of topology devoted to abstract causal spaces.

Etiology is the branch of knowledge concerned with causes.
a connected $C^1$ $n$-dimensional differentiable manifold $M$ on which a continuous oriented null-cone structure is defined. An oriented null-cone structure is a continuous linear mapping of the $n$-dimensional solid Euclidean half cone defined in Cartesian coordinates $\{x^1, \ldots, x^n\}$ by

$$x^n \geq \sqrt{\sum_{r=1}^{n-1} (x^r)^2},$$

into each fibre of the tangent bundle. Put another way we are providing each point of the manifold with a cone so we could also term this structure as a “Lorentzian cone field” on the manifold $M$. The image of the Euclidean half cone on each tangent space is the future null cone whereas the image of the opposite half of the Euclidean cone is the past null cone. Particular cases of space-time manifolds in this sense are the so-called conformal structures (see [87, 150]), while a generalization is given by the conal manifolds in which the Euclidean cone is replaced by a closed convex pointed cone $C \subset V$ where $V$ is an $n$-dimensional vector space [150, 94] (see also §7.3).

Once the null-cone structure is given, definitions of concepts such as timelike, null or spacelike vectors and future oriented continuous curves proceed along obvious lines so all these ideas will be taken for granted ([25] is systematic in the definition of these and other related concepts such as spacelike or timelike submanifolds, inextensible and maximal subsets, etc), as they are equivalent in an obvious sense to those we have already given. Here we are more interested in the study of binary relations arising from the causality in much the same way as we have done in the previous subsection.

**Definition 3.12 (Qualified causality and chronology relations)** Let $\mathcal{I}$ and $\mathcal{J}$ be two subsets of $M$ and consider an auxiliary set $\mathcal{U}$ also in $M$. We will say that $\mathcal{J}$ lies in the causal future of $\mathcal{I}$ with respect to $\mathcal{U}$, denoted as $\mathcal{I} <_\mathcal{U} \mathcal{J}$, if for every point $x \in \mathcal{J}$ there is a past-directed causal continuous semi-arc contained entirely within $\mathcal{U}$ and intersecting some point of $\mathcal{I}$. Similarly, $\mathcal{J} >_\mathcal{U} \mathcal{I}$ if for any $x \in \mathcal{J}$ there is a future directed causal continuous semi-arc contained entirely within $\mathcal{U}$ and intersecting some point of $\mathcal{I}$. The relations $\mathcal{I} \ll_\mathcal{U} \mathcal{J}$ and $\mathcal{J} \gg_\mathcal{U} \mathcal{I}$ are defined similarly.

The attribute “qualified” here means that the relations are defined through the reference to a subset $\mathcal{U} \subset M$. Unqualified relations are those in which $\mathcal{U}$ is the manifold $M$ itself. In this last case the subscript will be dropped from the binary causal relations.

Observe also that in definition 3.12 causal and chronological relations are defined between subsets of $M$ as opposed to the case previously considered in Kronheimer and Penrose’s work where only causal relations between points were defined. In the notation of section 2, and for Lorentzian manifolds, the previous sets can be defined as follows

$$\mathcal{I} <_\mathcal{U} \mathcal{J} \iff \mathcal{J} \subset J^+(\mathcal{I}, \mathcal{U}), \quad \mathcal{I} \ll_\mathcal{U} \mathcal{J} \iff \mathcal{J} \subset I^+(\mathcal{I}, \mathcal{U}),$$

and so on. Therefore, it should be noticed that according to definition 3.12, $\mathcal{I} <_\mathcal{U} \mathcal{J}$ and $\mathcal{J} >_\mathcal{U} \mathcal{I}$ have in general different meanings and the same happens with $\mathcal{I} \ll_\mathcal{U} \mathcal{J}$ and $\mathcal{J} \gg_\mathcal{U} \mathcal{I}$ (this is a difference with the Kronheimer and Penrose causal relations $<, \ll$ and $\rightarrow$). Only in the case in which the sets $\mathcal{I}$ and $\mathcal{J}$ consist of a single point can the statements $\{x\} <_\mathcal{U} \{y\}$ and $\{x\} \ll_\mathcal{U} \{y\}$ be read from right to left. In fact in this last case the unqualified relations are just the causal relations of Kronheimer and Penrose.
The notions of causal and chronological future can be given straightforwardly once the previous relations have been set. Let us simply remark that they received specific notation in [25]

\[ \mathcal{U}_\mathcal{I} \equiv \{ x \in M : \mathcal{I} \subseteq x \} = I^+(\mathcal{I}, \mathcal{U}), \quad \mathcal{U}_\mathcal{I} \equiv \{ x \in M : \mathcal{I} \subset x \} = J^+(\mathcal{I}, \mathcal{U}), \]

and similarly the qualified horismoidal future of \( \mathcal{I} \) was written as

\[ \mathcal{U}_\mathcal{I} = E^+(\mathcal{I}, \mathcal{U}). \]

This notation does not seem to have been used ever since. One can define, however, a new binary relation called horismos relation.

**Definition 3.13 (Qualified horismos relation)** For any two subsets \( \mathcal{I}, \mathcal{T} \subset M \), the qualified future and past horismos relation with respect to \( \mathcal{U} \subset M \) are defined by

\[ \mathcal{I} \subseteq \mathcal{T} \Leftrightarrow \mathcal{T} \subset E^+(\mathcal{I}, \mathcal{U}), \quad \mathcal{T} \supset \mathcal{I} \Leftrightarrow \mathcal{T} \subset E^-(\mathcal{I}, \mathcal{U}). \]

Carter also used new notation for the straightforward generalization of the Cauchy developments, namely

\[ \mathcal{I} \supset \mathcal{I} = D^+(\mathcal{I}, \mathcal{U}), \quad \mathcal{I} \supset \mathcal{I} = D^+(\mathcal{I}, \mathcal{U}), \quad \mathcal{I} \supset \mathcal{I} = D^-(\mathcal{I}, \mathcal{U}), \quad \mathcal{I} \supset \mathcal{I} = D^-(\mathcal{I}, \mathcal{U}) \]

which allow to define Cauchy causality relations between subsets of \( M \) whenever they are qualified on an open \( \mathcal{U} \) (or on timelike proper submanifolds [25]).

**Definition 3.14 (Qualified Cauchy causality relations)** Let \( \mathcal{U} \) be an open subset of \( M \). For any two sets \( \mathcal{I}, \mathcal{T} \subset M \) we say that \( \mathcal{T} \) lies in the future causal Cauchy development of \( \mathcal{I} \) with respect to \( \mathcal{U} \) if \( \mathcal{T} \subset \mathcal{D}^+(\mathcal{I}, \mathcal{U}) \). This is denoted by \( \mathcal{I} \supset \mathcal{I} \). The relation \( \mathcal{I} \subset \mathcal{I} \) is defined analogously using \( \mathcal{D}^-(\mathcal{I}, \mathcal{U}) \). Finally

\[ \mathcal{I} \supset \mathcal{I} \Leftrightarrow \mathcal{T} \subset \mathcal{D}^+(\mathcal{I}, \mathcal{U}), \quad \mathcal{T} \subset \mathcal{D}^-(\mathcal{I}, \mathcal{U}) \]

The main properties of all these binary relations between the subsets of \( M \) are summarized in the following proposition [25]

**Proposition 3.1 (Basic properties of Carter’s causal relations)** The following statements are true for any subsets \( \mathcal{U}, \mathcal{V} \subset M \) (here, the qualifying subset \( \mathcal{U} \) must be taken as either open or a timelike submanifold whenever this is necessary):

(i)

\[ \mathcal{U} \subset \mathcal{V} \Rightarrow \{ \mathcal{U} \subset \mathcal{V}, \quad \mathcal{U} = \mathcal{V}, \quad \mathcal{U} \supset \mathcal{V} \} \]

(ii)

\[ \mathcal{U} \subset \mathcal{V}, \quad \mathcal{V} \supset \mathcal{V} \subset \mathcal{V}, \quad \mathcal{V} \supset \mathcal{V}, \quad \mathcal{V} \supset \mathcal{V} \subset \mathcal{V}. \]

(iii) The relations \( \subset, \supset \) are reflexive only for sets \( \mathcal{I} \subset \mathcal{U} \). The corresponding unqualified relations are thus reflexive.

(iv) The relations \( \supset, \subset \) are all reflexive.

(v) All relations \( \subset, \supset, \supset \subset, \subset \supset \) are transitive.
3.2.1. Etiological spaces. Even though most of [25] assumes a manifold structure, Carter also considered the question of how one can use the previous binary relations in order to get an axiomatic definition of etiological space, which is a generalization of Kronheimer-Penrose’s causal space.

**Definition 3.15 (Etiological space)**  A topological space $X$ endowed with two binary relations $<$ and $\ll$ is an etiological space if the following axioms are fulfilled

**Axiom 1.** $\forall x \in X, x < x$.

**Axiom 2.** Both $<$ and $\ll$ are transitive.

**Axiom 3.** $\ll \subseteq <$ in the sense of binary relations.

**Axiom 4.** If $x \ll x$ for some $x \in X$ $\Rightarrow \exists y \in X, y \neq x$: $x \ll y, y \ll x$.

**Axiom 5.** The topology of $X$ is a refinement of the Alexandrov topology constructed from $\ll$.

The great advantage of this definition is that any space-time manifold $M$ (in the sense used in this section, that is, with a null-cone structure, hence also for those complying with the standard definition 2.1) is an etiological space. This is clear by identifying the binary relations $<$ and $\ll$ with the unqualified relations between subsets of definition 3.12 in an obvious sense, that is to say, the meaning of $x < x$ is simply $\{x\} < \{x\}$, and analogously $x \ll x \iff \{x\} \ll \{x\}$.

Clearly an etiological space is more general than a causal space. To start with, the former is defined in terms of just two binary relations, instead of three. Moreover definition 3.15 relies on a less number of axioms than definition 3.1. It is easily checked that points (i), (ii), (v) in definition 3.1 are covered by the first three axioms of etiological spaces. However, axiom 4 is more general than the corresponding points (iii) and (iv) in definition 3.1. So the definition of etiological space is also more general in this sense. It might seem, however, that Kronheimer-Penrose’s definition is less restrictive in another sense, as it does not require the background set $X$ to be a topological space. Nevertheless, as is manifest from definition 3.4, any causal space is always a topological space with the Alexandrov topology, and thus this requirement for the etiological space is not restrictive at all (provided, of course, that the chosen topology for $X$ is the Alexandrov topology.)

Etiological spaces include all possible Lorentzian manifolds, including those which violate points (iii) and/or (iv) of the definition 3.1 of causal space. As mentioned before, the failure of these two points may seem not very satisfactory from a physical point of view, as there is a general consensus that any model of the real physical world should actually comply them. Nevertheless, and given that such “vicious” spaces do appear in the theory of General Relativity (e.g., the Gödel universe [56], see section 4 and [87]), and there is still some controversy as to whether or not they might be valid in some extreme regime, it seems more appropriate from a scientific point of view not to exclude them by axiom, as was argued also in [25]. In this way, at least we can study them and draw conclusions about their predictions and their absurdity, if this is so. Carter’s paper contains an interesting discussion and a classification of space-time manifolds according to the “virtuousness” of the causal relations present in $M$ (of course these results can be translated to etiological spaces). Other relevant references are [64, 87] and references therein. We will discuss these issues in section 4 as we believe they have a closer relationship with the contents discussed there.
In any case, Carter’s paper thoroughly discusses which further axioms must be added to definition 3.15 in order to get a causal space in the sense of Kronheimer and Penrose. These are essentially two

**Causality principle.** There are no \( x, y, x \neq y \) such that \( x < y \) and \( y < x \).

**Strong transitivity.** For any \( x, y, z \in \mathcal{X} \) we have

\[
x < y, \ y \ll z \Rightarrow x \ll z, \ \ x \ll y, \ y < z \Rightarrow x \ll z.
\]

The first of these takes care of the previous discussion, as then points (iii) and (iv) of definition 3.1 hold. This is clear for point (iii) which is exactly the causality principle. Point (iv) is then a consequence of this and axiom 4. As a matter of fact, provided that the causality principle is assumed, one could substitute axiom 4 in definition 3.15 by the following

**Chronology principle.** There is no element \( x \in \mathcal{X} \) such that \( x \ll x \).

Concerning the second added axiom, the strong transitivity principle, it was (erroneously) argued in [89] that this is a property satisfied by all spacetimes. As pointed out by Carter [25], this is true only for sufficiently differentiable structures. We meet once more the question of the differentiability of the metric (or of the null-cone structure), that we have already discussed briefly in subsection 2.2. It will never be sufficiently stressed the importance of these differentiability matters (at least from a mathematical viewpoint), as many “trivial” or “obvious” results, which are taken for granted, do not actually hold or have not been proved.

All spacetimes with a \( C^2 \) metric, or with a differentiable and not merely continuous null-cone structure, satisfy the strong transitivity principle. In summary, all spacetimes are etiological, all causal spaces in the sense of Kronheimer-Penrose are etiological too (provided the Alexandrov topology is used), and all etiological Lorentzian manifolds with a \( C^2 \) metric and satisfying the causality principle are causal spaces. It should be stressed, however, that the causality principle does not remove all possible causal pathologies [25, 64, 87, 153], as will be discussed in section 4.

For further details and developments concerning causal and etiological spaces, consult [167].

In both the Kronheimer-Penrose and Carter constructions the notions of causal and etiological space are based on abstractions of certain simple relations which can be set between points and subsets of a spacetime due to the presence of a Lorentzian metric. However it is still not clear how one can go the other way round, namely, starting from the axiomatic relations present in a causal or etiological space, and trying to reconstruct the Lorentzian metric, or a class of Lorentzian metrics, compatible with the given causal binary relations. A number of investigations have tackled this question and we will give an account of them in section 4.

### 3.3. Physically inspired axiomatic approaches

References [89] and [25] adopt a set of axioms abstracted from the mathematical properties of Lorentzian manifolds and from them they derive general results. There is however, another way to proceed which is trying to motivate the introduction of Lorentzian manifolds as models of spacetimes right from objects and concepts which can be touched and experimented upon, which exist in every day’s common life, or which seem to be intuitively incontrovertible (taken as axioms) [34, 174, 147, 148]. This
is the idea behind the approach taken by Ehlers, Pirani and Schild [34], improved by Woodhouse [174], as well as in all approaches related to quantum physics, where the starting point is often a discrete set to be smoothed out by some procedure in order to produce the effective continuous spacetime that we see. We consider these two lines in the following subsections.

### 3.3.1. Ehlers-Pirani-Schild-Woodhouse axiomatic construction.

In [34] the authors took as primitive concepts “particles”, “light rays” and “events” in a set $M$ and imposing certain axioms they showed that $M$ must be a four dimensional Lorentzian manifold. An interesting and powerful generalization of this construction was performed by Woodhouse in [174]. An obvious difference between the two approaches is that Woodhouse speaks of “light signals” instead of “light rays”. More importantly, Woodhouse’s approach is more ambitious, for he pays a deeper attention to the chronological and causal relations of events in the space-time, inasmuch as he proves how to endow the set of events with the structure of a causal space in the sense of definition 3.1, and he also derives the topology from the axioms.

We outline next the essentials of both constructions by mixing them in an appropriate way. The axioms presented herein do not necessarily correspond to those of [34, 174] nor do they follow the same order as in those papers, rather we have produced a self-consistent version which takes benefit of the primordial construction performed in [174] first, and then follows along the lines of the original paper [34].

**Axiom 3.1** The physical spacetime is represented by a set $M$ whose elements are called events endowed with a particular subset $\mathcal{P} \subset \mathcal{P}(M)$ of the set of subsets of $M$. The elements $P, Q, \ldots \in \mathcal{P}$ are called particles, each of which has the structure of a $C^0$ one-dimensional manifold homeomorphic to $\mathbb{R}$. There is at least one particle through each event.

The homeomorphism of $P, Q, \ldots$ with $\mathbb{R}$ provides each particle with an orientation and thereby with an antireflexive and transitive binary relation denoted by $\ll |_P$. By combining then these relations one can define the chronology relation $\ll$ between the points $x, y \in M$ if there is a finite sequence of $\{\ll | P_i\}_{i=1, \ldots, m}$ such that $x \in P_1$ and $y \in P_m$. In order to comply with point (iv) in definition 3.1 then we assume

**Axiom 3.2** The orientation of all particles can be chosen in such a way that for all $x \in M$, not $x \ll x$.

Once we have the chronology $\ll$, we can endow $M$ with the Alexandrov topology of definition 3.4. Thus we have a topological set. Furthermore, it is possible to use definition 3.8 in order to define an almost causal space (see the comments that follow that definition) in the sense of Kronheimer and Penrose. To actually get a causal space we need a third axiom

**Axiom 3.3** Let $x, y \in M$. If $I^+(y) \subset I^+(z)$, $\forall z \in I^-(x)$ and $I^+(x) \subset I^+(z)$, $\forall z \in I^-(y)$ then $x = y$.

From this axiom one can prove [174] that $\ll$ is future and past distinguishing so that, according to the remark after equation 3.2, $(M, \ll, <, \rightarrow)$ is a causal space.

At this stage, and in order to incorporate the properties of light propagation, another axiom is needed [174]

**Axiom 3.4** Every $x \in M$ has a neighbourhood $N_x$ which is future and past reflecting.
Then, local “light signals” from \( p \in N_x \) to \( q \in N_x \) can be defined without ambiguity as equivalent to the relation \( p \rightarrow_{N_x} q \).

Let \( P \) be a particle and, for each \( p \in P \), take one of the past and future reflecting neighbourhoods \( U_p \). As \( U_p \) belongs to the Alexandrov topology we can always choose \( U_p = \Gamma^-(q) \cap \Gamma^+(r) \), \( q, r \in P \). Define a neighbourhood \( U_p \) of \( P \) as the union of all such \( U_p \) for \( p \in P \). Then, for \( e \in U_P \) there is a \( p \in P \) such that \( e \in U_p \), and we can define the mappings \( f^\pm : e \rightarrow P \) by

\[
\begin{align*}
  f^+(e) &= v \in P \text{ such that } e \rightarrow_{U_p} v, \\
  f^-(e) &= u \in P \text{ such that } u \rightarrow_{U_p} e,
\end{align*}
\]

(see figure). The functions \( f^\pm \) are called the messages. In [34], the definition of echoes is also given: the mapping from \( f^- : U \rightarrow P \) to \( f^+ : U \rightarrow P \).

Consider now a point \( e \in M \) and two nearby particles \( P, P' \). In a physical spacetime \( e \) can be determined by means of two echoes with the points \( u \in P \) and \( u' \in P' \) as the respective domain points and \( v \in P, v' \in P' \) as the image points. These points determine four real numbers which will change if we change the point \( e \) but keep \( P \) and \( P' \) fixed. We have thus constructed a bijective map \( x_{PP'} : U \subseteq M \rightarrow \mathbb{R}^4 \) in a neighbourhood \( U \) of the event \( e \) as shown in the picture (throughout this Review null rays are represented by lines forming 45° degrees with the horizontal plane).

**Axiom 3.5** The set of maps \( x_{PP'}|_U \) is a smooth atlas for \( M \). The coordinate charts defined by this atlas are called radar coordinates.

In [174] the differentiability of the previous functions \( f^\pm \) and \( x_{PP'} \) are derived from the previous axioms, while in [34] this is simply assumed as in the previous axiom 3.5. The word smooth in this context means “as smooth as the message functions are”, that is to say, as smooth as the manifold looks like if we perform physical experiments with light signals and particles. Axiom 3.5 tells us that \( M \) is a smooth differentiable manifold of dimension four (this 4-dimensionality is built in in the above construction and cannot be avoided), and as proved before \( M \) has also the structure of a causal space. To proceed further in the construction something more about how light rays, and not merely light signals in local neighbourhoods, propagate on \( M \) must be said. From now on we slightly depart from the construction in [174] using the original ideas in [34], which are more transparent.

As before, for any event \( p \) choose a particle \( P \ni p \) and a neighbourhood \( U_p \) of \( P \) such that any event \( e \in U_p - P \) defines the two points \( f^\pm(e) \in P \). Choose a coordinate \( t \) for \( P \) (that is to say, a parametrization on \( P \)) with \( t(p) = 0 \).
Axiom 3.6 The function $g : p \rightarrow t(f^+(e))t(f^-(e))$ is of class $C^2$ on $U_p$.

At this point, light rays can be defined as an obvious “not-bending” extension of local signals [174]. The following axiom is then needed in [34].

Axiom 3.7 The set of nonvanishing vectors on $T_p(M)$ which are tangent to light signals consist of two connected components.

Using axioms 3.6 and 3.7 the authors of [34] manage to show that, in a local coordinate system $\{x^1, x^2, x^3, x^4\}$, the derivatives $g^{ab} \equiv g_{ab} \mid_p$ of the above defined function $g$ form a tensor at $p$. Moreover light rays directions at $T_p(M)$ are characterized as those vectors $T^a$ such that $g_{ab}T^aT^b = 0$ and the signature of the tensor $g_{ab}$ is Lorentzian at every event of $M$. Nonetheless this only determines the metric up to a positive conformal factor because a rescaling of the coordinate $t$ of the particle used in axiom 3.6 leads to a metric $g_{ab}^{\prime}$ conformal to $g_{ab}$. Therefore from the axioms laid down so far we conclude that $M$ has a conformal structure $\mathcal{C}$ such that the null vectors are the tangent vectors of light rays. Recall that a conformal structure is an equivalence class of conformally related metrics through a strictly positive conformal factor. The authors also show that light rays are in fact $\mathcal{C}$-null geodesics (null geodesics of any metric of the conformal structure). As remarked in [174], this proves as a by-product that light rays are smooth (this smoothness was an axiom in [34].)

An important observation is in order here. Axioms 3.1 to 3.7 do not guarantee that the conformal structure $\mathcal{C}$ is unique because a definite family of particles must be chosen to carry out the procedure described in the preceding paragraphs. It might well happen that changes in these families would lead to another different conformal structure $\mathcal{C}'$.

Nevertheless, if the family of particles (and thereby, the light signals) are fixed once and for all, then the conformal structure is claimed to be unique in [34]. To that end, the authors appeal to a result (theorems 5.9) attributed to Hawking [65] (see also the related results in theorem 4.5 and in subsections 4.2 and 5.3): any bijective map $\phi : (M, \mathcal{C}) \rightarrow (\bar{M}, \bar{\mathcal{C}})$ between the $C^3$ strongly causal (see definition 4.1 below) manifolds $M, \bar{M}$ with $C^2$ conformal structures $\mathcal{C}$ and $\bar{\mathcal{C}}$ such that both $\phi$ and $\phi^{-1}$ preserve the causal relations is a $C^3$ conformal diffeomorphism. It is known due for instance to Proposition 2.1, that all Lorentzian manifolds are locally strongly causal. Thus, if we apply locally the mentioned result to the structures $(M, D, \mathcal{C})$ and $(\bar{M}, \bar{D}, \bar{\mathcal{C}})$ ($D$ and $\bar{D}$ stand for the differential structures of $M$) and the identity map $M \rightarrow M$ we can conclude that both structures are diffeomorphic and conformally related.

By the way, observe once more the necessity of having a $C^2$ metric in order to obtain definite and precise results.

Further axioms are needed to obtain the projective or metric structure from the conformal one [34]. Here, given that we are only interested in the causal structure and this has already been obtained, we will just mention the two main axioms without further comments, (the ideas behind both of them are more or less intuitive).

Axiom 3.8 For any event $p \in M$ and a $\mathcal{C}$-timelike direction on $T_p(M)$ there is one and only one particle $P$ passing through $p$ with such direction.

Axiom 3.9 (Law of inertia) For each event $p \in M$ there exists a local coordinate system $\{x^1, x^2, x^3, x^4\}$ on a neighbourhood of $p$ such that any particle through $p$ can be parametrized by $x^a(u)$ and

$$\frac{d^2 x^a}{du^2} \bigg|_p = 0.$$
3.3.2. Quantum approaches. Causal sets. An idea which has emerged from many approaches to gravity quantization, and which seems to be an inherent feature to the sought theory of “quantum gravity” [45], is that the spacetime cannot be continuous down to a scale known as Planck’s scale. This scale is defined naturally by the combination of three basic constants of nature, namely, Newton’s constant $G$, the speed of light $c$ and Planck’s constant $\hbar$ in a quantity with dimension of length called Planck’s length and given by

$$l_p = \frac{G^{1/2} \hbar^{1/2}}{c^{3/2}} \approx 1.61 \times 10^{-35} \text{m}.$$ 

Similarly we can combine these constants to obtain natural scales of time and energy (Planck’s time and Planck’s energy). These are the scales upon which a quantum theory of the spacetime is expected to take over the classical differentiable picture. Thus, the theory of “quantum gravity” has flourished in the last decades as one of the most active branches of theoretical physics in an attempt to merge successfully Quantum Physics and General Relativity. A popular alternative is the use of (super-) String Theory, where in particular the Maldacena conjecture has important applications of the theory of causal boundaries, see section 6. The whole quantum issue would require a long “topical review” by itself, and we cannot treat any of the important problems arising there in a fair manner here. Thus, we have limited ourselves to present a very brief summary of main lines and a list of references which, hopefully, will be useful to the readers interested in this matter. Accounts of the mentioned and other of the purported quantum theories of gravity can be found in excellent recent reviews, e. g. [3, 75] and references therein.

As mentioned before, a generalization of the Kronheimer-Penrose construction with applications to quantum physics can be found in [165, 166]. However, the most fruitful line of research deals with the idea of a discrete spacetime, which is perhaps the next obvious stage after having discovered that matter and energy are discontinuous or quantized. Despite this being a logical step, only in the nineties explicit models of discrete spacetimes, and of “quantum geometries”, were constructed, see [18, 134, 135, 84, 85, 104, 102], although early attempts can be found in the literature [156, 74, 37, 127]. There are currently two main approaches to this subject. The first one is the causal set approach started in [13] and briefly discussed in following paragraphs and the second one is loop quantum gravity. For a relation between the two approaches see [103, 157] and references therein. Loop quantum gravity manages to quantize Einstein’s theory in a background independent way and one of its most remarkable predictions is that spacetime is formed at the quantum level by a discrete structure called spin network. This is a very active field of research and there are already excellent reviews about this subject [3, 4, 141]; as the subject falls outside the scope of this review we will not add anything else about this theory here.

The foundations of the causal set approach to quantum gravity were laid in [13]. For a pedagogical introduction, see [137]. As already mentioned, the basic idea of this approach is to regard the would-be spacetime as a discrete set at small scales. Furthermore, only the relation $<$ between its elements (called also points) is kept at such scales. The specific definition of causal set as stated in [13] is reproduced here.

**Definition 3.16 (Causal set)** A causal set $C$ is any finite set partially ordered by
a binary relation $<$ with the additional condition that $<$ is noncircular, i.e., there are no elements $x, y \in C$, $x \neq y$ such that $x < y < x$.

A partially ordered set is commonly called a poset. This definition has common points with definition 3.1 of Kronheimer and Penrose, and also with definition 3.15 supplemented with the causality principle. Thus, only causal spacetimes (see definition 4.1 below) could eventually be described by averaging causal sets. Actually this restriction is even tighter, because there is also an important difference with definitions 3.1 and 3.15: the absence of binary relations of the sort of $\ll$ and $\rightarrow$. This can be put in correspondence with the construction of a causal space from the causality of definition 3.10. As discussed briefly there, and also in §4.2, the relation $<$ alone suffices to characterize the causal properties of the “averaged” spacetime only if this is distinguishing. In other words, a built-in constraint of the causal set approach is that spacetimes whose microscopic behaviour is likely to be mimicked by causal sets must be at least distinguishing. In our opinion, a successful quantization of gravity—if this exists—should pose no restrictions on the resulting background spacetime we are trying to quantize, however unphysical this background might seem to be. Nevertheless, these issues were not discussed in [13].

A fundamental requirement of any discrete theory is the prescription to be followed in order to get the picture of a smooth Lorentzian manifold. In short, how can a spacetime be recovered from a causal set $C$. To achieve this a proposal is given in [13]: construct an embedding $f : C \rightarrow V$, where $(V, g)$ is an $n$-dimensional Lorentzian manifold with metric tensor $g$ complying with the following properties

(i) $f(x) \in J^-(f(y)) \iff x < y$.

(ii) The embedded points are uniformly spread on $V$ with unit length. This means that units are chosen in such a way that the numerical value of the integral of the volume element $n$-form canonically defined by $g$ is proportional to the number of elements of $C$ contained in the region of integration to the power $n$.

(iii) The characteristic length $\lambda$ of the continuous geometry (the length over which $g_{ab}$ vary appreciably) is much larger than the mean spacing between embedded points.

Any embedding $f$ meeting the above properties is called a faithful embedding. In principle a faithful embedding in a Lorentzian manifold $(V, g)$ does not need to exist for a given causal set but if it does it should be essentially unique. This means that given a pair of faithful embeddings $f_1 : C \rightarrow (V_1, g_1)$, $f_2 : C \rightarrow (V_2, g_2)$ there should exist a diffeomorphism $h : V_1 \rightarrow V_2$ such that $f_2 = h \circ f_1$. In [13] it is argued that a causal set for which no faithful embedding in $(V, g)$ exists, could nonetheless be faithfully embedded when coarse grained. A coarse graining $C'$ is essentially a subset of the causal set $C$ with the partial order $<$ inherited from $C$.

Once a causal set is defined one needs to find its dynamics. Such dynamics must reduce to (say) Einstein’s field equations in the continuum limit. In [13] a method is sketched to recover the Einstein-Hilbert action from a quantum causal set. Other models were tried in the subsequent follow-ups. Causal set theory has been widely investigated and by now there is a vast literature about this subject, see e. g. [1, 19, 32, 68, 80, 81, 136, 138, 137, 139, 140] and references therein. In fact there is not a single causal set theory and plenty of variants abiding by the idea of a basic discreteness of the spacetime have been devised. Of course we cannot do justice here to all this work (a field outside our expertise). We would simply like to remark that
hitherto none of these theories accounts for a satisfactory continuum limit in which a Lorentzian manifold is recovered. This seems to be the crux of most of the current approaches to “quantum gravity”: mind-boggling theories are put forward but the key issue of General Relativity as a limit is often quoted as “under current research”.

4. Causal characterization of Lorentzian manifolds and their classification

Lorentzian manifolds may have rather different global causal properties depending on their Lorentzian metrics. More importantly, completely smooth regular manifolds (such as $\mathbb{R}^n$) carrying analytical metrics with perfectly good local properties and which look completely innocuous at first sight may have closed future-directed timelike loops: violation of the chronology hypothesis. This was not fully realized until the publication of Gödel's famous paper [56] where he constructed an example of cosmological spacetime with rotation containing timelike loops through every point of the manifold. Other physically acceptable spacetimes, such as the Kerr solution [64, 161], are extensible as solutions to Einstein’s vacuum field equations, and their maximal extensions may contain regions in which such causality violations arise. As is obvious, one would desire to rule out these curves on the grounds of their unphysical properties, because they lead to violations of the “free-will” principle and to other traditional paradoxes (they would meet their own past after a certain proper time), see the discussion in [87] and references therein. It has been recently claimed, however, that these pathologies arise in physically acceptable spacetimes [15, 16].

Timelike loops are not the only undesirable causal feature one wishes to rule out from a spacetime for there may also be causal or null loops, or other timelike/causal curves which form “almost a loop”, or get trapped inside a compact region of the spacetime (causal imprisonment, see [111, 64] for particular examples.) This means that the causal classification of Lorentzian manifolds in a hierarchy which measures their causal behaviour is not so simple as to put in a group those having closed timelike loops and those which do not in another group. This is the ultimate reason why several causality conditions, forbidding such loops or the related “almost loops” in order to achieve physically acceptable spacetimes, were devised in [25, 64, 128] and references therein. Many of these conditions turned out to be insufficient to rule out all causality pathologies, and this is why the final classification is longer than expected. The subject has been studied over the years and nowadays there is a well established hierarchy of causality conditions, which can be found in many textbooks of General Relativity [64, 112, 172]. The different good properties that each condition achieves, as well as those which does not forbid, are now well understood. A summary of this subject is presented next.

4.1. Standard hierarchy of causality conditions

**Definition 4.1** A Lorentzian manifold $(V, g)$ is said to be:

- **not totally vicious** if $I^+(x) \cap I^-(x) \neq \emptyset$, $\forall x \in V$.
- **Chronological** if $x \notin I^+(x)$ $\forall x \in V$.
- **Causal** if $J^+(x) \cap J^-(x) = \{x\}$ $\forall x \in V$.
- **Future distinguishing** if $I^+(x) = I^+(y)$ only if $x = y$ for $x, y \in V$. Future and past distinguishing Lorentzian manifolds are simply known as distinguishing.
• **Strongly causal** if for every neighbourhood $\mathcal{U}_x$ of $x$ there exists another neighbourhood $\mathcal{U}_x' \subseteq \mathcal{U}_x$ containing $x$ such that for every future-directed causal curve $\gamma$ the intersection $\gamma \cap \mathcal{U}_x'$ is either empty or a connected set.

• **Causally stable** if there exists a function whose gradient is timelike everywhere (called a time function).

• **Causally continuous** if it is reflecting and distinguishing.

• **Causally simple** if it is distinguishing and $J^\pm(x)$ are closed sets $\forall x \in V$.

• **Globally hyperbolic** if there exists an edgeless acausal hypersurface $S$ such that $D(S) = V$. $S$ is called a Cauchy hypersurface.

Conditions in definition 4.1 are given in an increasing order of specialization. This means that the hierarchy is built in such a way that spacetimes belonging to a certain class of definition 4.1 contain a causal feature regarded as better than those present in the classes lying above. Detailed explanations (which are beyond the scope of this review) about these features and the route followed to build the hierarchy can be found in e.g. [64, 6, 153]. Here we are more interested in the attempts of improving this classification either by adding more classes to the hierarchy or defining it in more abstract terms. This will be treated in the next subsections. However, we will present now a summary of the main ideas behind each condition in the hierarchy, and the relation between them, for the sake of clearness and completeness.

The worst behaved spacetimes are totally vicious ones. An equivalent characterization of them is $\exists x \in V$ such that $I^+(x) \cap I^-(x) = V$, from where one immediately derives that $I^\pm(\zeta) = V$ for any set $\zeta \subseteq V$, see also [86]. This terminology was put forward in [25], where probably the first detailed abstract classification was given. Carter uses the name “(almost) vicious” for those spacetimes with (causal) timelike future-directed loops. Accordingly, he used the names “(almost) virtuous” for spacetimes complying with the (chronology) causality condition. A well-known result [5] is that all compact spacetimes are vicious [6, 64, 128], see also [86, 109]. A large set of non-compact vicious spacetimes was described in [145] from a mathematical point of view, and a discussion about their occurrence in physically realistic spacetimes can be found in [15, 16] and references therein.

The distinguishing condition was devised to forbid these “vices”. An equivalent statement for future distinction is (see [153] for a proof): every neighbourhood of $x$ contains another neighbourhood $\mathcal{U}_x$ of $x$ such that every causal future-directed curve starting at $x$ intersects $\mathcal{U}_x$ in a connected set. From this it is obvious that all distinguishing spacetimes are vicious. Nevertheless, a spacetime can be distinguishing and still there may be future-directed causal curves starting nearby a point $x \in V$ but not at $x$—and intersecting all neighbourhoods of $x$ in a disconnected set. This is the reason behind strong causality, which forbids these behaviours. An equivalent statement for strong causality is: $\forall x \in V$ there is a neighbourhood $\mathcal{U}_x$ such that for all $p, q \in \mathcal{U}_x$ with $p \ll q$, $I^+(p) \cap I^-(q) \subset \mathcal{U}_x$. Trivially strong causality implies, and is stronger than, the distinguishing condition. Moreover, strong causality forbids the above mentioned pathology of imprisonment [6, 25, 64].

Carter was the first to realize that the strong causality condition was actually not strong enough to avoid all simple causal pathologies. To show this, in [25] a classification by means of new causal relations called $n$-degree causal relations was elaborated. First of all, one defines the sets

$$1_\neq \mathcal{S} \equiv \left\{ x \in V : \left( J^+(\mathcal{S}) \cap J^-(x) \right) \bigcup \left( J^+(\mathcal{S}) \cap J^-(x) \right) \neq \emptyset \right\} ,$$
Next, new binary relations between subsets of $V$ are arranged by:

$$I^1 < I \iff I \subset I^1, \quad I^1 > I \iff I \subset (I^1)'.$$

Of course all this is valid for sets consisting of a single point. These relations are used in the first step of an inductive chain of definitions

$$I^0 < I \equiv \{ x \in V : \overline{< I^0 >} \cap (x >) \cup \bigcup_{r=1}^{n-1} \overline{< I^0 >} \cap (x^{n-r}) \cup \bigcup_{r=1}^{n-1} < I^0, x^{n-r} > \neq \emptyset \},$$

$$I^0 > I \equiv \{ x \in V : \overline{< I >} \cap (x^n) \cup \bigcup_{r=1}^{n-1} \overline{< I >} \cap (x^{n-r}) \cup \bigcup_{r=1}^{n-1} < I, x^{n-r} > \neq \emptyset \},$$

where

$$< I, x^{n-r} > \equiv < I > \cap (x^{n-r}).$$

From these relations one constructs binary relations $<, >$ between subsets of $V$ in the same fashion as above. These are then used to generalize the concepts of virtuous and vicious (again, $x < y$ will be used for $\{x\} < \{y\}$ and so on):

**Definition 4.2.** A subset $I \subset V$ is said to be virtuous to the $n$th degree ($n \geq 0$) if no pair of different points $x, y \in I$ satisfies $x < y$ and $x < x$ with $r + s \leq n$. And $I$ is said to be sub-vicious by $n$ degrees ($n \geq 0$) if for any two points $x, y \in I$ we have $x < y, y < x$ with $r + s \leq n$.

According to this definition, the most vicious spacetimes are those sub-vicious by zero degree, which correspond to the totally vicious spacetimes of definition 4.1. From here, going up the ladder of causal virtue, the spacetimes virtuous to the $1$st degree are simply the distinguishing ones, and those virtuous to the $2$nd degree are the strongly causal spacetimes. The $n$th-degree vicious spacetimes have increasing virtue for larger $n$, still they do not remove all causal problems. As a matter of fact, there is not only an infinite number of such conditions, but also the concept of “unlimited virtue” as Carter called it cannot be obtained from definition 4.2. The process of gaining virtue can be continued indefinitely, and all such spacetimes are in some sense unstable with respect to their virtue: small perturbations may always produce $1$st-degree vicious spacetimes.

This raises the question as to whether there exist Lorentzian manifolds which are maximally virtuous in this sense, that is to say, so that small perturbations will not affect their virtuousness. We will come back to the question of maximum “good causal behaviour” in subsections 4.2 and 4.3. Carter’s paper proposed to consider the causally simple spacetimes as those with “stable virtue” (though the virtue may be larger or improved within the class.) As a matter of fact, there are intermediate conditions which capture the idea of stable virtue: the stable and the continuous causality conditions. The first of these was introduced by Hawking in [63] with this precise purpose, see also [152]. An equivalent formulation of this condition is: there is a continuous timelike vector field $\mathbf{v}$ such that $g + \mathbf{v} \otimes \mathbf{v}$ is strongly causal. This means that we can open up slightly the null cones and the spacetime remains causal, see for more details proposition 5.4. Hawking showed in [63] that causal stability defined in terms of the vector field $\mathbf{v}$ implies that there exists a cosmic time which is a function
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$f : V \to \mathbb{R}$ increasing along every causal future directed curve on $V$. A complete proof of the differentiability of such cosmic time has only been accomplished recently in [11] (earlier claims were made in [152] but this proof seems to have unclear points.) In [11] these results were carried a step forward: if a cosmic time exists then one can find a smooth function $f$ (a time function) with an everywhere future-directed timelike gradient $df$, see theorems 5.5 and 5.6. The hypersurfaces $f = \text{const.}$ are spacelike and acausal, and they foliate the spacetime (see also [63, 171, 83]). For the causal continuity hypothesis, see [67].

Nevertheless, causally stable and causally continuous spacetimes may fail to be causally simple. This may happen if $\partial J^+(x) \neq E^+(x)$ for some point $x$ (or its past version), whose negation may be taken as an equivalent definition of causal simplicity (subsection 5.1). A remarkable example of a non-causally-simple spacetime was given by Penrose [125] using the so-called plane waves in General Relativity [161], see paragraph 6.5.3. Finally, even the foliation $\{\Sigma_f : f = \text{const.}\}$ constructed with a time function $f$ in causally simple spacetimes does not necessarily define Cauchy hypersurfaces, as $D(\Sigma_f) \neq V$ can certainly happen. Explicit typical examples are anti de Sitter spacetime [64] and the maximal analytic extension of the Reissner-Nordström solution [23]. Therefore, globally hyperbolic spacetimes, which were originally introduced by Leray (see [64, 128]) using the condition that $J^+(x) \cap J^-(x)$ be compact for all $x \in V$ (see subsection 5.1), is regarded as the strongest causal restriction considered hitherto. See subsection 5.2 for further details.

4.2. Mappings preserving causal properties

In this subsection we explore a very interesting approach for the causal classification of Lorentzian manifolds and, more generally, etiological/causal spaces. We may ask ourselves when two different such spaces can in some sense be termed as causally equivalent, or bearing the same causal structure. The basic idea, of course, is to define mappings between them which in a precise sense preserve the causal properties and regard those spaces which can be put into correspondence by means of one of these mappings as “causally isomorphic” or sharing the same causality. These sort of mappings have been considered many times in the literature, e. g. [176, 89, 21, 65, 100, 170, 169, 120]. Kronheimer and Penrose introduced this subject in [89], and then Budic and Sachs used an improvement in a paper [21] devoted to generalising the construction of the causal boundary (see §6.3.1 for more details) and hence they did not use these mappings to classify spacetimes in terms of their causal properties. Although the nomenclature in these and others papers is sometimes conflicting with each other, we have tried to unify the terminology with an up-to-date perspective.

**Definition 4.3** Let $(Z, <, \ll)$ and $(X, <, \ll)$ be etiological spaces (including in particular the causal spaces in the sense of Kronheimer and Penrose and the Lorentzian manifolds) and let $\theta : X \to Z$ be a mapping. $\theta$ is said to be

- **chronal preserving** if $x \ll y$ implies $\theta(x) \ll \theta(y)$.
- **causal preserving** if $x < y$ implies $\theta(x) < \theta(y)$.
- **a chronal isomorphism** if $\theta$ is bijective and $\theta$ and its inverse $\theta^{-1}$ are chronal preserving.
- **a causal isomorphism** if $\theta$ is bijective and both $\theta, \theta^{-1}$ are causal preserving.
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(\(Z, <, \ll\)) and (\(X, <, \ll\)) are causally isomorphic if there is a causal and chronal isomorphism between them.

Perhaps the first reference dealing with these mappings, prior to [89, 21], is Zeeman’s paper [176], where the causal isomorphisms (called “causal automorphisms” in that paper) of flat Minkowski spacetime were studied. The set of such causal isomorphisms clearly forms a group, called the causality group of flat spacetime, and Zeeman was able to prove that this group is generated by the orthochronous Lorentz group, the group of translations, and what he called the dilatation group (multiplication of the flat metric by scalars.) This is a preliminary result which was soon to be generalized in several ways. For instance, if the etiological spaces of definition 4.3 are in fact Lorentzian manifolds then Malament [100], elaborating on previous results in [65], was able to prove the following very important theorem (see also theorem 5.11 and subsection 5.3).

**Theorem 4.1** Let (\(V, g\)), (\(V', g'\)) be a pair of Lorentzian manifolds and \(f : V \to V'\) a bijection. If either of the following two conditions hold

(i) both \(f\) and \(f^{-1}\) preserve continuous timelike curves, or
(ii) (\(V, g\)) and (\(V', g'\)) are distinguishing and \(f\) is a chronal isomorphism,

then \(f\) is a smooth conformal isometry.

Clearly, a bijection \(f\) with the first of these properties is a chronal isomorphism, but the converse is not true in general. This is why further requirements upon \(V\) must be imposed. As Malament showed explicitly in [100], there are examples of chronic isomorphisms between non past-distinguishing spacetimes which do not preserve continuous timelike curves. Thus, the distinguishing hypothesis is essential in the second point of theorem 4.1. For distinguishing spacetimes, however, a map \(f : V \to V'\) preserving the chronology preserves also the continuous timelike curves, the ultimate reason for this is that for distinguishing spacetimes a curve \(\gamma\) is timelike if and only if, \(\forall\ p, q \in \gamma, p \ll q \) [47]. An important consequence of the above is that, for distinguishing spacetimes, the chronological relation \(\ll\) determines the Lorentzian metric up to a conformal factor because if two such Lorentzian manifolds are **chronally isomorphic** then by theorem 4.1 they must be conformally related. Once this is understood, the following statements proving the invariance of the hierarchy of causality conditions under chronal or causal isomorphisms [170, 169, 120] become rather obvious.

**Theorem 4.2** Let (\(V, g\)) and (\(V', g'\)) be chronally isomorphic spacetimes. Then, (\(V, g\)) is globally hyperbolic, causally simple, causally continuous, reflecting, strongly causal, distinguishing, causal, chronological, or totally vicious, if and only if so is (\(V', g'\)).

If (\(V, g\)) and (\(V', g'\)) are causally isomorphic, and one of them is causally stable, so is the other.

From the above we realize that the true applicability of these mappings to the classification of Lorentzian manifolds is rather limited in the relevant cases as we can only compare spacetimes **conformally related** to each other. This is too strong a restriction, as we are going to argue in what follows. For example, it prevents us from being able to give a meaning to the **local equivalence** of causal structures. This question was addressed by Kronheimer and Penrose in their seminal paper [89]. They showed explicitly a negative result which is often ignored: they proved, in a precise sense, that
the statement “any Lorentzian manifold has a causal structure locally equivalent to that of flat spacetime” is wrong as it stands—despite being considered as obvious too many times—if the local equivalence is of conformal type.

Of course, in order to give a meaning to the previous assertion between quotes, and to claim that is false, one must first of all provide a definition for “equivalent” causal structures. This is precisely what was given in [89], where a definition which appeared to be “natural” at first sight was put forward. In [89] the authors were only interested to discuss as to what extent a Lorentzian manifold regarded as a causal space is locally similar, from the causality point of view, to flat Minkowski spacetime, so the definition they gave was stated as follows

For each point \( x \) of the \( n \)-dimensional manifold \( V \) we can find a small open neighbourhood \( U_x \) of \( x \) together with a homeomorphism \( h \) of \( U_x \) onto an open subset \( U \) of \( n \)-dimensional Minkowski spacetime such that \( h \) is a causal isomorphism.

A counterexample of this assertion is explicitly constructed in [89]. Here the set \( U_x \) acquires its causal structure either as a causal subspace of \( V \) or as a Lorentzian submanifold of \( V \). The authors argue that the above quoted assertion is in general false on the grounds that the causal structure of a Lorentzian manifold is determined by its Lorentzian cone, hence the causal structure of an arbitrary Lorentzian manifold and that of flat spacetime will differ even locally unless the spacetime is locally conformally flat. Of course, this is not the case generically, even for the simplest examples.

In the next paragraph we are going to show, however, that an assertion such as “any spacetime is locally equivalent to flat spacetime from the causal viewpoint” does make sense (see theorem 4.4) if we agree to abandon the concept of causal isomorphism of definition 4.3 as the basic ingredient to be used for that purpose, and thereby we will see that the conformal structure is not the appropriate causal structure of a Lorentzian manifold for these matters.

4.2.1. Causal relationship. Isocausal Lorentzian manifolds. To surmount the mentioned difficulties, the present authors developed a new approach to the subject in [47]. The idea is now to consider mappings preserving the causal relations whose inverses do not necessarily do so. To that end, we apply the preservation of the causal properties at the first possible level: the algebraic level studied in subsection 2.1. From this the preservation of the rest of the levels (local and global), and of the binary causal relations, will follow.

**Definition 4.4** Let \( \Phi : V \rightarrow W \) be a global diffeomorphism between two Lorentzian manifolds. We say that \( W \) is causally related with \( V \) by \( \Phi \), denoted \( V \prec_{\Phi} W \), if for every future-directed \( X \in T(V) \), \( \Phi X \in T(W) \) is future directed too. \( W \) is said to be causally related with \( V \), denoted simply by \( V \prec W \), if there exists \( \Phi \) such that \( V \prec_{\Phi} W \). Any diffeomorphism \( \Phi \) such that \( V \prec_{\Phi} W \) is called a causal mapping.\(^\S\)

Of course, a similar definition in which past-directed vectors are mapped into future-directed ones (anticausal mapping) can also be given. All the results described below hold likewise for causal and anticausal mappings although we only make them explicit for causal mappings. In short, the idea behind this definition is that the image by \( \Phi \) of the future null cone at every \( x \in V \) is contained within the future null cone at \( \Phi(x) \).

\(^\S\) We used the term “causal relation” for these mappings in [47], but we have preferred to use here the name causal mapping to avoid confusion with the binary causal relations.
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Let $g$ and $\tilde{g}$ be the Lorentzian metrics of $V$ and $W$, respectively. The condition imposed in definition 4.4 is very easy to check for a fixed diffeomorphism $\Phi : V \rightarrow W$, because

$$V \prec_\Phi W \iff \tilde{g}(\Phi' \tilde{X}, \Phi' \tilde{Y}) = \Phi^* \tilde{g}(\tilde{X}, \tilde{Y}) \geq 0, \quad \forall \text{ future-directed } \tilde{X}, \tilde{Y} \in T(V)$$

which means that $\Phi^* \tilde{g}$ is a future tensor according to definition 2.2. It can be proven [47] that this is the characterization of causal and anti-causal mappings. In other words, as explained in subsection 2.1, $\Phi^* \tilde{g}$ must satisfy the dominant energy condition at every point of the manifold $V$. The algebraic conditions that make this happen are well known (see e.g. [64] for a presentation in four dimensions and [49, 46] for its generalization to arbitrary dimension), and several criteria to ascertain if a given tensor is future or not can be found in [154, 9, 47, 49, 46].

In general, all causal objects are preserved, in a precise sense, by causal mappings. A summary of these preservations is given next [47].

**Proposition 4.1** If $V \prec_\Phi W$, then

(i) all contravariant (resp. covariant) future tensors of $V$ (resp. $W$) are mapped by $\Phi$ to contravariant (resp. covariant) future tensors on $W$ (resp. $V$).

(ii) all timelike future-directed vectors on $V$ are mapped to timelike future-directed vectors. And if the image $\Phi' \tilde{X}$ of a future vector $\tilde{X}$ is null, then $\tilde{X}$ is a null future-directed vector.

(iii) every continuous future-directed timelike (causal) curve is mapped by $\Phi$ to a continuous future-directed timelike (causal) curve.

(iv) for every set $\zeta \subseteq V$, $\Phi(I^+(\zeta)) \subseteq I^+(\Phi(\zeta))$, $\Phi(J^+(\zeta)) \subseteq J^+(\Phi(\zeta))$, and $D^+(\Phi(\zeta)) \subseteq \Phi(D^+(\zeta))$.

(v) if a set $S \subseteq W$ is acausal (achronal), then $\Phi^{-1}(S)$ is acausal (achronal).

(vi) if $\Sigma \subseteq W$ is a Cauchy hypersurface, then $\Phi^{-1}(\Sigma)$ is a Cauchy hypersurface in $V$.

(vii) $\Phi^{-1}(F)$ is a future set for every future set $F \subseteq W$; and $\Phi^{-1}(\partial F)$ is an achronal boundary for every achronal boundary $\partial F \subseteq W$.

From point (iv) follows that any causal mapping is in particular a chronal-preserving and causal-preserving diffeomorphism (see definition 4.3). Notice, however, that the inverse of a causal mapping does not need to be a causal mapping. This opens up a whole new vista which allows to investigate the causal structures, and the causal equivalence, from new perspectives with improved results. Of course, the chronal and causal isomorphisms are particular cases of causal mappings, and therefore the previous results on this subject are included by default under the more general causal relationship of definition 4.4.

Next theorem proven in [47] provides yet more support to the idea that causally related manifolds share some global causal properties.

**Theorem 4.3** If $V \prec W$ and $W$ is globally hyperbolic, causally stable, strongly causal, distinguishing, causal, chronological, or not totally vicious then so is $V$.

Therefore a Lorentzian manifold cannot be causally related to another belonging to a different causality class. In [47] we did indeed show that sometimes two diffeomorphic spacetimes are not causally related, namely, there is no diffeomorphism $\Phi : V \rightarrow W$ such that $V \prec_\Phi W$. This is symbolized by $V \not\prec W$. A very simple example is
flat spacetime which cannot be causally related with anti-de Sitter spacetime because the former is globally hyperbolic and the latter is not. In view of proposition 4.1, theorem 4.3 and many other related results, we argued in [47] that this impossibility of putting in causal relation two Lorentzian manifolds is due to the existence of a global causal property in one of the spacetimes which is absent in the other one. A more interesting example is provided by de Sitter spacetime and the Einstein static universe [64, 161]. As is known, every inextensible timelike curve $\gamma$ in de Sitter space has a non-empty $\partial I^+(\gamma)$ (a “particle horizon”) and this is not so for the Einstein static universe [64, 161]. One can then prove [47] that this property implies that de Sitter universe is not causally related with the Einstein static universe. Observe that the base manifolds for these two spacetimes are the same ($\mathbb{R} \times S^{n-1}$), and more importantly, both of them are globally hyperbolic. As a matter of fact, the Einstein universe is causally related with the de Sitter spacetime.

Given the above comments, an interesting property of causal mappings is that they define a binary relation, the relation $\prec$, in the set of all diffeomorphic Lorentzian manifolds. Clearly $\prec$ is reflexive and transitive and hence it is a preorder. Nevertheless, as mentioned above, there are simple examples (such as de Sitter and Einstein static spacetimes) where a Lorentzian manifold $W$ is causally related with another $V$ but not the other way round, namely, $V \prec W$ but $W \not\prec V$. Thus, the relation “$\prec$” is not symmetric. Furthermore, “$\prec$” is not antisymmetric either, as there certainly are Lorentzian manifolds such that both $V \prec W$ and $W \prec V$ hold [47]. From all the discussion so far, we deduce that those Lorentzian manifolds in which causal relations exist in both ways can be expected to share common causal properties and hence a special name is reserved for them.

**Definition 4.5** Two Lorentzian manifolds $V$ and $W$ are called causally equivalent or isocausal if $V \prec W$ and $W \prec V$. The relation of causal equivalence is denoted by $V \sim W$.

Note that we talk about isocausality as a property between Lorentzian manifolds, and not as a property of a particular mapping. As a matter of fact, there are no explicit particular mappings used here, and more importantly there are two mappings implicitly used. Observe that in general the diffeomorphisms setting the relation $V \prec W$ will be unrelated to those establishing $W \prec V$. In this way $V$ and $W$ only need to be diffeomorphic in order to test their causal equivalence and no geometric conditions such as conformal equivalence are required or implied, contrary to what happened with the approaches reviewed above. This is a consequence of working with causal mappings whose inverse is not necessarily a causal mapping. See also theorem 4.5 below.

With the definition 4.5, which certainly makes sense, and using the fundamental proposition 2.1, we have one of the previously discussed sought results:

**Theorem 4.4** Any two Lorentzian manifolds are locally isocausal.

In other words, the definition given in [89] and quoted in the previous subsection can be forced to make sense if modified as follows:

- For each point $x \in V$ we can find a small open neighbourhood $U_x$ of $x$ and an open subset $U$ of flat $n$-dimensional Minkowski spacetime such that $U_x$ and $U$ are isocausal: $U_x \sim U$. 
Let us stress that isocausality here and in theorem 4.4 is used in the sense of definition 4.5. Thus, there are two chronal-preserving diffeomorphisms: \( \Phi : U_2 \rightarrow U \) and \( \Psi : U \rightarrow U_2 \), say.

The particular case that a causal mapping has an inverse which is a causal mapping too can be characterised as a conformal diffeomorphism.

**Theorem 4.5** For a diffeomorphism \( \varphi : (V, g) \rightarrow (W, \tilde{g}) \) the following properties are equivalent

1. \( \varphi^* \tilde{g} = \lambda g, \lambda > 0 \).
2. \( (\varphi^{-1})^* g = \mu \tilde{g}, \mu > 0 \).
3. \( \varphi \) and \( \varphi^{-1} \) are both causal (or anticausal) mappings.

### 4.3. Ordered chains of causal structures

The relation \( \sim \) is a binary relation on the set of diffeomorphic Lorentzian manifolds. This relation is obviously reflexive, transitive and symmetric, that is to say, is an equivalence relation. Thus we can collect the Lorentzian manifolds in equivalence classes and, following standard procedures, partially order them. This is the subject of this subsection.

To start with, choose a given a differentiable manifold \( M \) meeting the conditions of theorem 5.1. Recall that the set of Lorentzian metrics which can be defined on \( M \) is designed by \( \text{Lor}(M) \). Clearly, for any fixed \( M \) we will find metrics with quite different causal properties. For instance, in \( M = \mathbb{R}^4 \) we can define the flat Minkowskian metric which is globally hyperbolic, or the anti de Sitter metric, which is just causally simple, or even the Gödel metric [56, 64, 161], which is totally vicious. The nice thing about the relations \( \sim \) and \( \preceq \) is that all of these metrics can be sorted by means of the equivalence relation \( \sim \), and then orderly classified by the appropriate generalization of the preorder \( \preceq \).

**Definition 4.6 (Causal structure)** A causal structure on the differentiable manifold \( M \) is any element of the quotient set \( \text{Lor}(M)/\sim \). Each of these causal structures is denoted by \( \text{coset}(g) \) where \( (M, g) \) is any representative of the equivalence class, that is

\[
\text{coset}(g) = \{ \tilde{g} \in \text{Lor}(M) : (M, \tilde{g}) \sim (M, g) \}.
\]

Obviously the equivalence classes depend on \( M \) so that, in case of possible confusion one should use the notation \( \text{coset}_M(g) \) making explicit the base differentiable manifold. Observe that, due to theorem 4.5, these causal structures include the conformal ones in the sense that all metrics globally conformally related to a given \( g \) are elements of \( \text{coset}(g) \). However, the causal structures are much richer and larger than the conformally related metrics, and this is a desirable property which allows to, for example, say that all Lorentzian manifolds have the same causal structure locally (theorem 4.4). Or more interestingly, it permits to give a precise truthful meaning to the sentence “asymptotically flat spacetimes [64, 128, 122, 123, 172] (see subsection 6.1) have the causal structure of flat spacetime asymptotically”, see Example 9 in [47]. Concerning this issue, see also [93, 99, 116].

Causal structures are naturally ordered by the binary relation \( \preceq \) defined on \( \text{Lor}(M)/\sim \) by

\[
\text{coset}(g_1) \preceq \text{coset}(g_2) \iff (M, g_1) \prec (M, g_2).
\]
This is a reflexive, antisymmetric and transitive relation, that is, a partial order. The property measured and classified by this partial order is in some sense the quality of the causal behaviour. This is intuitive from theorem 4.3, which can be re-written loosely providing an order of part of the hierarchy of standard causality conditions as

\[
\text{glob. hyp.} \preceq \text{c. stable} \preceq \text{strongly c.} \preceq \text{disting.} \preceq \text{causal} \preceq \text{chron.} \preceq \text{tot. vicious}.
\]

But the order provided by “\(\preceq\)” is indeed finer than this, because not every pair of globally hyperbolic spacetimes based on the same \(M\) are causally equivalent (recall the example of de Sitter and Einstein spacetimes), and the same happens for pairs of causally stable, strongly causal, et cetera, Lorentzian manifolds. This means that each of the subsets defined by the standard hierarchy, such as the globally hyperbolic metrics on a given \(M\), are themselves also partially ordered by \(\preceq\). Hence we can build abstract chains in the form

\[
\ldots \preceq \text{coset}(g_1) \ldots \preceq \text{coset}(\tilde{g}_1) \ldots \preceq \ldots \preceq \text{coset}(g_2) \preceq \ldots \preceq \ldots \preceq \text{coset}(g_m) \preceq \ldots
\]

Interesting questions are (i) the length of the partial order \(\preceq\), that is to say, the size of the longest possible chain of the above type; and (ii) the existence of lower or upper bounds for these chains which could represent Lorentzian manifolds with the better or worse causal properties, respectively. This would give an answer to the question of whether or not is possible to define the best causally behaved metric on a given manifold. One can give reasons to accept that totally vicious spacetimes are always placed at the rightmost of a causality chain but it is still unknown what type of globally hyperbolic Lorentzian manifold (if any) should be at the leftmost of these chains, of if the chains continue indefinitely to the left. Observe that, from this point of view, the coset \(\mathbb{R} \times S^{n-1}\) containing de Sitter spacetime is “better” than that defined by Einstein’s universe.

5. Causality and topology

In this section we will discuss the interplay between the causality of a set in any of the senses presented in previous section and its topology, if any. Topology and causality keep a close relationship in the case of Lorentzian manifolds because the existence of a Lorentzian metric is not compatible with all topologies of a differentiable manifold. Conversely if we put a certain topology in our manifold certain Lorentzian metrics are not allowed on it or some causal curves with certain topological properties may not be present. Therefore we will start with a brief account of some classical results settling down these issues for the case of Lorentzian manifolds and proceed afterwards to more general cases involving abstract causal spaces. Other interesting investigations can be found in [26, 114].

5.1. Some classical results and their generalizations

The simplest question is which manifold topologies are compatible with the existence of a Lorentzian metric. This question is answered by the next characterization: a differentiable manifold admits a Lorentzian metric if and only if there exists on \(M\) a nowhere zero vector field [87, 117]. Then, the following theorem, whose proof can be found in [160, 117], follows.
Theorem 5.1 If $M$ is a smooth manifold then the following statements are equivalent

(i) $M$ admits a Lorentzian metric.

(ii) Either $M$ is non-compact or it is compact with zero Euler characteristic.

As explained in subsection 4.1, compact manifolds always fail to be chronological—they always contain closed timelike curves—, so that the restriction put by this theorem is very mild in physical terms.

Once we have a Lorentzian manifold the next step is to study the topological properties of the basic chronological sets $I^\pm$, $J^\pm$, etc as well as those of causal curves. The main results were presented in section 2, §2.3.2. Further results will be presented next.

Some of the conditions of the standard hierarchy of definition 4.1 are in fact topological conditions on relevant sets of causality theory. Others admit an alternative formulation in terms of topological concepts. Recall that a set valued function $I : M \to \mathcal{P}(M)$ (where $\mathcal{P}(M)$ is the power set of $M$) is outer continuous at $x$ if for every compact set $K \subseteq \text{ext}(I(x))$ we can find a neighbourhood $U_x$ of $x$ such that $K \subseteq \text{ext}(I(y))$, $\forall y \in U_x$. Similarly we can define inner continuity at $x$ using the set $I(x)$ instead of $\text{ext}(I(x))$. The set-valued functions defined by $I^\pm$ are inner continuous [67]. On the other hand, their outer continuity is a property of causally continuous spacetimes:

Proposition 5.1 A spacetime is

(i) Globally hyperbolic if and only if it is strongly causal and $J^+(x) \cap J^-(y)$ is compact for all $x, y \in V$.

(ii) Causally simple if and only if it is distinguishing and $\partial J^\pm(x) = E^\pm(x)$, for all $x \in V$.

(iii) Causally continuous if it is distinguishing and the set-valued functions $I^+$ and $I^-$ are outer continuous.

There are also a number or results concerning the domain of dependence [53, 6, 64, 128, 153].

Theorem 5.2 Let $\zeta \subset V$ be a closed achronal set of a Lorentzian manifold $(V, g)$,

(i) if strong causality holds on $\overline{J^+(\zeta)}$, then $H^+ \left[ \overline{E^+(\zeta)} \right]$ is non-compact or empty.

(ii) $(\text{int}D(\zeta), g)$ is globally hyperbolic.

The first of these results is crucial in the proof of the Hawking-Penrose singularity theorem [66, 64, 153]. The second states in particular that if an acausal set without edge $\Sigma$ fails to be a Cauchy hypersurface, still its total domain of dependence is globally hyperbolic with $\Sigma$ as Cauchy hypersurface. This is also of paramount importance for the proof of the singularity theorems, due to the following property of globally hyperbolic domains. In a globally hyperbolic domain of a spacetime, the sets $C(a, b)$ are compact with respect to the $C^0$ topology introduced in definition 2.6. This implies that the length functional on curves attains its maximum on $C(a, b)$ in globally hyperbolic domains. This argument together with the fact that, under certain assumptions, the longest causal curves between points are timelike geodesics with no conjugate points allows one to prove the incompleteness of causal geodesics [128, 66, 64, 6, 172, 87, 153], for one can prove under physically reasonable assumptions
that all inextensible causal curves must have conjugate points, and this contradicts the global hyperbolicity unless the curves are extensible. In this last case the reasoning is a little bit more involved and uses essentially point (i) in theorem 5.2, see [66, 64] for details and section 6 (p. 707) in [153] for an intuitive description of the proofs.

In [158] Sorkin and Woolgar put forward a generalization of the ordinary causal relation \(<\) with the aim of extending the compactness of \(C(a, b)\) on globally hyperbolic Lorentzian manifolds to the case in which the metric tensor is only \(C^0\). This new relation is called \(K^+\) and it is defined as the smallest relation containing \(I^+\) that is transitive and topologically closed as a subset of \(V \times V\) where \((V, g)\) is the Lorentzian manifold. Using this relation the authors generalize concepts such as causal curve and global hyperbolicity.

5.2. Splitting theorems for globally hyperbolic spacetimes

Globally hyperbolic spacetimes being the best causally behaved Lorentzian manifolds, as well as the arena on which singularity theorems were founded upon, have received a great deal of attention over the years. Moreover, global hyperbolicity is often assumed as a physically realistic restriction, because physicists expect that a sensible theory must be capable of defining well-posed initial value problems, or Cauchy problems, for the physical fields and this is only the case in globally hyperbolic spacetimes. Thus, there are many results concerning these class of spacetimes, which are of a very special type topologically speaking.

In this section we deal with these topological constraints on globally hyperbolic Lorentzian manifolds. These are usually known as “splitting theorems”, for the results prove that the manifold can be foliated by spacelike hypersurfaces which are, all of them, \(C^0\) Cauchy hypersurfaces. The pioneering classical result is due to Geroch [53].

**Theorem 5.3** Any globally hyperbolic n-dimensional Lorentzian manifold \((V, g)\) is homeomorphic to the topological product \(\mathbb{R} \times S\) where \(S\) is a \((n - 1)\)-dimensional topological manifold. The image of \(S\) under the homeomorphism is a Cauchy hypersurface on \((V, g)\).

Geroch’s splitting theorem is the first of a series attempting to translate the Cheeger-Gromoll splitting theorem in proper Riemannian geometry [27] to the Lorentzian case. This theorem states: “Let \(M\) be a complete and connected proper Riemannian manifold with non-negative Ricci curvature. If \(M\) contains a complete geodesic realizing the distance between any two of its points (a “line”) then \(M\) splits isometrically as \(M' \times \mathbb{R}\).” Partial results towards this direction were obtained in [36, 7, 43] later generalized in the next result proven by Galloway [44].

**Theorem 5.4** Let \((V, g)\) be a connected globally hyperbolic n-dimensional spacetime whose Ricci tensor \(\text{Ric}\) satisfies the condition \(\text{Ric}(\vec{X}, \vec{X}) \geq 0\) for all timelike vectors \(\vec{X}\). If \((V, g)\) contains a complete timelike geodesic \(\gamma\) which is maximal between any two of its points, then it is isometric to \((\mathbb{R} \times S, dt^2 \oplus (-g_1))\) where \((S, g_1)\) is an \((n - 1)\)-dimensional complete proper Riemannian manifold and \(\gamma\) is represented by the factor \((\mathbb{R}, dt^2)\).

Traditionally, these types of manifolds were called a “flat extension” of \((S, g_1)\), see e.g. [143], so the conclusion in the theorem can be re-formulated by saying that \((M, g)\) is a timelike flat extension of an \((n - 1)\)-dimensional complete proper Riemannian manifold. Another simpler way of stating the same is that there exists a “global
Gaussian time coordinate”, see [112, 128, 153] for the use of Gaussian coordinates and its relation with maximal geodesics. Recently, theorem 5.4 has been employed in a relatively simple proof of the positive mass theorem [28].

The previous generalizations of the Cheeger-Gromoll splitting theorem require, as the original theorem itself, the existence of a “line”—an inextensible maximal/minimal geodesic. The conclusion is thus stronger than in theorem 5.3. However, the original Geroch’s result can be substantially improved to obtain an orthogonal splitting of the type appearing in theorem 5.4. This has been achieved in recent work by Bernal and Sánchez [10, 11], who first of all were able to prove that the homeomorphism in Geroch’s theorem can be

smoothed out to be a diffeomorphism, and the Cauchy hypersurfaces are also smooth.

**Theorem 5.5** Any globally hyperbolic spacetime admits a smooth spacelike Cauchy hypersurface \( S_0 \) and it is diffeomorphic to \( \mathbb{R} \times S_0 \).

More importantly, these authors [11] have managed to prove the following fundamental result, probably the most general and powerful splitting theorem available so far.

**Theorem 5.6** Any globally hyperbolic Lorentzian manifold \((V, g)\) is isometric to the smooth product spacetime with base manifold \( \mathbb{R} \times S \) and Lorentzian metric

\[
g = \beta \, dt \otimes dt - \bar{g}
\]

where \( S \) is a smooth spacelike Cauchy hypersurface, \( T : \mathbb{R} \times S \to \mathbb{R} \) is the natural projection, \( \beta \) is a smooth positive function on \( V \), and \( \bar{g} \) is a rank-2 degenerate symmetric tensor field on \( V \), such that the following properties are fulfilled

(i) \( dt \) is timelike and future-directed everywhere (therefore \( T \) is a differentiable time function).

(ii) All hypersurfaces \( S_T : \{ T = \text{const.} \} \) are mutually diffeomorphic spacelike Cauchy hypersurfaces—with, say, \( S_0 = S \).

(iii) The radical of \( \bar{g} \) is one-dimensional everywhere on \( V \) and given by \( \text{Span}\{ \partial_T \} \).

Clearly this theorem supersedes theorems 5.3, 5.4 and 5.5 and settles a long history of guesses and hunches about a possible generalization of Geroch’s result, the existence of differentiable time functions, and the smoothness of the Cauchy hypersurfaces. For an account of this history, consult [10, 11] and references therein.

### 5.3. Topologies on causal spaces and Lorentzian manifolds

We now present a brief summary of the topologies which can be defined on a causal space starting with the results presented in [89]. These results hold for general causal spaces and so they carry over to causal Lorentzian manifolds. The Alexandrov topology \( T^\ast \) already presented in definition 3.4 is the most natural topology for a causal space but there are other topologies which can be defined on a causal space or a Lorentzian manifold. Kronheimer and Penrose defined the topology \( T^\ast \) as the smallest topology in which each set \( J^\ast(x) \) is closed. If \( X \) is a manifold then its manifold topology is denoted by \( T^\ast \).

**Proposition 5.2** If \((X, T^\ast)\) is Hausdorff and the causal space \( X \) is full then \((X, < <)\) is distinguishing.
Recall that any Lorentzian manifold is, in particular, a full causal space, and given that for all \( x, y \in X \), \( I^+(x) \cap I^-(y) \) are always open in \( \mathcal{T}^\text{man} \), in general \( \mathcal{T}^* \) is smaller than \( \mathcal{T}^\text{man} \). The previous proposition states that \( \mathcal{T}^* \) is strictly smaller than \( \mathcal{T}^\text{man} \) in non-distinguishing spacetimes. As a matter of fact, \( \mathcal{T}^* \) and \( \mathcal{T}^\text{man} \) are equivalent topologies if and only if the spacetime is strongly causal [8 9, 128]. Other equivalent statements are gathered next [89].

**Theorem 5.7** The following statements are equivalent.

(i) \( \mathcal{T}^* = \mathcal{T}^\text{man} \).
(ii) The topological space \((X, \mathcal{T}^*)\) is Hausdorff.
(iii) If \( \forall x \in I^- (a) \) and \( \forall y \in I^+ (b) \) we have \( x < y \), then \( b \neq a \) unless \( a = b \).
(iv) If \( b \to a \), and \( \forall x \in I^- (a) \) and \( \forall y \in I^+ (b) \) we have that \( x \ll y \), then \( a = b \).
(v) If \( X \) is a Lorentzian manifold then it is strongly causal.

Most of the topological properties of the chronological sets discussed in section 2 are carried over to full causal spaces using the Alexandrov topology. Here, we use \( A^\text{int}^* \), \( A^{\text{cl}}^* \) and \( A^\text{bdy}^* \) to mean the interior, closure, and boundary with respect to the Alexandrov topology, respectively.

**Theorem 5.8** If \( X \) is a full causal space then [89]

(i) \( x \in I^-(x)^{\text{cl}} \cap I^+(x)^{\text{cl}} \).
(ii) \( I^+(x)^{\text{cl}} = \{ y \in X : I^+(y) \subset I^+(x) \} \) and dually for the past.
(iii) For any subset \( A \subset X \), \( J^+(A) \subset I^+(A)^{\text{cl}} \) and \( I^+(A) = J^+(A)^{\text{int}} \).
(iv) The following statements are equivalent

- \( X \) is a future-reflecting \( \mathcal{B} \)-space
- \( J^+(x) = I^+(x)^{\text{cl}} \) for all \( x \in X \)
- If \( A \) is compact with respect to \( \mathcal{T}^+ \) then \( J^+(A) = I^+(A)^{\text{cl}} \) and \( J^+(A)^{\text{bdy}} = E^+(A) \)
- If \( A \) is compact with respect to \( \mathcal{T}^+ \) then \( J^+(A)^{\text{bdy}} = E^+(A) \)

(v) \( \mathcal{T}^+ \) is smaller than \( \mathcal{T}^* \) if and only if \( X \) is a past-reflecting \( \mathcal{B} \)-space.

Other topologies on Lorentzian manifolds can sometimes provide a deeper insight into certain global causal properties of spacetimes. Of particular importance in this sense is the paper by Hawking, King and McCarthy [65] where a new topology for Lorentzian manifolds was studied. This topology, called the path topology and denoted by \( \mathcal{P} \), is the finest topology such that the induced topology on every timelike curve agrees with the topology induced from the manifold. Intuitively we can think of \( \mathcal{P} \)-homeomorphisms as transformations mapping bijectively timelike curves to timelike curves. In [65] the following result was proved:

**Proposition 5.3** For strongly causal spacetimes \( \mathcal{P} \)-homeomorphisms are homeomorphisms of the manifold mapping null geodesics into null geodesics.

This is an intermediate result useful to prove the following theorem 5.10. To that end a result of Hawking’s is invoked [65]:

**Theorem 5.9 (Hawking’s theorem)** Any homeomorphism of a spacetime which takes null geodesics into null geodesics is a \( C^\infty \) diffeomorphism.

From proposition 5.3 and theorem 5.9 the next theorem easily follows.
Theorem 5.10 In strongly causal spacetimes, $\mathcal{P}$-homeomorphisms are smooth conformal diffeomorphisms.

This result can be generalized in two directions. The first generalization is theorem 4.1 in which no causality conditions upon the spacetime are required and the result holds for bijective mappings with no further topological properties. The second generalization is the result of Huang’s paper [76] and it only takes into account null geodesics in its formulation.

Theorem 5.11 Let $(V, g)$ be a strongly causal space-time of dimension greater than three and let $f : V \to V$ be a bijection such that both $f$ and $f^{-1}$ take null geodesics into null geodesics. Then $f$ is a homeomorphism and, by Hawking’s theorem 5.9, a smooth conformal transformation.

5.4. Topology on $\text{Lor}(V)$.

Many times it is interesting to introduce topologies in the set $\text{Lor}(V)$ of Lorentzian metrics over the differentiable manifold $M$ in order to have a definition of “metric closeness”, which is of relevance for concepts such as causal stability and related conditions. The most studied topologies are the fine $C^r$ topologies whose definition taken from [6] we reproduce here. Other related results can be found in [12].

Definition 5.1 ($C^r$ topologies on $\text{Lor}(V)$) Let $g_1, g_2 \in \text{Lor}(V)$ and define a fixed locally finite covering of $V$ by coordinate neighbourhoods whose closure lies in a coordinate chart. Set a continuous function $\delta : V \to (0, \infty)$. Then we say that $g_1$ and $g_2$ are $\delta$-close in the $C^r$ topology, written $|g_1 - g_2| < \delta$, if all the corresponding components of $g_1$ and $g_2$ and of their $j^{th}$ derivatives $D^j g_1$, $D^j g_2$ ($0 \leq j \leq r$) up to order $r$ satisfy $|D^j g_1|_p - |D^j g_2|_p| < \delta(p)$, $\forall p \in V$.

The sets

$$\{g_1 \in \text{Lor}(V) : |g_1 - g_2| < \delta\}$$

form a basis of the fine $C^r$ topology of $\text{Lor}(V)$ which can be shown to be independent of the chosen covering. An interpretation of these topologies is: metrics close in the fine $C^0$ topology have light cones which are close; if the metrics are close in the fine $C^1$ topology then their geodesics are close; if the metrics are close in the fine $C^2$ topology, their curvature tensors are close; and so on. In all these cases, closeness of the light cones, of geodesics and of curvature tensors are defined in an appropriate intuitive sense. A simple application of these ideas is given next.

Proposition 5.4 The spacetime $(V, g)$ is causally stable if and only if there is a fine $C^0$ neighbourhood $U(g)$ of the metric $g$ such that for any $g_1 \in U(g)$ the spacetime $(V, g_1)$ is causal.

This result tells us that the stable causality condition is nothing but the stability against fine $C^0$ perturbations of causal spacetimes. Actually fine $C^r$ topologies are employed to study the stability of certain properties of spacetimes such as the existence of complete and incomplete curves or the causality conditions (cf. chapter 7 of [6]).

6. Causal boundaries

In this section we discuss one of the most studied issues in causality theory with important applications in related fields of General Relativity such as singularity theory
or asymptotic properties of fields and gravitation, and more recently even in quantum aspects of gravity (through string theory, supergravity, or conformal field theories) via Maldacena’s Conjecture [101, 75]. This is certainly a very remarkable example of the profitability of studying causal boundaries and causal completions: Maldacena’s Conjecture requires the very concept of causal boundary (at infinity) in its rigorous formulation.

On mathematical grounds it has always been fruitful to attach a boundary to a topological set $X$ in order to make it “closed” or “complete”. If $X$ is a metric space then there is a canonical way to accomplish this by constructing its Cauchy completion $\overline{X}$. Hence the boundary $\partial X$ of $X$ is defined by $\partial X \equiv \overline{X} - X$. An important example of this is a proper Riemannian manifold which as is well known can be transformed into a metric space with the standard notion of geodesic distance so the issue of attaching a boundary to a proper Riemannian manifold can be addressed with no further complications. Quite often, however, there are several inequivalent but sensible ways of attaching a boundary to a given set, be it because it has no metric-space structure, or because we want to make abstraction of this. This will certainly be the case for spacetimes, as they are not metric spaces, so that many boundaries are feasible (with good properties) and we will have to decide about which particular properties and objects we want to describe or study by means of the completion and the boundary. The driving idea in almost all of the boundaries conceived so far is to obtain completions which are causal sets or Lorentzian manifolds but in such a way that the completion is consistent with the original causal structure. This means that causal relations between points of the completed manifold must agree with those of the original spacetime.

The concept of such a “causal boundary” for spacetimes was first introduced by Penrose more than forty years ago and it turned out to be one of the most prolific ideas in General Relativity as we hope to make plain in this section. In essence, Penrose’s idea is to embed the spacetime under study into another bigger Lorentzian manifold conformally, so that the causal properties are trivially kept, and obtain properties of the initial spacetime by examining its boundary on the encompassing spacetime. After the initial success of this construction in simple but physically relevant examples, the construction was generalized in several directions by a number of authors, sometimes keeping the idea of the embedding, sometimes not, but usually with the goal of trying to remove the use of conformal transformations, as they are almost impossible to find in generic situations. The most important generalizations are reviewed in this section. As a pre-conclusion, unfortunately we must say that in general either they are too theoretical, and cumbersome to be tested in explicit examples, or they present some undesirable features. However, there have been some recent advances and developments [149, 58, 59, 47, 106] which might help in the final achievement of a definite and generally accepted definition of causal boundary. These are treated in subsections 6.3.4, 6.3.6, 6.4.4 and 6.4.5.

6.1. Penrose conformal boundary

Penrose first introduced his idea of conformal boundary in [122] and further developed it in [123, 124]. His aim when defining the conformal boundary was to study from another point of view questions related with the behaviour and radiative properties of spacetimes which are asymptotically flat. In fact the very definition of asymptotic flatness can be formulated naturally in the framework of the conformal boundary.
Let \( \tilde{g} \) be the metric tensor of a \( n \)-dimensional spacetime \( \tilde{M} \) (in purity, Penrose had General Relativity in mind and considered only the case \( n = 4 \), but we will leave the dimension of the spacetime free) and suppose that we can set a conformal correspondence of this spacetime with a finite region \( M \) of another \( n \)-dimensional Lorentzian manifold with metric \( g \) (this is sometimes called the “unphysical spacetime”). This means that at every point of such region we have the relation

\[
g = \Omega^2 \tilde{g}.
\]

If \( M \) was judiciously chosen (say such that \( \tilde{M} \) has compact closure in the unphysical spacetime) then the whole of “infinity” in the physical spacetime can be brought to finite values of the coordinates of the unphysical spacetime. Some other properties of the original spacetime which nevertheless are not part of it, such as its singularities, can also be read off from their guessed places on \( M \). Thus, “infinity” and “singularities” are typically parts of the boundary \( \partial M \) of \( \tilde{M} \) in the unphysical spacetime, and thus we can extract properties of the physical spacetime, its global structure and its singularities, by just analyzing the properties of the boundary \( \partial \tilde{M} \). The set \( \partial \tilde{M} \) is called the conformal boundary of \( \tilde{M} \). In some situations, such as asymptotically flat spacetimes, the part of the conformal boundary describing infinity of \( \tilde{M} \) can be defined as the set of \( x \in M \) where \( \Omega = 0 \) (and if one wishes to talk about “null infinity”, see below, the condition \( d\Omega \neq 0 \) is added). The part of the conformal boundary representing infinity is denoted by \( \mathcal{I} \) and called sometimes conformal infinity. Therefore, in these cases a substantial part of the boundary are \((n-1)\)-dimensional hypersurfaces of the unphysical spacetime \( M \). However, in general nothing can be said about \( \mathcal{I} \) in this sense, and it can be disconnected, or discrete, or 1-, 2-dimensional, et cetera, as well as a combination of hypersurfaces and these, and so on.

The procedure just described is called the conformal compactification of \( \tilde{M} \) relative to \( M \). It depends on \( M \) and the chosen conformal factor, but it is generally accepted that a complete boundary (\( \partial \tilde{M} \) has compact closure on \( M \)) is essentially unique [39]. The crucial point here, and for the whole construction, is that the physical spacetime is conformally related to its image on the enlarged manifold, and therefore the causal properties of \( \tilde{M} \) have been kept. Besides, the boundary acquires causal properties itself as a set of \( M \), so that it may be given attributes such as spacelike, timelike, or null, or it can have parts to the future/past of \( \tilde{M} \) (so that is to the future/past of the entire spacetime!), et cetera.

The conformal compactification can be carried out explicitly in the case of flat Minkowski spacetime (see [124] for details). A suitable choice for the unphysical spacetime is the Einstein static universe. Many other choices are possible, but this particular one has the virtue of making \( \partial \tilde{M} \) complete. To see this we write the flat Minkowski metric in spherical coordinates

\[
ds^2 = dt^2 - dr^2 - r^2 d\Omega^2,
\]

where \( d\Omega^2 \) stands for the standard round metric on the sphere \( S^{n-2} \), and perform the coordinate transformation (the angular part is simply omitted for shortness)

\[
t = \frac{1}{2} \left( \tan \left( \frac{\bar{t} + \bar{x}}{2} \right) + \tan \left( \frac{\bar{t} - \bar{x}}{2} \right) \right),
\]

\[
r = \frac{1}{2} \left( \tan \left( \frac{\bar{t} + \bar{x}}{2} \right) - \tan \left( \frac{\bar{t} - \bar{x}}{2} \right) \right),
\]
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with coordinate ranges \(-\pi < \bar{t} + \bar{x} < \pi, -\pi < \bar{t} - \bar{x} < \pi, 0 < \bar{x} < \pi\). This
transformation brings Minkowski's line element into the form

\[
ds^2 = \frac{1}{4 \cos^2 \left(\frac{\bar{t} + \bar{x}}{2}\right) \cos^2 \left(\frac{\bar{t} - \bar{x}}{2}\right)} \left(d\bar{t}^2 - d\bar{x}^2 - \sin^2 \bar{x} \, d\Omega^2\right),
\]

from which we see that it is conformal to a certain region of Einstein static universe (which has the line-element in brackets). This is depicted in the figure below taken from [123] where it is shown the Einstein static cylinder \((n - 2)\) spatial dimensions suppressed, so that each horizontal circle of the cylinder represents a \(S^{n-1}\) sphere) and in red the region conformal to Minkowski spacetime.

In the picture we see that the conformal boundary is split in different regions called by Penrose \(\mathcal{I}^+, \mathcal{I}^-, I^+, I^-\) and \(I^0\) (many times \(i^+, i^-\) and \(i^0\) are used nowadays). \(i^+, i^-\) and \(i^0\) are points whereas \(\mathcal{I}^\pm\) are \((n - 1)\)-dimensional null hypersurfaces of topology \(S^{n-2} \times \mathbb{R}\) (to make clearer these topologies the original Penrose paper [122] presents a different picture but the regions names are still the same). The sets \(\mathcal{I}^+\) (\(\mathcal{I}^-\)) are formed by “endpoints” of the inextensible future-directed (past-directed) radial null geodesics and \(i^+\) (\(i^-\)) are the “endpoints” of inextensible timelike future-directed (past-directed) geodesics. Finally \(i^0\) is the endpoint of all spacelike geodesics, as well as infinity for some spacelike slices. Several issues related with this conformal compactification are discussed in Penrose’s papers [122, 124] being some of them seminal ideas for important lines of research developed in later years, see for a review [39]. One of the first applications was the definition of asymptotic flatness by means of conformal compactifications. The idea is simply that the conformal boundary of any asymptotically flat spacetime must resemble that of Minkowski spacetime just described.

Penrose suggested that a 4-dimensional spacetime \(\tilde{M}\) is asymptotically flat if an unphysical spacetime \(M\) exists such that the conformal infinity \(\mathcal{I}\) can be decomposed in two null three dimensional hypersurfaces \(\mathcal{I}^+\) and \(\mathcal{I}^-\) with the topological properties described in the figure above. As a matter of fact, it can be proven that, if the vacuum Einstein equations hold on a neighbourhood of \(\mathcal{I}\), then \(\mathcal{I}\) is a smooth null hypersurface with two connected components, among other results, see [64, 39, 162]. Later this definition was refined and the concepts of asymptotic simplicity and weakly asymptotic simplicity [124, 64] were introduced in the literature. It is also possible to study the gravitational radiation of asymptotically flat systems (isolated bodies) using these techniques. Indeed, Penrose was able to give explicit expressions
for the gravitational power radiated “at infinity” by an isolated system as an integral over $\mathcal{I}^\pm$ [124, 126] (similarly there are expressions for the incoming radiation using $\mathcal{I}^-$), see for further details [39, 162]. These results were known previously from the work of Bondi, Trautman, Pirani, Sachs and others [144, 14] but the technique of the conformal boundary gave them a fuller, completely coordinate-independent, geometrical significance.

The conformal compactification can be carried out for other spacetimes such as de Sitter, anti-de Sitter or some Robertson-Walker geometries [64]. This showed that singularities (such as the big bang) may be part of the boundary, that the conformal boundary can have new unexpected properties (an example of this is the non-differentiability of the metric at $i^0$), and that one can give definite properties to $\partial M$ as a region of the unphysical spacetime. In general, however, the full conformal compactification is very difficult to achieve, usually impossible. The good news is that, in certain particular but relevant cases, it is possible to perform the conformal compactification of a two-dimensional piece of the metric retaining the important information. This is what happens for instance in spherically symmetric spacetimes where the non-angular part of the metric (2 dimensions) is conformally flat and thus liable to be conformally compactified. In this case we can draw two dimensional pictures, called Penrose diagrams [64, 126], and define $\mathcal{I}^\pm$, $i^\pm$ for the conformal boundary so obtained if they exist. These extremely useful representations of spherically symmetric spacetimes will be further treated in section 6.5.1. Similarly, sometimes one can take a particularly relevant 2-dimensional surface (ergo conformally flat) of a given spacetime and draw its Penrose diagram. This does not give a full insight into the properties of the spacetime, but it certainly enlightens our comprehension of some of its features. A paradigmatic example of this situation is given by the Penrose diagram of Kerr’s spacetime, which is just the diagram of only its axis of symmetry, first found by Carter [24], see [64].

Despite the fact that the conformal compactification is almost a chimera in generic spacetimes, Friedrich [41, 42] has been able to establish a procedure such that it is possible to write down a set of equations (conformal equations) in which $\Omega$ and $g$ are part of the unknowns. In principle this could help to perform the conformal embedding of a spacetime, but even if not so, it allows to talk about (and find some of) the properties of the conformal boundary. Furthermore, the analysis of spacelike infinity, which is one of the more intricate issues in conformal completions, can also be recast in a form were $i^0$ is given internal structure so that its properties become more transparent. For a review about all these matters, see [39].

Penrose's idea of attaching a “boundary” to a spacetime was, and still is!, very attractive and it was pursued by several authors over the years with rather assorted techniques and aims. The question as to why one should be interested in enlarging or generalizing the conformal boundary construction to something else has many answers. First of all, one would like to avoid objects that are foreign to the spacetime under analysis, such as $\Omega$ and the fictitious manifold $M$. Thus, a construction of the boundary using objects of the spacetime exclusively (as is the case of the Cauchy completions for metric spaces) has been systematically sought. On the other hand, the conformal compactification is the simplest, and truly very appealing, way of attaching a boundary to a Lorentzian manifold, so that sometimes the idea has been to look for ways to avoid the technical problems in finding the unphysical spacetime and the conformal factor—which may be very difficult to find in closed form. A third reason why the definition of a boundary can be useful is the analysis of singularities.
Singularities are not part of the spacetime since they are related with diverging quantities, or with incompleteness of curves, or with lack of tangent vectors. Another way to look at this is by saying that a singularity is a set of points “where” the spacetime itself ends, or blows up, so it is sensible to think that a singularity will lie “at the boundary” of the spacetime. And this is certainly the case in the simple examples constructed with Penrose diagrams, and in general in all examples known so far. Thus, a suitable definition of boundary should allow to tell apart which of its parts are singularities, and which are at infinity, among other possibilities, if they arise. If this were achieved, questions otherwise meaningless such as the shape of the singularity or its causal character would make perfect sense.

In the forthcoming subsections we review what we believe are the most relevant attempts in these directions presented so far in the literature giving accounts of the motivations lying behind each construction.

6.2. Geroch, Kronheimer and Penrose construction

One of the most famous attempts towards the construction of a causal boundary was performed by Geroch, Kronheimer and Penrose in [54] where they put forward a scheme to attach a “boundary” to any spacetime fulfilling certain causality restrictions. The method followed involved advanced topological constructions based only on global causal objects present in Lorentzian manifolds. They tried to show that a causal boundary could be associated to certain spacetimes (i) without invoking “external” concepts such as the unphysical spacetime or the conformal factor \( \Omega \), and also (ii) by using only causal concepts with no attention to the existence of a Lorentzian metric. Only distinguishing spacetimes were covered by this method (henceforth called GKP construction or \( c \)-boundary).

Instigating ideas for the GKP boundary were previously published by Seifert in [151]. He proposed a scheme to attach a causal boundary to space-times by basically assigning a future and past endpoint to any inextensible causal curve on a space-time \( V \). The set of such points would be the causal boundary \( \partial V \). One can then introduce an ordering \( J^+ \) on the completed space-time \( \bar{V} \) which is an extension of the usual causal relation \( < \) on the space-time \( V \). These ideas were much improved in [54].

Before entering into the details of the GKP construction we need to define certain concepts dealing with future and past sets. Recall that future and past sets (definition 2.8) are those sets of the manifold \( M \) whose chronological future (past) is contained in the set itself. A future or past set is called indecomposable (IF or IP resp.) if it is not empty and cannot be written as the union of two subsets which are themselves future or past sets. Roughly speaking indecomposable future and past sets can be divided into sets which are of the form \( I^+(p) \) for \( p \in M \) and those sets which cannot be written as the chronological future or past of any point of the manifold. These distinction gives rise to proper indecomposable future and past sets (abbreviated PIF’s and PIP’s) and terminal indecomposable future and past sets (TIF’s and TIP’s) respectively. Let us denote the collection of IP’s by \( \hat{M} \) and the set of all IF’s by \( M \). For distinguishing spacetimes the manifold \( M \) is the simplest example of TIP and TIF, but as we show next terminal indecomposable sets are easily characterised (we only formulate the result for the past case) [54, 64, 88].

**Theorem 6.1** Any IP is of the form \( I^-(\gamma) \) where \( \gamma \) is a future-directed timelike curve. If \( \gamma \) has a future endpoint \( p \in M \) then the IP is the PIP \( I^-(p) \), while if \( \gamma \) is future
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endless then the IP is a TIP.

Therefore timelike future-endless curves give rise to TIP’s according to this theorem. As an example we can take Minkowski spacetime. In this case it is known that TIP’s and TIF’s are respectively the chronological past and future of timelike curves with constant acceleration plus the whole manifold itself (there are no other terminal indecomposable sets in Minkowski spacetime), and these have endpoints at \( \mathcal{I} \) in the conformal boundary. Actually, one can set a clear correspondence between the TIP’s and TIF’s and the different regions of the conformal boundary described in section 6.1: TIP’s and TIF’s represent \( \mathcal{I}^+ \) and \( \mathcal{I}^- \) respectively and the TIP or TIF defined by the manifold itself represents \( i^\pm \). Observe that \( i^0 \) is missing in this picture. As shown in [54], it is also possible to determine the set of IP’s for asymptotically simple spacetimes getting a similar structure to that obtained by means of the conformal compactification. Turning to the general case, theorem 6.1 suggests that we may regard the TIP \( I^- (\gamma) \) as a sort of future ideal end point attached to the curve \( \gamma \). The attribute ideal means that the point does not belong to the manifold \( M \) but rather to a larger set containing \( M \) as a proper subset. We should therefore try to construct a new manifold consisting of the points of \( M \) plus the ideal points and call it the completion \( \hat{M} \) of \( M \). The set \( \hat{M} - M \) would then be the causal boundary and would contain ideal points only. How to construct this larger set, and to endow it with a topology, is the next task.

Clearly there is an obvious correspondence between PIP’s or PIF’s and the points of the manifold in a distinguishing spacetime. This correspondence is set by the injections \( I^+ : M \to \hat{M}, I^- : M \to \hat{M} \) so the set of PIF’s is \( I^- [M] \) and the set of PIP’s is \( I^+[M] \). Neither \( I^+ \) nor \( I^- \) are bijections, as the TIP’s and TIF’s are not included (TIP’s and TIF’s are ideal points) so the sets \( \hat{M} \) and \( \hat{M} \) are “bigger” than \( M \) which means that they can provide us with the sought enlargement of \( M \). Consequently, an obvious starting point to construct the enlarged manifold is the set \( \hat{M} \cup \hat{M} \). Now we try to find a natural injection from \( M \) into \( \hat{M} \cup \hat{M} \), but the problem is that each point \( p \in M \) naturally corresponds to two elements, \( I^+(p) \) and \( I^-(p) \), of \( \hat{M} \cup \hat{M} \) so it is not clear how we can map \( M \) into \( \hat{M} \cup \hat{M} \) in an injective way. To surmount this difficulty, the proposal in [54] was that a set of identifications must be carried out among the elements of \( \hat{M} \cup \hat{M} \). There are some identifications which are obvious (for instance \( I^+(p) \) and \( I^-(p) \) must be identified) but, unfortunately, sometimes identifications among the ideal points are also needed. A simple example of this happens when we try to construct the completion for spacetimes whose conformal boundary \( \mathcal{I} \) contains timelike regions. In this case any ideal point in one of such regions could stem from a TIP as well and a TIF, so they should be identified (see figure).
A further problem which arises in the construction of the enlarged manifold is the definition of a suitable topology and, a posteriori, its differentiable structure. These two problems, the identification rules and the construction of “good” topologies and manifold structures, are the conundrum in the whole GKP construction.

The answer provided in [54] was proven later not to be appropriate for some cases and a number of generalizations were tried to improve this point as we will review in forthcoming sections. Let us nevertheless present a summary of the idea in [54] as it became the background on which the generalizations were founded. As explained above there are obvious identifications, so we define the set $M^\natural$ as $\bar{\mathcal{M}} \cup \hat{\mathcal{M}}$ with the elements $I^+(p)$, $I^-(p)$ identified $\forall p \in M$. For any element $P$ of $\bar{M} \cup \hat{M}$ we write $P^*$ for the corresponding element of $M^\natural$. Assuming now the stronger condition of $M$ being strongly causal in order to ensure that the Alexandrov topology agrees with the manifold topology (theorem 5.7) the “extended Alexandrov topology” is defined on $M^\natural$ as the coarsest topology such that for each $A \in \hat{\mathcal{M}}$, $B \in \bar{\mathcal{M}}$ the four sets $A^\text{int}$, $B^\text{ext}$, $B^\text{int}$, $A^\text{ext}$ are open sets where

$$A^\text{int} = \{P^* : P \in \bar{M} \text{ and } P \cap A \neq \emptyset\},$$

$$A^\text{ext} = \{P^* : P \in \hat{M} \text{ and } \forall S \subset M \ P = I^-(S) \Rightarrow I^+(S) \not\subset A\}.$$ 

The sets $B^\text{int}$ and $B^\text{ext}$ have similar definitions with the roles of past and future interchanged. The set $M^\natural$ becomes a topological space which in general is not Hausdorff. To avoid this awkward feature, an equivalence relation $R_H$ is defined on $M^\natural$ by the intersection of all the equivalence relations $R \subset M^\natural \times M^\natural$ such that $M^\natural/R$ is Hausdorff. This new Hausdorff topological space is then taken as the desired enlarged manifold $\bar{M}$. If the spacetime $M$ is strongly causal, it was claimed in [54] that the identifications performed when passing from $M^\natural$ to $\bar{M}$ will never occur between elements of $M^\natural$ representing original points of $M$. Moreover there exists a natural, dense, topological embedding of $M$ into $\bar{M}$. Stronger related statements will be presented later in proposition 6.1.

The GKP c-boundary is very appealing for only simple causal properties are used in its definition and it recovers the structure found with conformal techniques in relevant cases, such as asymptotically simple spacetimes. However, a number of issues are left open in this construction aside from the construction of the rule $R_H$ already commented. First of all the different regions of the causal boundary were not studied for general cases (only spacetimes asymptotically simple are treated) and second the construction of causal relations between points at the boundary was not investigated in [54]. These features were the subject of subsequent papers published by other authors who took the GKP scheme as their starting point. Some of these are reviewed next.
6.3. Developments of the Geroch, Kronheimer and Penrose construction

In this section we review the different attempts carried out to fill in the unfinished steps of the GKP construction. We must say right from the start that each intended improvement was sooner or later found to bear undesirable features rendering all of them, as well as the original GKP, unsuitable to be regarded as boundaries for spacetimes in general situations. Only the main ideas of each construction are given as the details and examples usually result in rather technical statements.

6.3.1. The Budic and Sachs construction. The paper by Budic and Sachs [21] develops the GKP construction for causally continuous spacetimes. In this case the authors are able to extend the causal structure and the topology of the spacetime $M$ to the enlarged spacetime $\hat{M}$ by means of the definition of a suitable relation $R_H$.

Before explaining how this relation is constructed the following definition is needed [67].

**Definition 6.1 (Common past and future)** For any set $U \subset M$ the chronological common past and the chronological common future are respectively:

$$\downarrow U = I^- \{ x \in M : x \ll p, \forall p \in U \}, \quad \uparrow U = I^+ \{ x \in M : x \gg p, \forall p \in U \}.$$  

Clearly $\downarrow I^-(p) \subseteq I^-(p)$ and $\uparrow I^+(p) \subseteq I^+(p)$. We denote by $\mathcal{F}$ and $\mathcal{P}$ the collection of future and past sets respectively. A **hull pair** on $\mathcal{P} \times \mathcal{F}$ is any element $(P, F)$ such that $P = \downarrow F$ and $F = \uparrow P$. A very important result proved in [21] is that $(I^-(p), I^+(p))$ is a hull pair for any $p \in M$ if $M$ is causally continuous (this is too strong a restriction; it is known that $(I^-(p), I^+(p))$ is a hull pair if and only if $M$ is just reflecting [67]). If we recall that the sets $I^+(p)$ and $I^-(p)$ must be identified in the GKP construction, and given that this identification takes place naturally in causally continuous spacetimes through the relations $\downarrow$ and $\uparrow$, it seems appropriate to introduce a binary relation on $\hat{M} \times \check{M}$ defined by the elements $(P, F)$ which form a hull pair. This is an equivalence relation “$\sim$” for causally continuous spacetimes and it is used to construct the quotient $\hat{M} \cup \check{M} / \sim$ which play the role of the enlarged manifold $\check{M}$. In order to avoid working with a quotient set of $\hat{M} \cup \check{M}$ the authors define the set

$$M = \check{M} \cup \hat{M} - \downarrow (\check{M} \cup \hat{L}),$$

which is called the **causal completion** of $M$. Here set $\hat{L}$ is the **future hull lattice** (the past hull lattice is defined dually) and its definition reads

$$\hat{L} \equiv \{ X \in \mathcal{F} : X = \uparrow U, \quad \text{U open} \}.$$  

The causal completion can be endowed with a causal structure by means of two binary relations $<$ and $\ll$ defined on the set $\mathcal{P} \cup \mathcal{F}$ making it a causal space in the sense of Kronheimer and Penrose, definition 3.1. It is also possible to define a topology $\mathcal{T}$ on $\mathcal{P} \cup \mathcal{F}$ (also called extended Alexandrov topology) as follows: a set $\mathcal{C} \subset \mathcal{P} \cup \mathcal{F}$ is called an enlargement of $M$ if it contains either $I^+(x)$ or $I^-(x)$ for all $x \in M$. Then the extended Alexandrov topology $T$ on the enlargement $\mathcal{C}$ is the smallest topology such that for all $C \in \mathcal{C}$ the subsets $I^+\{C\}, I^-\{C\}, C - J^-\{C\}, C - J^+\{C\}$ are open. Since $M$ is an enlargement of $M$, the causal completion inherits the extended Alexandrov topology. Therefore $M$ is both a causal and a topological space. The
causal boundary is now defined as the set \( \partial M = M - \bar{I}M \) where \( \bar{I} : M \to M \) is the mapping

\[
\bar{I} : x \mapsto \bar{I}(x) = I^-(x), \quad \forall x \in M.
\]

It can be shown that for causally continuous spacetimes \( \bar{I} \) is a dense embedding of \( M \) into \( \hat{M} \), see also proposition 6.1.

Some other general properties of \( \hat{M} \) are studied [21]. For example an interesting result is that \( \hat{M} \) is globally hyperbolic if and only if for every \( \bar{x} \in \hat{M} \), either \( \bar{I}^-(\bar{x}) \) or \( \bar{I}^+(\bar{x}) \) is the empty set (here \( \bar{I}^\pm \) are calculated with respect to the causal relations introduced in \( M \)).

6.3.2. Rácz’s generalizations In [132] Rácz gave a modification of the topology defined by Geroch, Kronheimer and Penrose for the set \( M \cup \bar{M} \) as the coarsest topology \( A^* \) in which for each PIF \( F \) and PIP \( P \) the four sets \( F^{\text{int}}, F^{\text{ext}}, P^{\text{int}} \) and \( P^{\text{ext}} \) are open. Here

\[
F^{\text{int}} = \{ A \in \hat{M} \cup \bar{M} : A \in \hat{M} \text{ and } A \cap F \neq \emptyset \text{ or } A \in \bar{M} \text{ and } \forall S \subset M : I^+(S) = A \Rightarrow I^-(S) \cap F \neq \emptyset \},
\]

\[
F^{\text{ext}} = \{ A \in \hat{M} \cup \bar{M} : A \in \hat{M} \text{ and } A \nsubseteq F \text{ or } A \subset \bar{M} \text{ and } \forall S \subset M : A = I^-(S) \Rightarrow I^+(S) \nsubseteq F \}.
\]

The sets \( P^{\text{int}} \) and \( P^{\text{ext}} \) have a similar definition. Here the topology \( A^* \) is set up directly on \( \hat{M} \cup \bar{M} \) without introducing the intermediate set \( M^\sharp \). Next an identification rule \( R \) is defined on \( \hat{M} \cup \bar{M} \) yielding the completion \( \hat{M} \) and the topology \( \hat{A} \) as the quotients \( \hat{M} \cup \bar{M} / R \) and \( \hat{A}^* / R \), respectively. The minimal requirement which \( R \) must comply with is that the sets \( I^+(p) \) and \( I^-(p) \) be identified. For any such relation \( R \) the first point in proposition 6.1 holds, so that next goal is finding a relation \( R \) such that the topology \( \hat{A} \) is Hausdorff. To achieve this, a technical causal condition on \( M \) is imposed: for each \( p \in M \) there exist \( a, b \in M \) such that \( a \in I^-(p), b \in I^+(p) \) and there is no set \( S \) satisfying both \( I^+(S) \subset I^+(a) \) and \( I^-(S) \subset I^-(b) \) for which \( I^+(S) \) is a TIF or \( I^-(S) \) is a TIP.

Rácz also developed the GKP construction for the case of causally stable spacetimes in [133], where an explicit identification rule and topology were constructed. As an example, in [132] the singular portion of the causal boundary of Taub plane-symmetric static vacuum spacetime was shown to be a one-dimensional set under this construction whereas in the original GKP scheme this turned out to be a point. Further details about this and other drawbacks of the c-boundary construction will be discussed in §6.3.5.

6.3.3. The Szabados construction for strongly causal spacetimes. Almost simultaneously, Szabados tried to address similar questions for strongly causal spacetimes in a couple of important papers [163, 164]. The first of them, which is the work discussed in this section, is a truly penetrating study on general identification rules on the set \( M^\sharp \) defined by Geroch, Kronheimer and Penrose, where a new version of the completion \( \hat{M} \) together with a causal structure for \( \hat{M} \) was presented.

Szabados calls future preboundary points to the TIF’s (past preboundary points if they are TIF’s) and they are collected in sets denoted by \( \partial^+ \), \( \partial^- \) respectively. The remaining elements of \( M^\sharp \) are the identified pairs \( (I^+(p), I^-(p)) \), \( p \in M \) which are regarded as the image of the injection \( i : M \to M^\sharp \), \( i : p \mapsto i(p) \). Szabados pointed out that the elements of \( \partial^\pm \) are sometimes termed as the ”endpoints” of inextensible
causal curves in $M$, but this statement needs a clarification because the concept of endpoint is purely topological (see definition 2.5) so the assertion only makes sense if a topology $\mathcal{T}$ has been defined on $M$. More importantly the causal endpoints must be consistent with topological endpoints which means for instance that the TIP $I^-(\gamma)$ has to be the future endpoint in $M$ of the future inextensible causal curve $\gamma$. These problems were neatly resolved in [163] constructing an appropriate topology $\mathcal{T}$ on $M$ from the Alexandrov topology $\mathcal{I}$ on $M$—which coincides with the manifold topology for strongly causal spacetimes cf. theorem 5.7.

The needed topology $\mathcal{T}$ is constructed as the quotient topology of $\mathcal{T}^2$ by a certain equivalence relation $\mathcal{R}$ on $M^2$. The topology $\mathcal{T}^2$ and the equivalence relation $\mathcal{R}$ are determined at a later stage. A basic consistency requirement is to impose that $i : (M, \mathcal{T}) \rightarrow (M^2, \mathcal{T}^2)$ be an open dense embedding and that the elements of $M^2$ given by $P = I^-(\gamma), F = I^+(\gamma)$ be respective past and future endpoints of $i \circ \gamma$ in the topology $\mathcal{T}^2$. Under these assumptions an important conclusion can be drawn.

**Proposition 6.1** Let $\mathcal{R}$ be any equivalence relation on $M^2$ which is trivial on the subset $i(M)$ and define the canonical projection $\pi : M^2 \rightarrow M^2/\mathcal{R}$. If a topology $\mathcal{T}^2$ on $M^2$ with the properties explained above is set then

(i) the mapping $\pi \circ i : (M, \mathcal{T}) \rightarrow (M^2/\mathcal{R}, \mathcal{T}^2/\mathcal{R})$ is an open dense embedding.

(ii) $\pi(P)$ and $\pi(F)$ are future and past endpoints of the curve $\pi \circ i \circ \gamma$ where as above $P = I^-(\gamma)$ and $F = I^+(\gamma)$.

From this result we conclude that any equivalence relation acting as the identity on $i(M)$ renders the preboundary points as topological endpoints of inextensible causal curves. This is a good result but not enough for our purposes because there are relations $\mathcal{R}$ such that the endpoints are not unique for a single inextensible causal curve. This can be restated as saying that the quotient topology $\mathcal{T}^2/\mathcal{R}$ is not Hausdorff so in order to get rid of this feature one would have to search for another equivalence relation $\mathcal{R}$ making the quotient topology Hausdorff. In fact, the prescription originally given by Geroch, Kronheimer and Penrose is just “take the minimal $\mathcal{R}$ such that $\mathcal{T}^2/\mathcal{R}$ is Hausdorff”, but as Szabados shows in [163] there are explicit examples in which no such equivalence relation exists.

Szabados suggested then to consider quotient topologies with milder restrictions. Recall that two points $x_1, x_2$ of a topological space are $T_1$-separated if there is an open neighbourhood of $x_1$ which does not contain $x_2$ and there is an open neighbourhood of $x_2$ not containing $x_1$. If this property holds for all pairs $x_1, x_2$ then the set is said to be $T_1$-separated. $T_2$ separation is the standard Hausdorff separation property. The next result was proven in [163].

**Proposition 6.2** Let $\mathcal{T}^2$ be the GKP topology of the set $M^2$ and $\mathcal{R}$ any equivalence relation defined on $M^2$. Then, each inner point and each boundary point of the topological space $(M^2/\mathcal{R}, \mathcal{T}^2/\mathcal{R})$ are $T_1$-separated.

After these important considerations, an explicit identification rule $\mathcal{R}$ was put forward. This rule relies on the concept of naked TIP’s and TIF’s introduced in [129]. A TIP $P$ is said to be naked if for some point $p \in M$ we have $P \subset I^-(p)$. The naked counterpart of $P$ is a TIF $F$ such that for every $q \in F$ the property $P \subset I^-(q)$ holds. If there is no naked counterpart of $P$ containing $F$, then $F$ is a maximal naked counterpart of $P$. In general, there are naked TIPs with several maximal naked counterparts. However, [163]
Proposition 6.3 Any naked \( P \in \partial^\pm \) possesses a maximal naked counterpart. Moreover, \( \uparrow P = \cup_\alpha F_\alpha \) where \( F_\alpha \) are the maximal naked counterparts of \( P \).

Of course there is a dual formulation in terms of TIFs and their maximal naked counterparts. Naked TIPs and TIFs play the role of points lying in “timelike” parts of \( \partial^\pm \) which are in essence the points requiring identification. Therefore, a binary relation “\( \sim \)” can be set up as follows: \( P \sim F \) if \( P \) and \( F \) are maximal naked counterparts of each other. If \( F \) is a naked counterpart of the naked TIP \( P \) then we can always find sets \( F_0 \) and \( P_0 \) such that \( F \subseteq F_0 \) and \( P \subseteq P_0 \) where \( F_0 \) and \( P_0 \) are maximal naked counterparts of each other, so \( P_0 \sim F_0 \). The relation “\( \sim \)” can then be extended to the whole \( M^3 \) in the following way: \( X \sim X, \forall X \in M^3 \) and if \( B, B' \in \partial^+ \cup \partial^- \), \( B \neq B' \) we say that \( B \sim B' \) if for a finite number of preboundary points \( B_1, \ldots, B_r \), the chain of relations \( B \sim B_1 \sim \ldots \sim B_r \sim B' \) holds. Interestingly, if \( (P, F) \) is a hull pair then \( P \sim F \) which means that Szabados identification is a generalization of the Budic and Sachs identification.

The space \( \bar{M} \) can also be endowed with a chronology relation \( \ll \) and from this with a causal relation by means of definition 3.8, see [163] for details. However, this may not agree with the original manifold chronology, see [106]. Once the necessary identifications between TIPs and TIFs has been carried out, still different TIPs (or TIFs) may need identification. This is taken care of in the second paper [164]. As yet another final positive result, an asymptotic causality condition is then introduced, which provides the uniqueness of endpoints of causal curves in the completed manifold \( \bar{M} \).

6.3.4. The Marolf and Ross recipe. Recently Marolf and Ross [106] have proposed a new identification rule partly based on Szabados work. They start from the observation that sets which are maximal naked counterparts of each other can be related in an alternative way as follows: the relation \( R_{pf} \subseteq \bar{M} \cup \bar{M} \) is the set of pairs \((P, F)\) such that \( F \) is a maximal subset of \( \uparrow P \) and \( P \) a maximal subset of \( \downarrow F \).

The Szabados relation “\( \sim \)” can be obtained as the smallest equivalence relation containing \( R_{pf} \). Observe that “\( R_{pf} \)” is defined directly on the set \( \bar{M} \cup \bar{M} \) whereas “\( \sim \)” is in principle defined only on \( M^3 \). Marolf and Ross argue that causal completions need not arise from equivalence relations and quotient spaces because sometimes these relations enforce the identification of ideal points which should not be identified on causal grounds. To avoid this, they suggest characterizing the points of the causal completion \( \bar{M} \) by two sets representing their future and their past and, as a key point, in the case of ideal points one of these sets may be empty. Explicitly: the causal completion \( \bar{M} \) of a spacetime \( M \) is the set of pairs \((P, P^*)\) such that

(i) \( (P, P^*) \in R_{pf} \subseteq \bar{M} \cup \bar{M} \) or
(ii) \( P = \emptyset \) and \( P^* \) is not an element of any pair in \( R_{pf} \) or
(iii) \( P^* = \emptyset \) and \( P \) is not an element of any pair in \( R_{pf} \).

The elements of \( \bar{M} \) are denoted by \( \bar{P} \equiv (P, P^*) \). As discussed in previous cases, a first important property is that the sets \( I^-(p) \) always appear in the pair \((I^-(p), I^+(p))\) in \( \bar{M} \). Therefore there exists a natural embedding \( \Phi: M \to \bar{M}, \Phi(p) = (I^-(p), I^+(p)) \), which shows the similarity of this construction to the Szabados scheme.

As shown in [106], there is a natural chronology in the completion \( \bar{M} \)

Theorem 6.2 The binary relation \( \ll \) defined by \( \bar{P} \ll \bar{Q} \iff P^* \cap Q \neq \emptyset \) is a chronology on \( \bar{M} \).
The definition of a causal relation $< \,$ seems more conflictive and the authors discuss different approaches, none giving a definite satisfactory answer adapted to all the examples considered in [106]. The construction of a topology upon $\bar{M}$ is also rather technical and, in fact, more than a single topology is considered. The main result regarding these topologies is the fact that $M$ is homeomorphic to its image on $\bar{M}$ under the natural embedding, and its image is dense on $\bar{M}$. Finally, separation properties of the topologies are also considered.

6.3.5. Caveats on the GKP-based constructions. After this account of the attempts based on the Geroch, Kronheimer and Penrose paper it may be a little bit disappointing to learn that the majority of the GKP-based schemes present unsatisfactory features—see, however, subsection 6.3.6. Here we will limit ourselves to give brief comments about the unconvincing aspects referring the interested reader to the literature for the precise details. Let us remark that the main driving force behind the successive GKP-modifications was in fact to give an answer to the successive drawbacks found in certain well-devised examples.

To start with, Penrose [129, 130] classified the points of the $c$-boundary as infinity points and finite points. In the first case the $\infty$-TIP or $\infty$-TIF representing the point is a set of the form $I^-\gamma \,$ or $I^+\gamma \,$, respectively, where $\gamma \,$ is a complete future (past) inextensible causal curve. The case of finite ideal points has the same definitions but now the curve $\gamma \,$ is incomplete. Unfortunately, this classification does not provide a boundary with the “right shape” in some examples, as we show next. The paradigmatic counterexample is the GKP boundary of Taub’s plane-symmetric static vacuum spacetime, whose line-element is

$$ds^2 = z^{-1/2}(dt^2 - dz^2) - z(dx^2 + dy^2), \quad z > 0, \quad -\infty < t, x, y < \infty.$$  

In [90] the $c$-boundary of this spacetime is explicitly constructed and it is shown that its singular part in the sense of Penrose consists of a single point. However, the Penrose diagram of the appropriate 2-dimensional subregion spanned by the coordinates $\{t, z\}$ (see subsection 6.5.1) shows that the singularity $z = 0$ should be a one-dimensional set. Other counterexamples with analogous behaviours are also presented in [90]. Thus, a first type of bad behaviour is that the “shape” of the boundary is not the expected one in certain examples. The modifications of the $c$-boundary explained in subsections 6.3.2, 6.3.3 and 6.3.4 correct this problem for the Taub spacetime, though.

Unfortunately, there are other type of problems in the GKP-based constructions dealing with the topology of the completed spacetime. These problems were illuminatingly pointed out in [91, 92] where they became apparent through carefully conceived examples in which the $c$-boundaries are explicitly constructed. The failure shown in all examples is similar in nature: the topology of the causal boundary $\partial M$ using Rácz or Szabados versions for the $c$-boundary does not match the natural topology of the completed spacetime $\bar{M}$. For instance, many examples are just two- or three-dimensional Minkowski spacetime with certain regions removed. The causal completion of the excised spacetime is then performed and it is proven that the topology obtained according to each of the $c$-boundary prescriptions under surveillance is different from the topology of the closure in Minkowski spacetime of the excised region. This is shown typically using sequences whose limit differ on each topology.

The authors conclude in [92] that the attempt to describe the singularity structure via the completion of the spacetime and the definition of a causal boundary may
be inappropriate. As a matter of fact, all attempts have encountered undesirable properties so far in certain simple cases, so that one wonders what could happen in more complicated cases where there is no obvious “right” answer to be obtained or compared to.

For further problems see [142]. On the other hand, for a positive albeit moderate result regarding the future c-boundary, see next subsection.

6.3.6. Harris’ approach to the GKP boundary using categories. Harris adopted a different approach to the subject in the companion papers [58, 59]. Apparently, his goal was to show that the GKP boundary may be regarded as natural in a categorical sense under some well-established restrictions. Thus, instead of seeking yet another construction for the c-boundary, the universality of the GKP boundary construction was settled “in a categorical manner”.

As part of the problem resides in that completions of a spacetime may fail to be a manifold, in [58] only “chronological sets”, which are similar to the causal spaces of definition 3.1 but just having the chronology relation \( \ll \) are considered. Given a chronological set \( X \), the GKP procedure of attaching a future causal boundary to a space-time is carried over to \( X \), provided certain conditions are met. A crucial point here is that only the future completion is defined. This is a clever choice, because many of the problems arising with the GKP construction have their roots in the intricate problems inherent to the identification of ideal points in the past boundary \( \partial^- \) with ideal points in the future boundary \( \partial^+ \). Actually, one can show that the topology generated for the Minkowski spacetime does not coincide with that of its conformal completion of section 6.1, see [59].

The future-completed chronological set is denoted by \( X^+ \) and the future chronological boundary by \( \partial^+(X) \). There are examples with \( X^+ = X \), called future-complete chronological sets. A map \( f: X \to Y \) between two chronological sets \( X \), \( Y \) is called future-continuous if \( f \) is chronal preserving in the sense of definition 4.3 and if the image \( f(x) \) of the limit of a future chain on \( X \) is the future limit of the (necessarily future) image chain on \( Y \). Important properties are: (i) for any future continuous \( f \) there exists an extension \( f^+: X^+ \to Y^+ \); (ii) the natural inclusion \( \iota^+_X: X \to X^+ \) fulfills the property \( f^+ \circ \iota^+_X = \iota^+_Y \circ f \). Using this, the universality principle is proved: if \( Y \) is future-complete, for any future-continuous map \( f: X \to Y \) \( f^+ \) is the unique future-continuous extension of \( f \) to \( X^+ \) such that \( f^+ \circ \iota^+_X = f \). This result is very important because it says that any future completion of \( X \) can be compared with its GKP version preserving the chronology of \( X \), that is to say, the GKP construction is universal (in the appropriate category, [58] states all results in the language of categories and functors). As remarked before, Harris also points out that the total c-boundary, both future and past with appropriate points identified, does not seem to have an analogous universality property.

A very readable account of this line of research with examples and a summary of the relevant results can be consulted in [60], by the same author. Recently, there has appeared a new paper where the relation between the GKP boundary of a quotient spacetime by a set of isometries and the induced quotient of the GKP boundary is analyzed [61] from this point of view.
6.4. Other independent definitions

In this section we summarise other attempts to attach a “causal boundary” to any spacetime. As happens with the case of the c-boundary, some of these constructions were found to be unsatisfactory in certain examples. We start with the cases were no external objects but only the intrinsic structure of the spacetime are needed (subsections 6.4.1–6.4.3), and then we consider new approaches where the idea of “enlargement” or “embedding” into larger manifolds has been reformulated and improved (subsections 6.4.4 and 6.4.5).

6.4.1. g-boundary. The g-boundary was devised by Geroch in [51] and was probably one of the first attempts to attach a boundary to Lorentzian manifolds. The main idea behind the construction was to try and deal with singularities (understood as inextensible incomplete geodesics), and to provide them with causal and metric properties. At the time [51] was published, there was no generally accepted definition of singularity in General Relativity although, generally speaking, geodesic incompleteness was already agreed to indicate the presence of singularities (be them removable or essential singularities, see §6.4.4). A (essential) singularity should never be considered as part of the spacetime and so it should be placed “at its boundary”. Thus, Geroch tried to attach a boundary to each incomplete inextensible causal geodesic.

The g-boundary is outdated nowadays, since the definition of singularities requires not only caring about geodesic incompleteness, but about that of other causal curves as well. This was actually remarked by Geroch himself in a very interesting paper [52] where an explicit future-incomplete inextensible timelike curve with bounded acceleration was explicitly exhibited in an otherwise geodesically complete spacetime. However, we have considered interesting to include a summary of the g-boundary here due to its historical interest and for the sake of completeness.

The g-boundary is formed by equivalence classes of incomplete inextensible geodesics under a certain equivalence relation. Essentially the whole idea is to collect in classes geodesics which stay close to each other. Let \((M, g)\) be an \(n\)-dimensional spacetime and denote by \(G\) the tangent bundle \(T(M)\). As is well known \(T(M)\) is a differentiable manifold of dimension \(2n\). Furthermore each point \((p, \vec{\xi}) \in G, p \in M, \vec{\xi} \in T_p(M)\) determines one and only one geodesic \(\gamma\) and conversely so we can speak of the points of \(G\) as the geodesics on \(M\). Next one defines the \((2n + 1)\)-dimensional manifold \(H = G \times (0, \infty)\) and the subsets

\[
H_+ = \{(p, \vec{\xi}, a) \in H : \varphi(p, \vec{\xi}) > a\}, \quad H_0 = \{(p, \vec{\xi}, a) \in H : \varphi(p, \vec{\xi}) = a\},
\]

\[
G_I = \{(p, \vec{\xi}) \in G : \varphi(p, \vec{\xi}) < \infty\},
\]

where \(\varphi(p, \vec{\xi})\) is the affine length of the geodesic \((p, \vec{\xi})\), which is infinity if the geodesic is complete. We need also the mapping \(\Psi : H_+ \to M\) defined by the rule \(\Psi(p, \vec{\xi}, b) = \) the point \(q \in M\) resulting after travelling an affine distance \(b\) along the geodesic \((p, \vec{\xi})\).

The next step is topologizing \(G_I\) as follows: for any open set \(O \subset M\) we define the set \(S(O)\) as the subset of \(G_I\) such that

\[
\exists U \subset H \text{ open containing } (P, \vec{\xi}, \varphi(P, \vec{\xi})) \in H_0 \text{ and } \Psi(U \cap H_+) \subset O.
\]

In plain words \(S(O)\) consists of the incomplete geodesics entering \(O\) and never leaving it. The collection of sets \(S(O)\) with \(O\) ranging over all open sets of \(M\) serves as a basis for a topology on \(G_I\) called the open set topology.
We are now ready to define the equivalence relation on $G_I$ leading to the $g$-boundary. Any two elements $\alpha, \beta \in G_I$ are related (written $\alpha \approx \beta$) if every open set in $G_I$ containing $\alpha$ also contains $\beta$ and vice-versa||. The relation “$\approx$” is an equivalence relation on $G_I$ and the set of equivalence classes, denoted by $\partial$, form the $g$-boundary of $M$. Furthermore the topology of $G_I$ induces a topology on the quotient set $\partial$ in the standard way. The completed space-time $\bar{M}$ is then the union of $M$ plus the singular points $\partial$, and can be endowed with a topology whose basis is formed by pairs of open sets $(O, U) \in M \times \partial$ with the added property that $U \subset S(O)$. The restriction of this topology to $M$ and $\partial$ is consistent with the topologies previously introduced.

An interesting feature of the $g$-boundary is the possibility of regarding $\partial$ as if it were a hypersurface, so that concepts such as spacelike or timelike $\partial$ can be defined. Thus, $\partial$ is spacelike at $e \in \partial$ if there exists a neighbourhood $(O, U) \subset M$ such that for every $e' \in U$ there is another neighbourhood $(O', U') \subset M$ of $e'$ with the property $[I^+(e, (O, U)) \cup I^-(e, (O, U))] \cap (O', U') = \emptyset$. Similarly, $\partial$ is timelike at $e$ if for every neighbourhood $(O, U)$ of $e$ in $\bar{M}$ we can find two points $e', e'' \in U$ such that $I^+(e', (O, U)) \cap I^-(e'', (O, U))$ contains an open neighbourhood of $e$ in $\bar{M}$. Here, $I^\pm$ are the natural extensions of $I^\pm$ to $\bar{M}$.

Some applications of this construction discussed in [51] deal with the study of space-time extensions and the relationship between the $g$-boundary and the conformal boundary of Penrose. An important limitation of the $g$-boundary is that by construction $\partial$ only includes singularities and not points “at infinity”. Furthermore, as remarked above, only some singularities are included in $\partial$. These and other unsatisfactory features of the $g$-boundary were considered in [55] for certain examples.

6.4.2. b-boundary The $b$-boundary construction invented by Schmidt [146] was also motivated by the problem of singularity definition in General Relativity. At the time Schmidt’s paper was published, relativists were already aware that not only inextensible incomplete geodesics were to be taken into account when constructing the boundary of singular points, rather all causal (and even non-causal) inextensible curves had to be considered, [52].

To deal with this problem, Schmidt worked with the bundle of frames $L(M)$ constructed from the spacetime $M$ and it is actually in this manifold where the completion is carried out obtaining the set $L(M)$ and its boundary $\partial L(M) = L(M) - L'(M)$. From this the $b$-boundary of the manifold is defined as $\partial M = \pi(\partial L'(M))$ where $L'(M)$ is one of the connected components in which $L(M)$ splits for any orientable manifold and $\pi$ is a suitable extension of the projection of the frame bundle onto its base space. In order to achieve the completion of the frame bundle $L(M)$ the author defines a proper Riemannian metric on this manifold and perform its standard Cauchy completion.

Let us summarize next how the Riemannian metric is constructed on $L(M)$ (the reader is assumed familiar with fibre bundle theory). A frame bundle is a particular case of a principal bundle so one can use in this case standard concepts which are specific for them. As is well known, a connection on a principal bundle defines a subspace in the tangent space of any point $u \in L(M)$ (horizontal subspace) which is complementary to the vertical space. The standard horizontal vector fields $\{B_i\}_{i=1, \ldots, n}$

|| Other equivalence relations on $G_I$ are also considered in [51] but “$\approx$” is the weakest in the sense that points of $G_I$ are identified if they cannot be distinguished topologically.
are the only horizontal vector fields fulfilling the property
\[ \pi' B_i|_u = X_i, \quad u = X_1, \ldots, X_n, \]
where \(\{X_1, \ldots, X_n\}\) is a frame of \(L(M)\). Introduce also a set of 1-forms \(\{\theta^1, \ldots, \theta^n\}\) dual to the family \(\{B_1, \ldots, B_n\}\)
\[ \theta^i(B_k) = \delta^i_k. \]
The connection can be characterized by means of a 1-form \(\omega\) with values in \(gl(n, \mathbb{R})\), the Lie algebra of the structural group \(Gl(n, \mathbb{R})\). In this sense horizontal vector fields are characterized by the condition
\[ \omega(X) = 0 \iff X \text{ is horizontal.} \]
The 1-form \(\omega\) can be written in components with respect to a basis of the Lie algebra \(gl(n, \mathbb{R})\), \(\{E_i^k\}_{1 \leq i, k \leq n^2}\), as
\[ \omega = \omega^i_k E_i^k, \]
where the coefficients \(\omega^i_k\) are 1-forms on \(L(M)\) called the connection forms. A proper Riemannian metric \(g\) on \(L(M)\) is given by
\[ g(X, Y) = \sum_i \theta^i(X)\theta^i(Y) + \sum_{i,k} \omega^i_k(X)\omega^i_k(Y), \]
for any pair of vector fields \(X, Y\) on \(L(M)\). This is called the bundle metric. Notice that once we get the completion of the frame bundle the projection \(\pi\) must be extended as well from \(L(M)\) to \(\overline{L(M)}\) (we still use the same symbol \(\pi\) for this extended projection). This is done through the extension of the right action of the structural group \(Gl(n, \mathbb{R})\) on the frame bundle.

At first glance this construction may seem rather abstract with no definite relation with the points of a boundary for \(M\). To start digging the intimate relationship with incompleteness on \(M\), we must realize that points of the \(b\)-boundary are equivalence classes of Cauchy sequences in \(L(M)\) with respect to the distance defined by the metric \(g\), so it would be interesting to have an interpretation for the length of curves in \(L(M)\) with respect to the bundle metric. For curves \(u(t)\) such that their tangent vector \(\dot{u}\) is horizontal (horizontal curves) this length takes the form
\[ L = \int_0^1 \left( \sum_{i=1}^n \theta^i(\dot{u})\theta^i(\dot{u}) \right)^{1/2} dt. \]
As \(u(t)\) is horizontal, the frame \(\{X_1(t), \ldots, X_n(t)\}\) determined by \(u(t)\) is parallel propagated along the curve \(x(t) = \pi(u(t))\) whose tangent vector is
\[ \dot{x} = \pi'(\dot{u}) = \theta^i(\dot{u})X_i. \]
From this we conclude that \(L\) is nothing but the “Euclidean length” of the curve \(x(t)\) measured with respect to a parallel propagated frame. In particular, the previous reasoning applies to any geodesic because the lift of any geodesic is a horizontal curve. From this it is not difficult to conclude the following important result

**Theorem 6.3** If the bundle metric is complete, then the connection is geodesically complete.
Causal structures and causal boundaries

Therefore geodesically incomplete connections entail incomplete bundle metrics or in other words any incomplete geodesic on $M$ determines a point of the $b$-boundary. The advantage of the $b$-boundary is that the converse of theorem 6.3 is not true, and in fact incomplete curves which are not geodesics also determine points of the $b$-boundary.

Equation 6.1 can be seen as an appropriate notion of length valid for any $C^1$ curve of $M$ (any $C^1$ curve admits an horizontal lift). This new length has some indeterminacies, but if it is finite for a given choice of frame, then it is finite for any other choice. So, one can speak of $b$-completeness of curves without ambiguity, and without resorting to frame bundle theory in fact. In this sense a spacetime is said to be singularity-free if it is $b$-complete, see [64, 153] for details about this.

Schmidt’s paper also discusses some issues related to the topology of the $b$-boundary. As happened in other cases, this is the tricky point and in fact there are examples in the literature where the $b$-boundary is constructed explicitly and shown to have strange topological properties [82, 17].

6.4.3. Meyer’s metric construction. In the interesting paper [110], Meyer describes a boundary construction based on a definition of metric distance for the spacetime $(M, g)$. Firstly, for any set $U \subset M$ its height $d(U)$ is the supreme of the length of all the future-directed causal curves $\gamma \subset U$. Using the notation $A \Delta B \equiv (A - B) \cup (B - A)$ for the symmetric difference, the distance between two past sets $A$ and $B$ is

$$D(A, B) \equiv d(A \Delta B),$$

and analogously for future sets. If $p, q \in M$ one introduces the quantities $D^-(p, q)$ and $D^+(p, q)$ by

$$D^-(p, q) = D(I^-(p), I^-(q)), \quad D^+(p, q) = D(I^+(p), I^+(q))$$

(the same notation is used for the distance between future sets and past sets). Finally the distance between two points is simply

$$D(p, q) = D^+(p, q) + D^-(p, q).$$

The pair $(M, D)$ is then a metric space but the distance $D(p, q)$ need not be finite for all the pairs $p, q$. To get rid of this and other awkward features, only spacetimes of finite timelike diameter [6] are considered. This is the main disadvantage of Meyer’s construction, as it leaves out many simple obvious spacetimes. Under the mentioned assumption $D(p, q)$ is finite for any pair or points and in addition to this the function $D$ is continuous on $M \times M$. Furthermore the topology induced by this distance agrees with the manifold topology, and the continuity of $D$ implies causal continuity of $(M, g)$. The metric space $(M, D)$ can now be completed to $(\bar{M}, \bar{D})$ by means of the Cauchy completion and one gets the boundary as $\bar{M} - M$. The causal structure can then be extended to the completion $\bar{M}$.

The relation between this boundary and the $c$-boundary is analyzed, and the new boundary is explicitly constructed for some examples, in [110].

6.4.4. $a$-boundary. The abstract boundary or $a$-boundary, first introduced by Scott and Szekeres in [149], was also devised with the study of singularities in mind. However, while the $g$- and $b$-boundaries did only use objects intrinsic to the Lorentzian manifold to be completed, the $a$-boundary comes back to the original idea on which
the conformal compactification and conformal boundary were founded: embeddings into larger sets. Even more, the aim in [149] was to put forward a general scheme of how to envelope any differentiable manifold, be it Lorentzian or semi-Riemannian or without metric, or with only a connection..., into a larger one, and then how to obtain all possible boundaries that such a manifold admits. On a second stage, the abstract construction accommodates itself very well with a general definition of singularity in the case of pseudo-Riemannian manifolds, or manifolds with an affine connection. Perhaps it could be fair to say that the a-boundary put in rigorous terms some concepts which were used more or less vaguely many times by relativists. Further developments of the a-boundary can be found in [2].

Before giving an account of this construction some definitions are in order. The definition of curve is similar to that presented in definition 2.3 with the difference that half-open intervals \( I = (a, b) \), \( -\infty < a \) are used. The parameter of the curve is said to be bounded if \( b < \infty \) and unbounded otherwise. A family of curves \( C \) on a manifold \( M \) has the bounded parameter property (b.p.p.) if the following conditions are met

(i) for any point \( p \in M \) there is at least one curve \( \gamma \) of the family passing through \( p \).

(ii) If \( \gamma \) is a curve of the family then so is any subcurve of \( \gamma \).

(iii) Any pair of curves \( \gamma, \gamma' \subset C \) related by an allowed parametrization change have both a bounded parameter or an unbounded parameter.

This third point is important at this stage as we do not have a notion of affine, proper or other well-behaved lengths such as (6.1).

Important concepts in the a-boundary are the following.

**Definition 6.2 (Envelopment)** The differentiable manifold \( \hat{M} \) is said to be an envelopment of the manifold \( M \) if there exists a \( C^\infty \) embedding \( \phi : M \to \hat{M} \) and both \( M, \hat{M} \) have the same dimension. Envelopments will be denoted by the triple \( (M, \hat{M}, \phi) \).

**Definition 6.3 (Boundary points and sets)** The point \( p \in \hat{M} \) is a boundary point of the envelopment \( (M, \hat{M}, \phi) \) if it belongs to the topological boundary of \( \phi(M) \) in \( \hat{M} \). A boundary set is a set consisting of boundary points.

One can also say that a curve \( \gamma : I \to M \) approaches a boundary set \( B \) of the envelopment \( (M, \hat{M}, \phi) \) if \( \phi \circ \gamma \) has some element of \( B \) as an endpoint.

If different envelopments of the same manifold \( M \) are given, one needs to consider the relation between their boundary points and sets. For example, let \( (M, \hat{M}, \phi) \) and \( (M, \hat{M}', \phi') \) be two different envelopments of the same differentiable manifold \( M \) and let \( B, B' \) be respective boundary sets. Then \( B \) “covers” \( B' \) if for every open neighbourhood \( U \) of \( B \) in \( \hat{M} \) there exists an open neighbourhood \( U' \) of \( B' \) in \( \hat{M}' \) such that

\[
\phi \circ \phi'^{-1}(U' \cap \phi'(M)) \subset U.
\]

This definition can be applied to boundary sets consisting of a single point too. An equivalence relation \( \sim \) between boundary sets on different envelopments can be defined as \( B \sim B' \) if \( B \) covers \( B' \) and \( B' \) covers \( B \). Any of the equivalence classes of this equivalence relation is called an abstract boundary set denoted by \([B]\). Clearly single points also give rise to equivalence classes, which are called abstract boundary points.

**Definition 6.4 (Abstract boundary)** The abstract boundary (or in short a-boundary) of a differentiable manifold \( M \) is denoted by \( \mathcal{B}(M) \) and formed by the set of all its abstract boundary points.
It is also interesting to study the properties of boundary points related to the curves of \( M \). To do this one must select a family \( C \) of curves having the b.p.p. seen before. Then, a boundary point \( p \) of the envelopment \((M, \hat{M}, \phi)\) is a \( C \)-boundary point or \( C \)-approachable if it is a limit point of some curve of the family \( C \). Boundary points which are not \( C \)-boundary points for any family \( C \) are called unapproachable. Explicit examples of unapproachable boundary points are shown in [149]. This definition can be extended to abstract boundary points and so those elements of the abstract boundary which are approachable are called abstract \( C \)-boundary points.

Up to this point only the differentiable properties of \( M \) have been used to construct the \( a \)-boundary, which can in principle be defined for any differentiable manifold with no further structures required. If we consider now pseudo-Riemannian manifolds admitting a \( C^k \) metric \( g \) of arbitrary signature, then new natural concepts which are in agreement with traditional ones can be given.

**Definition 6.5 (Metric extension)** A pseudo-Riemannian manifold \((\hat{M}, \hat{g})\) is a \( C^l \) metrical extension \((1 \leq l \leq k)\) of the pseudo-Riemannian manifold \((M, g)\), with \( g \) of class \( C^k(M) \), if \( \hat{g} \) has class \( C^l(\hat{M}) \) and there exists an envelopment \((M, \hat{M}, \phi)\) such that on \( M \)

\[
\phi^*(\hat{g}) = g.
\]

Extensions are denoted by \((M, g, \hat{M}, \hat{g}, \phi)\).

Boundary points of any envelopment of a pseudo-Riemannian manifold can then be further classified according to whether they can be forced to be regular in some metric extension or not. Thus, a boundary point \( p \) of an envelopment \((M, \hat{M}, \phi)\) is said to be \( C^l \) regular if there exists a \( C^l \) metric extension \((M, g, \hat{M}, \hat{g}, \phi)\) (same \( \phi \)!) of \((M, g)\) such that \( \phi(M) \cup \{p\} \subseteq \hat{M}_1 \subseteq \hat{M} \). In other words, the pseudo-Riemannian manifold \((M, g)\) can be continued through the boundary point \( p \). Whether or not a boundary point is regular is independent of any family of b.p.p. curves \( C \).

As a matter of fact, regular points need not be approachable by certain fixed, previously chosen, families of curves such as geodesics. This is shown in [149] by explicit examples, but it is already known from the previously mentioned examples of [52]. This is important because boundary points can then be classified according to whether they are approachable by different families \( C_1, C_2, \ldots \) or not, and then depending on their approachability for each particular family. This classification, with some additional details, is spelt out in [149] resulting in a very elaborated scheme which shall not be reproduced here. If nevertheless one chooses a particular family \( C \) of b.p.p. curves, then a definition of singularity is put forward in [149]: a boundary point \( p \) of the envelopment \((M, \hat{M}, \phi)\) is \( C^l \) singular, also called a \( C^l \) singularity, with respect to the family \( C \), if \( p \) is not a \( C^l \) regular boundary point and \( p \) is \( C \)-approachable with bounded parameter —i.e. there exists a curve in the family \( C \) which approaches \( p \) with bounded parameter.

We see that the concept of singularity depends on the envelopment and on the family of curves. One may wonder if there are cases in which a singularity is present for any envelopment of the manifold \((M, g)\), given a fixed family \( C \). This is resolved by calling \( C^m \) removable singularity to any \( C \)-singularity \( p \) that can be covered by a \( C^m \) non-singular boundary set of another envelopment. Otherwise the point is called an essential singularity with respect to \( C \). An important result is that essential singularities have been proved to be stable [2]. Essential singularities can be further classified depending on the properties of the curves approaching them. One of us
[153] put forward a definition of singularity based on the above by understanding the inextensibility of b.p.p curves as \( h \)-completeness. Thus a concept of singular extension was also introduced. Still the subject presents too many subtleties to be sketched here.

Despite the fact that the \( a \)-boundary provides a general framework for the study of completions and boundaries of general manifolds by using envelopments, it is too general to provide definite properties such as shape, causal structure, topology, metric, et cetera of the boundary. It can be used, however, as an appropriate starting point for any try, and as a background for other more definite constructions, see next subsection.

6.4.5. Causal relationship and the causal boundary. After this journey through all the attempts to construct a valid causal boundary for generic spacetimes, one often reaches the state in which, on intuitive grounds, prefers the original plain and elementary definition of the Penrose conformal compactification. It is clear, simple, productive and provides all the required properties: shape, causal structure, metric, causal character, probably distinction between infinity and singularities, and so on and so forth. The only problem was, in fact, the actual impossibility of finding the conformal embedding explicitly in general.

This is why we tried to improve the conformal compactification in [47], where a new definition of causal boundary was presented. The idea is very simple: now that we have come to know that an appropriate definition of isocausal Lorentzian manifolds (definition 4.5) is more general than the conformal one, and that the causal structure can be defined as in definition 4.6, we can simply repeat the whole conformal-boundary construction by replacing any “conformal” mappings by the causal mappings of definition 4.4. This simple generalization provides us with the notions of causal extensions, causal boundaries and causal compactification in an obvious way. This program has many advantages: on the one hand, it keeps all the good properties of the conformal scheme, which is in fact contained as a particular case. Thus, one can give attributes such as shape, causal character, dimensionality, connectivity, topology, et cetera, to the boundaries, and furthermore the traditional Penrose diagrams can be generalized to get intuitive pictures of complicated spacetimes, see §6.5.2 and [47]. On the other hand, it avoids to a certain extent the only problem with the compactification process, which was looking for the conformal factor and embedding. Now we only have to look for two mutual causal mappings, which are certainly easier to find as the restriction to be a causal mapping is much milder (see the discussion after definition 4.4) than that of being a conformal one.

To be precise, let us start with the following definition. Recall the equivalence relation \( "\sim" \) of definition 4.5.

**Definition 6.6 (Causal extension)** A causal extension of a Lorentzian manifold \((V, g)\) is any envelopment \((V, \tilde{V}, \Phi)\) into another Lorentzian manifold \((\tilde{V}, \tilde{g})\) such that \(V\) is isocausal to \(\Phi(V)\): \(V \sim \Phi(V)\).

Observe that a causal extension for \((V, g)\) is in fact a causal extension for the causal structure, that is to say, for the whole equivalence class \(\coset{V}{g} \in \text{Lor}(V)/\sim\) based on \(V\) of which \((V, g)\) is a representative, see definition 4.6.

Notice that the causal extension is different from the traditional metric extensions (definition 6.5) in which the metric properties of \((V, g)\) are kept, and also from the conformal embeddings of subsection 6.1, where the conformal metric properties were maintained. Here we only wish to keep the causal structure of \(V\), in the sense of
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We would like to remark, however, that obviously all metric extensions and all conformal embeddings are a particular type of causal extension, hence they are trivially included in definition 6.6: any conformal embedding is a causal extension with the particular choice that the causal equivalence between $V$ and $\Phi(V)$ is of conformal type.

We arrive at the main definition.

**Definition 6.7 (Causal boundary)** Let $(V, \tilde{V}, \Phi)$ be a causal extension of $(V, g)$ and $\partial V$ the topological boundary of $\Phi(V)$ in $\tilde{V}$. Then, $\partial V$ is called the causal boundary of $(V, g)$ with respect to $(\tilde{V}, \tilde{g})$. A causal boundary is said to be complete if $\Phi(V)$ has compact closure in $\tilde{V}$.

Note again that all the members in coset$_V(g)$ have the same causal boundary with respect to a given causal extension. In principle, however, the causal boundaries of coset$_V(g)$ may depend on its causal extensions. At this stage, this is a problem more serious here than for conformal completions, as in the latter case it seems reasonable to conjecture that the main properties of a causal boundary will be unique if the boundary is complete [39], and this is not the case for the former case, see the long discussion in Examples 12 and 13 in [47]. We have persuaded ourselves, though, that this problem can be easily overcome by adding the necessary restrictions to definition 6.7. This is one of the open questions about definition 6.7.

With regard to how to distinguish between points at infinity or singularities at a causal boundary one can use the ideas of the a-boundary in an effective way. Thus, in [47] we gave the following primary classification

**Definition 6.8** Let $\partial V$ be the causal boundary of $(V, g)$ with respect to the causal extension $(V, \tilde{V}, \Phi)$. A point $p \in \partial V$ is said to belong to:

(i) a singularity set $S \subseteq \partial V$ if $p$ is the endpoint in $(\tilde{V}, \tilde{g})$ of a curve which is endless and incomplete within $(V, g)$.

(ii) future infinity $I^+ \subseteq \partial V$ if $p$ is the endpoint in $(\tilde{V}, \tilde{g})$ of a causal curve which is complete to the future in $(V, g)$. And similarly for the past infinity $I^-$. 

(iii) spacelike infinity $i^0 \subseteq \partial V$ if $p$ is the endpoint in $(\tilde{V}, \tilde{g})$ of a spacelike curve which is complete in $(V, g)$.

Let us remark that the traditional $i^\pm$ of the conformal compactification have been included here in the general sets $I^\pm$. We can also collect all past and future infinities in a set called *causal infinity*. Hitherto, it has not been proved that all points in a causal boundary belong to one of the possibilities of the previous definition, nor that the different possibilities are disjoint in general. These are other interesting open questions.

To summarize, our scheme provides in principle a simple way to attach a causal boundary to any Lorentzian manifold. Its practical application in specific cases still suffers from the problem of finding the causal extensions $(V, \tilde{V}, \Phi)$ for generic Lorentzian manifolds $(V, g)$, but this is certainly easier than finding the conformal completions. Besides, it is our opinion that a systematic way to find out whether or not two Lorentzian manifolds are isocausal does exist, and so this problem could be fully resolved. We thus believe that it is worth exploring this new line and try to answer the open questions mentioned in this subsection.

For some examples of causal diagrams constructed using causal extensions see §6.5.2 and [47].
6.5. Examples and applications

We now briefly present some remarks and comments about the traditional Penrose diagrams (§6.5.1), its comparison to the new causal diagrams (§6.5.2), and some recent interesting applications and results which have been found in Lorentzian manifolds of pp-wave type (§6.5.3), a subject of renewed interest for both mathematicians and physicists, specially with regard to string theory.

6.5.1. Penrose diagrams. Perhaps one of the most severe difficulties faced by the conformal boundary consists in the high efforts needed to construct the unphysical spacetime from a given physical metric. Friedrich’s conformal equations are a first step towards this direction but they seem to be too complex to allow for an analytical solution in interesting physical cases such as isolated bodies.

Despite this difficulty we can still extract useful information out of the conformal methods in many particular cases. A paradigmatic example is the case of spherically symmetric spacetimes: the group $SO(n-1)$ acts multiply transitively on the spacetime and the transitivity surfaces are spheres $S^{n-2}$. The most general form of the line element is

$$ds^2 = A(R,T)dT^2 - 2B(R,T)dRdT - C(R,T)dR^2 - L(R,T)d\Omega^2. (6.2)$$

The $T-R$ part of the metric can always be brought into an explicitly conformally flat form by means of a suitable coordinate change $T = T(t,r), R = R(t,r)$ yielding

$$ds^2 = F(t,r)(dt^2 - dr^2) - \Xi^2(t,r)d\Omega^2. \quad (6.3)$$

Due to the spherical symmetry the $t-r$ part of the metric contains a great deal of the relevant information about the global causal properties of the spacetime and so we can dismiss the angular part in a first approximation. This is very useful, since this $t-r$ part can always be conformally embedded in two-dimensional Minkowski spacetime in the obvious way. Thus, if we had a finite conformal diagram for the two-dimensional Minkowski spacetime, we could represent on them the conformal boundary for (the $t-r$ part of) the spacetime under study. The places where the factor $\Xi$ vanishes or diverges, which are typically either infinity or (removable or essential) singularities can also be represented.

The required compactification for two-dimensional Minkowski spacetime is easily achieved. Let us start with its canonical line-element $ds^2_0 = dt^2 - dx^2, -\infty < t < \infty, -\infty < x < \infty$, and let us perform the following coordinate change, bringing infinity to finite values of the new coordinates,

$$t = \frac{1}{2}(\tan(\bar{t} + \bar{x}) + \tan(\bar{t} - \bar{x})), \quad x = \frac{1}{2}(\tan(\bar{t} + \bar{x}) - \tan(\bar{t} - \bar{x})),$$

under which the line-element becomes

$$ds^2_0 = \frac{1}{4\cos^2(\bar{t} + \bar{x})\cos^2(\bar{t} - \bar{x})}(d\bar{t}^2 - d\bar{x}^2).$$

The ranges of the new coordinates are $-\pi/2 < \bar{t} + \bar{x} < \pi/2, -\pi/2 < \bar{t} - \bar{x} < \pi/2$ so the conformal embedding of two-dimensional Minkowski spacetime in the plane $\mathbb{R}^2$ is depicted as shown below.
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Conformal compactification of two-dimensional Minkowski spacetime.
The coloured zone corresponds to the physical spacetime and the different regions of the conformal boundary are also marked. Note the disconnected structure of spatial infinity $i^0$.

Once we have the Penrose diagram for flat spacetime in 2 dimensions, the “relevant” $t - r$ part of the line-element given by (6.3) for any spherically symmetric spacetime can also be represented in two-dimensional pictures like the above figure. However, of course, each point of the new picture (with the exception of the region with $\Xi = 0$ if it is not an essential singularity) represents a $(n - 2)$-sphere. The diagram and the boundary can adopt many different shapes, and in some occasions the whole $t - r$ region cannot be depicted in one single portion of the type of the figure, see e.g. [23, 64]. The pictures just described, called Penrose diagrams, are very useful, ubiquitous in the relativity literature, and they were made popular by Carter [23, 24] and the textbook [64]. Perhaps the simplest example is the diagram of $n$-dimensional Minkowski spacetime which, given that $r > 0$ for such spacetime, can be obtained from the above figure by cutting it through the $t$-axis and discarding any of the two symmetric halves. Most of the relevant causal information about $n$-dimensional Minkowski spacetime is kept by its Penrose diagram. Penrose diagrams of many relevant spherically symmetric solutions of Einstein equations can be found in [64]. Many others can be found by reading any journal concerning gravity and relativity.

Perhaps a very instructive example is the case of anti de Sitter spacetime, which is also of present relevance concerning the already mentioned Maldacena’s conjecture (or anti-de-Sitter/Conformal-Field-Theory correspondence). The inextensible anti de Sitter line-element has the form (6.2) as follows

$$\begin{align*}
ds^2 &= \cosh^2 R \, dt^2 - dR^2 - R^2 d\Omega^2
\end{align*}$$

where $0 < R < \infty$ and $-\infty < t < \infty$. It is very simple to check that one cannot conformally embed this spacetime in a compact region of Minkowski spacetime or Einstein static universe. This is why the traditional diagram for anti de Sitter spacetime is a non-compact part of Einstein static universe with two parallel lines as infinity, and one adds artificially two points $i^\pm$ to the picture, see e.g. [64]. However, one can draw a true Penrose diagram following the comments of the previous paragraph. One forgets about the angular part of the metric and the following change of coordinate

$$\begin{align*}
\tilde{t} &= \frac{1}{2} \ln \left( \frac{R + \sqrt{R^2 + 4}}{R - \sqrt{R^2 + 4}} \right), \\
\tilde{R} &= R
\end{align*}$$

This transformation will transform the $t - r$ coordinates of the original spacetime into the $\tilde{t} - \tilde{R}$ coordinates of the new spacetime, and the resulting Penrose diagram will have the desired properties. The result is a Penrose diagram for anti de Sitter spacetime that is a non-compact part of Einstein static universe with two parallel lines as infinity, and it is very similar to the traditional diagram for anti de Sitter spacetime.
\[ r = \arctan(\sinh R) \text{ brings the line-element to its form (6.3)} \]

\[ ds^2 = \frac{1}{\cos^2 r} (dt^2 - dr^2) + \arg \sinh^2 (\tan r) d\Omega^2 \]

where now \( 0 < r < \pi/2 \). Therefore, a finite Penrose diagram can be drawn now by just taking the part of 2-dimensional flat spacetime defined by \( 0 < x < \pi/2 \), compare to [168]. This proves that infinity is timelike everywhere. A 3-dimensional representation of anti de Sitter spacetime can then be given as its Penrose diagram; this is shown in the following figure taken from [47].

In this picture we show a Penrose-like diagram for anti-de Sitter spacetime. In this case we have preferred to draw a 3-dimensional diagram to get a clearer picture of the causal infinity. Every \( t = \) const. slice (in red) has been reduced to an open horizontal disk, so that every point in the diagram represents a \((n-3)\)-sphere except for the vertical axis which is the origin of coordinates. The boundary of the picture represents the conformal infinity \( \mathcal{I} \) of anti-de Sitter. It is remarkable that this boundary has precisely the shape of the Einstein universe. Thus, one is tempted to say that the causal boundary of \( n \)-dimensional anti-de Sitter spacetime is the \((n-1)\)-dimensional Einstein universe. Notice also the timelike character of \( \mathcal{I} \) and the non-global hyperbolicity of the anti-de Sitter spacetime.

6.5.2. Causal completions and causal diagrams. In §6.4.5 we introduced the generalization of conformal embeddings called causal extensions, and defined the causal boundary. Given that the Penrose diagrams are based on the conformal compactification, we can generalize these diagrams by using causal mappings. In this way, as with Penrose diagrams, we can try to analyze and understand the properties of complicated spacetimes by studying the simpler ones to which they are isocausal, and by drawing pictures which will generally include the causal boundary—showing some of its properties.

To draw a causal diagram for \((V, g)\), we must look for a causal extension \((V, \tilde{V}, \Phi)\) (definition 6.6). There is no general rule to find an appropriate \( \Phi \), but the fact that we are dealing with causal mappings instead of conformal relations may be many times an advantage. This is so because less restrictive conditions are needed for the isocausality condition \( \Phi^* \tilde{g} |_\gamma \in \coset \gamma (g) \), than for a conformal relation \( \Phi^* \tilde{g} |_\gamma = \Omega^2 g \).

In [47] we presented different explicit examples for which the causal extension was performed explicitly and the causal diagrams drawn. These diagrams provide a shape and a causal character for the boundary. A remarkable example which shows
how this scheme works and its applicability is given for instance by the Kasner-type spacetime

\[ ds^2 = dt^2 - \sum_{j=1}^{n-1} t^{2p_j} (dx^j)^2, \quad 0 < t < \infty, \quad -\infty < x^j < \infty \]

where \( p_j \) are constants. If \( p_j < 1 \) then it is shown in [47] that the above spacetime is isocausal to

\[ ds^2 = dt^2 - (B + e^{-kt})^2 \sum_{j=1}^{n-1} (dx^j)^2, \quad k = \max\{1 - p_j\}, \quad B > 1, \quad 0 < t < \infty. \]

But this a Robertson-Walker line-element! Thus this can be easily written in explicitly conformally flat form by means of the simple coordinate transformation \( \tau = \log(1 + Be^{kt})/(kB) \) getting

\[ ds^2 = \frac{B^2 e^{2k\tau}}{(e^{k\tau} - 1)^2} \left( d\tau^2 - \sum_{j=1}^{n-1} (dx^j)^2 \right), \quad 0 < \tau < \infty. \]

From here we see that this spacetime can be conformally embedded in (the upper half of) Minkowski spacetime and this very conformal embedding is a causal extension for our original Kasner-type model. Performing the conformal compactification of Minkowski spacetime if desired we can obtain a complete causal boundary. The picture of this conformal embedding is called a causal diagram of the original Kasner-like spacetime and it conveys most of the causal information about this spacetime despite that Kasner and FRW models are not conformally related.

The case with one of the Kasner exponents \( p_1 = 1 \) and the rest as before was also treated, and an interesting diagram found in [47]. Other examples of causal diagrams can be found in [47].

6.5.3. Causal properties of the Brinkmann spacetimes: pp-waves. Plane fronted waves with parallel rays (pp-waves) have attracted a lot of interest in recent years specially within the string theory community. The main reason is that all the scalar curvature invariants are zero in these spacetimes and so they are exact solutions (solutions “at any order in the expansion of the string action”) of the classical theory, providing backgrounds upon which string theorists can explicitly try to study quantum phenomena involving gravity. Besides this modern interest, the relevance of these spacetimes is clear as they describe under certain circumstances the simplest (though highly idealized) models of exact gravitational waves. A general study of the global causal properties and the causal completions of these and some related spacetimes has been systematically carried out only recently. The main results are the subject of this subsection.

The spacetimes with all scalar curvature invariants vanishing have been explicitly found in [30] for arbitrary dimension, see also references therein and [155]. They include in particular all spacetimes characterized by the existence of a null vector field \( K \) which is parallel (also called covariantly constant). The most general local line-element for such a spacetime was discovered by Brinkmann [20] by studying the Einstein spaces which can be mapped conformally to each other: there exists a local coordinate chart in which the line element takes the form [20]

\[ ds^2 = 2du dv + H(x, u)du^2 - A_i(x, u)dx^i du - g_{ij}(x, u)dx^i dx^j, \quad x = \{x^1, \ldots, x^{n-2}\}. \quad (6.4) \]
where the functions $H$, $A_i$ and $g_{ij} = g_{ji}$ ($\det(g_{ij}) \neq 0$) are independent of $v$, otherwise arbitrary, and the parallel null vector field is given by $\vec{k} = \partial/\partial v$. This null vector field has then vanishing shear, rotation and expansion so the hypersurfaces orthogonal to $k^a$ (wave fronts) are locally planes. However, they are not planes!

The Brinkmann spacetimes include to the so-called pp-waves [33, 161], and nowadays it has become commonplace to consider the existence of a covariantly constant null vector field as the “definition of pp-waves”. Thus, many times the whole family in (6.4) are called pp-waves. However, this is a clear misunderstanding. The term pp-wave arises as a shorthand for “plane fronted” gravitational waves with “parallel” rays, [33, 161], and it should be reserved to those cases contained in (6.4) for which $g_{ij}$ can be reduced to $g_{ij} = \delta_{ij}$ for each fixed $u$. In other words, for the cases where $(M, g_{ij})$ (with $g_{ij}$ at at fixed $u$) is a flat Riemannian manifold, $M$ being the manifold with coordinates $\{x^i\}$. The reason is that the Brinkmann metrics certainly contain parallel rays (the null geodesics defined by $\vec{k}$), and these rays are orthogonal to $(n-2)$-surfaces defined locally by $u =\text{const}$. However, these surfaces, whose first fundamental form is determined by the $g_{ij}$ at the fixed $u$, are not planes nor flat in any sense in general. As a matter of fact, they can have any possible geometry, see e.g. [22]. This misunderstanding arises because, in General Relativity (that is, for $n = 4$ exclusively), all solutions of the Einstein vacuum or Einstein-Maxwell equations have in appropriate coordinates $g_{ij} = \delta_{ij}$ [33, 161]—and, as a matter of fact, $A_i = 0$ too—. In other words, all vacuum or Einstein-Maxwell solutions in General Relativity with a parallel null vector field are pp-waves. This is not so in general, though, and it does not hold in $n = 4$ for metrics with other type of Ricci tensor, or for other values of $n$. If, nonetheless, one wishes to keep a semantic relation to pp-waves, a more adequate term would be “Mp-waves”, or something similar, indicating by the ‘M’ the Riemannian manifold(s) defined by the $g_{ij}$ at constant $u$.

Keeping these remarks in mind, important particular cases of the line-element (6.4) are given by: (i) as already mentioned, line elements with $A_i = 0$ and a flat metric $g_{ij}(x)$ (ergo reducible to $\delta_{ij}$) which are called pp-waves; (ii) pp-waves with the additional restriction that $H(x, u) = f_{ij}(u)x^ix^j$ are called plane waves; and (iii) if the plane wave has also $\partial f_{ij}/\partial u = 0$, so that the $f_{ij}$ are constants, are locally symmetric spacetimes in the classical sense [117] that the curvature tensor is parallel (that is to say, covariantly constant). These locally symmetric plane waves have sometimes been termed as “homogeneous plane waves”, but this could be misleading as there is a more general class of plane waves with a group of symmetries acting transitively on the spacetime which are thus traditionally called homogeneous, see [161] and references therein.

In the classical paper [125] Penrose proved that plane waves are in general not globally hyperbolic. These results have been enlarged recently by a number of authors with many results of interest. Marolf and Ross [105] performed a general study of the causal boundary for locally symmetric plane waves. After a rotation in the coordinates $\{x^i\}$ the line element for these waves can be reduced to

$$ds^2 = 2dudv - du^2 \sum_{i=1}^{n-2} \epsilon_i \mu_i^2 x_i^2 - \delta_{ij}dx^i dx^j,$$

where $\epsilon_i$ are signs given by $\epsilon_i = 1$ for $i \in \{1, \ldots, j\}$ and $\epsilon_i = -1$ for $i \in \{j+1, \ldots, n-2\}$, and the $\mu_i$ are constants. A straightforward calculation shows that the Weyl curvature tensor vanishes if and only if $j = n$ or $j = 0$, that is to say, if all the $\epsilon_i$ have the same
sign. These are the only cases in which finding a conformal embedding into Minkowski or other conformally flat spacetime will not be too hard a task.

Remarkably these embeddings are known and the result depends on the sign of all the $\epsilon_i$. The case with $\epsilon_i = 1$ for all $i = 1, \ldots, n$ was addressed in [8] for $n = 10$ where the conformal embedding of this particular plane-wave spacetime into 10-dimensional Einstein static universe was constructed. The conformal boundary $\mathcal{I}$ consists of a one-dimensional null line which winds around the compact dimensions. In a three-dimensional picture this line would look like an helix contained on the Einstein cylinder, see the figure. The case of negative $\epsilon_i$ for arbitrary $n$ can be conformally embedded in $n$-dimensional Minkowski spacetime resulting in a sandwich region limited by two parallel null planes.

These results were generalized by the same authors in [105, 107] for all locally symmetric plane waves with at least one positive $\epsilon_i$, and in [78] for general plane waves and some pp-waves, but now instead of the conformal boundary, the GKP boundary is constructed explicitly. A very important remark was noted in [107]: as these plane waves have a non-vanishing but constant Weyl curvature tensor, the conformal completion is actually impossible. This is an explicit case where a complete conformal boundary cannot be defined (observe that the causal completion and a complete causal boundary in the sense of §6.4.5 and §6.5.2 can certainly be looked for). This and other results in [107, 78, 77] are very interesting and can be summarized as follows:

(i) The $c$-boundary for locally symmetric plane waves with at least one $\epsilon_1 = 1$ is again a one-dimensional null set plus two points (identified as $i^\pm$) [105].

(ii) However, “spatial” infinity — or the part of the boundary approached by spacelike curves — cannot be included in the total boundary as the completion $\bar{M}$ is non-compact in the topologies introduced in [106], see §6.3.4. A new topology is introduced and an analysis of the points that should be added to the $c$-boundary to include spatial infinity is performed in [107] with no definite conclusion.

(iii) For the case of plane waves the behaviour at infinity is determined by the properties of the functions $f_{ij}(u)$. The case with $\text{tr}f_{ij} \geq 0$ and $f_{ij}(u)$ regular $\forall u \in \mathbb{R}$ was treated in [78] with the outcome that if any of these $f_{ij}(u)$ approaches zero for large values of $u$ fast enough then the GKP boundary is of dimension higher than one. This case includes for instance Minkowski spacetime in which all the $f_{ij}(u)$ are zero. On the contrary, if these functions exhibit a polynomial, trigonometric or hyperbolic behaviour, such that the geodesics exhibit an oscillatory regime for large values of $u$, or even if the $f_{ij} \rightarrow 0$ for large $u$ as a rational function of $u$, then it can be shown that the boundary is again a one-dimensional null line.

(iv) In [77], by using techniques of geodesic connectivity, the result that every point of any general plane-wave spacetime can be joined to infinity (not increasing the value of $u$ too much) is proved regardless of the properties of the functions $f_{ij}(u)$. In fact, tighter results can be proved, see [22, 38]. This was interpreted
in [77] by saying that plane waves cannot have “event horizons”. A stronger and
to more precise version of this result can be found in [155], where the complete
absence of closed trapped surfaces in the general spacetime with vanishing
curvature invariants—including in particular all Brinkmann metrics (6.4)— was
demonstrated.

(v) The c-boundary for pp-waves was also investigated in [78]. Now, we have a more
diverse set of behaviours since the function \( H(x, u) \) may be singular in both the
coordinate \( u \) and the transverse coordinates \( x^i \). Some conditions under which
the GKP boundary is again one-dimensional (and null) are presented in [78],
usually under the assumptions that the spacetime is geodesically complete and
distinguishing.

Linking to this last comment, the question of the degree of causal “virtue”
of general pp-waves is of great interest. After the seminal paper [125] mentioned
before, a recent important advance in the study of the causal properties of “Mp-
waves” with \( A_i(x, u) = 0 \) (see above) follows from the work of Flores and Sánchez
[38]. Thus, a classification according to the standard hierarchy of causality conditions
(definition 4.1) was found for these spacetimes. Assuming that \( H(x, u) \) is smooth,
the classification depends on the behaviour of \( H(x, u) \) at large values of the
transverse coordinates \( x^i \), and in this way the notions of subquadratic, quadratic
or superquadratic behaviour at spatial infinity are introduced: \( H(x, u) \) behaves
\textit{subquadratically} at spatial infinity if there exists \( \bar{x} \in M \) (recall that \( (M, g_{ij}) \)
is the transverse Riemannian manifold with constant \( u \)) and continuous functions
\( R_1(u), R_2(u) \geq 0, p(u) < 2 \) such that

\[
H(x, u) \leq R_1(u) d^p(u)(x, \bar{x}) + R_2(u), \forall (x, u) \in M \times \mathbb{R},
\]

where \( d(x, y) \) is the canonical distance function on \( (M, g_{ij}) \). If \( p(u) = 2 \) then \( H(x, u) \)
is said to behave \textit{quadratically} at spatial infinity. Finally \( H(x, u) \) is \textit{superquadratic} if
there exists a sequence \( \{y_n\} \subset M \) and a point \( \bar{x} \in M \) such that \( d(\bar{x}, y_n) \to \infty \), when
\( n \to \infty \) and

\[
H(y_n, u) \geq R_1 d^{2+\epsilon}(y_n, \bar{x}) + R_2, \forall u \in \mathbb{R},
\]

for some quantities \( \epsilon, R_1, R_2 \in \mathbb{R} \) with \( \epsilon, R_1 > 0 \). The results in [38] can be summarized
in the next theorem.

**Theorem 6.4** All general Mp-wave spacetimes with \( A_i(x, u) = 0 \) are causal. If in
addition

(i) the proper Riemannian manifold \( (M, g_{ij}) \) is complete, \( H(x, u) \geq 0 \) and \( H(x, u) \)
behaves superquadratically, then they are non-distinguishing.

(ii) \( H(x, u) \) behaves at most quadratically at spatial infinity then they are strongly
ciausal.

(iii) \( H(x, u) \) behaves subquadratically at spatial infinity and the Riemannian distance
on \( M \) is complete, then they are globally hyperbolic.

Notice that the cases presented in this theorem are not mutually exclusive and other
behaviours of \( H(x, u) \) not covered by this result may result.

In [22], a thorough analysis of the geodesic properties and geodesic connectivity
of Mp-waves was performed, and again the sub- or super-quadratic properties of
\( H(x, u) \) revealed themselves as essential to classify the different possibilities. More
related results were found in [79] for cases with an explicitly non-smooth $H(x,u)$, but only for pp-waves: with a flat $(M,g_{ij})$. If $H(x,u)$ satisfies the inequality $H(x,u) \leq A_{ij}(u)x^i x^j, \forall x^i$ (in the above terminology $H(x,u)$ would be at most quadratic) for certain functions $A_{ij}(u)$ which may have singularities, then the pp-waves are causally stable. All these results and theorem 6.4 are thus very interesting and have physical implications, because the critical at most quadratic behaviour is in fact the one usually relevant in General Relativity, for this is the behaviour of the plane waves which are exact solutions of the vacuum or Einstein-Maxwell equations.

7. Causal transformations and causal symmetries

Transformations preserving the “causal structure” or the causal relations have been already described in this review (section 4.2). As remarked there, it seems that the right concept of causal structure is that of definition 4.6, arising from the idea of isocausality (definition 4.5) and causal mappings (definition 4.4), and that this structure is more general than the conformal one: the causal structure is defined by a metric up to causal mappings. This idea can be pursued further and in such way we can try to generalize the group of conformal transformations, and the conformal Killing vector fields [161], by defining and studying sets of transformations whose elements are mappings preserving the causal relations (or the causal properties of Lorentzian manifolds). In general however, the algebraic structures stemming from these transformations are no longer groups, but monoids, as the inverse of a causal-preserving map does not need to be a causal preserving map. A monoid is a set $G$ endowed with an internal associative operation “$\cdot$” admitting a neutral element $e$. If there is no such neutral element the pair $(G,\cdot)$ is called a semigroup. Semigroups (specially Lie subsemigroups) and (sub-)monoids have been largely studied in mathematics (standard references are [98, 71, 72]) due to their independent interest. Along this section we will try to convey the idea that these algebraic structures are truly the ones needed to describe the “submonoids of causal-preserving mappings”. Furthermore, we will define the set of causal transformations, obtain its algebraic structure, derive the infinitesimal generators of one-parameter submonoids, and find their characterization in terms of the Lie derivative of the metric.

7.1. Causal symmetries

In this section we are interested in finding the structure and properties of the set of transformations preserving the Lorentzian cones of a Lorentzian manifold. The starting point for this was given in sections 4.2 and 2.1 where transformations preserving causal relations and the Lorentzian cone on a manifold were introduced (this was made explicit in definitions 4.3 and 4.4).

**Definition 7.1 (Causal symmetries)** Causal mappings for which both the domain and the target spacetimes are the same differentiable Lorentzian manifold $(V,g)$ are called causal transformations or causal symmetries. The set of all causal transformations is denoted by $\mathcal{C}(V,g)$. As we saw in section 4.2, a transformation $\phi$ is a causal symmetry if and only if $\phi^* g$ is a future tensor. From the properties of causal mappings it is clear that the composition of transformations is an internal operation in $\mathcal{C}(V,g)$ which is associative with identity element (the identity transformation). Therefore,
**Proposition 7.1** The set \( C(V, g) \) is a submonoid of the group of diffeomorphisms of the manifold \( V \).

However, \( C(V, g) \) is not a subgroup because the inverse of a causal symmetry is not in general a causal symmetry. As a matter of fact, only conformal transformations of \((V, g)\) will have such an inverse, see theorem 4.5. Recall that if \( S \subset G \) is a proper submonoid of a group \( G \), its group of units \( H(S) \) is given by \( S \cap S^{-1} \). The group of units is the maximal subgroup contained in \( S \) in the sense that there is no other bigger subgroup of \( G \) contained in \( S \) possessing \( H(S) \) as a proper subgroup. See [71] for the proof of this and other properties of monoids and semigroups. Denoting by \( \text{Conf}(V, g) \) the set of all conformal transformations of \((V, g)\) we have then

**Proposition 7.2** The maximal subgroup of \( C(V, g) \) is the group of conformal transformations, that is to say, the group of units of \( C(V, g) \) is

\[
C(V, g) \cap C(V, g)^{-1} = \text{Conf}(V, g).
\]

The causal symmetries which are not conformal transformations are given the name of proper causal symmetries. Obviously, \( C(V, g) \) depends on the background metric \( g \), but the set \( C(V, g) \) is conformally invariant, a desirable property.

**Proposition 7.3** \( C(V, g) = C(V, \sigma g) \) for any positive smooth function \( \sigma \) on \( V \).

The set \( \text{coset}_V(g) \) is not well-defined, however, so that we cannot say that the causal symmetries are the same for a given causal structure in the sense of definition 4.6. Nevertheless, given a causal structure, the causal symmetries of any of its metric representatives are in bijective correspondence [49]:

**Proposition 7.4** For any \( g_1, g_2 \in \text{coset}(g) \) there is a one-to-one correspondence between the sets \( C(V, g_1) \) and \( C(V, g_2) \).

The most extensive account dealing with causal symmetries are the papers [48, 49] published recently (the nomenclature and notation followed here is taken from that papers) although similar ideas under different terminology can also be found in the literature [57, 62, 121].

### 7.2. Causal-preserving vector fields

Our aim now is to obtain infinitesimal generators for one-parameter submonoids of causal transformations and try to find out the differential conditions fulfilled by these generators in much the same way as it has been done with isometries, homotecies or conformal symmetries [175, 161]. Despite causal transformations not forming a group, infinitesimal generators for them can still be defined. This is accomplished by considering local one-parameter submonoids of causal symmetries defined by the condition

\[
\{ \varphi_s \}_{s \in I \cap \mathbb{R}^+} \subset C(V, g),
\]

where \( \{ \varphi_s : V \to V \}, s \in \mathbb{R} \) is a one-parameter group of global diffeomorphisms and \( I \subset \mathbb{R} \) is a connected interval of the real line containing zero. The most interesting case occurs when \( I \cap \mathbb{R}^+ = \mathbb{R}^+ \) in which case we say that \( \{ \varphi_s \}_{s \in \mathbb{R}^+} \) is a global one-parameter submonoid of causal symmetries. It can be easily seen [48, 49] that \( \varphi_0 = \text{Id} \) is the only conformal transformation contained in \( \{ \varphi_s \} \) unless the submonoid is in fact a subgroup of conformal transformations. Therefore, there cannot be realizations of \( S^1 \) as a one-parameter submonoid of proper causal symmetries.

We can give our main definition in this section.
Definition 7.2 (Causally preserving vector fields) A smooth vector field \( \vec{\xi} \) defined on an entire Lorentzian manifold is said to be causal preserving if the local one-parameter group generated by \( \vec{\xi} \) complies with (7.1) for some interval \( I \).

These causal-preserving vector fields are a strict generalization of conformal Killing vectors, which are particular cases of them.

The next step is to derive the necessary and sufficient conditions for a vector field \( \vec{\xi} \) to be causal preserving. To that end, we need to classify such vectors in terms of the set of null directions which are kept invariant under the mappings of the one-parameter submonoid. These are called canonical null directions of the submonoid, or in short, of \( \vec{\xi} \). The set of canonical null directions, denoted by \( \mu_{\vec{\xi}} \), only depends on the specific submonoid if the metric tensor \( g \) is analytic on \( V \) and indeed we can calculate it from the vector field \( \vec{\xi} \) (this is the reason for the chosen notation). Whenever \( \mu_{\vec{\xi}} \neq \emptyset \), these null vectors are the part of the null cone preserved by the submonoid so we can regard causal transformations in this case as partly conformal transformations. The canonical null directions can be calculated explicitly by means of the condition

\[
\mu_{\vec{\xi}} = \{ \vec{k} \text{ null} : L_{\vec{\xi}} g(\vec{k}, \vec{k}) = 0 \}.
\]

Then we have the main result in [48, 49].

Theorem 7.1 Let \( \vec{\xi} \) be a smooth complete vector field and suppose there exists a function \( \alpha \) such that \( L_{\vec{\xi}} g - \alpha g \) is a future tensor field with the same algebraic type at every point of the manifold. Then, if

(i) \( \mu_{\vec{\xi}} = \emptyset \), \( \vec{\xi} \) is a causal preserving vector field (with no canonical null direction).

(ii) \( \mu_{\vec{\xi}} \neq \emptyset \) and \( L_{\vec{\xi}} \Omega \propto \Omega \) where \( \Omega \) is a \( p \)-form constructed as the wedge product of a maximal set of linearly independent elements of \( \mu_{\vec{\xi}} \). \( \vec{\xi} \) is a causal preserving vector field with \( \mu_{\vec{\xi}} \) as the set of its canonical null directions.

Observe that if \( p = n \), then \( \mu_{\vec{\xi}} \) contains all possible null directions and the causal preserving vector field is a conformal Killing vector. Note also that, for the case (i) of this theorem, the condition that \( L_{\vec{\xi}} g - \alpha g \) be a future tensor can be replaced by \( L_{\vec{\xi}} g(\vec{k}, \vec{k}) > 0 \) for all null vectors \( \vec{k} \), making no mention of the function \( \alpha \). There is an analogous statement for the case (ii) of the theorem which is more involved, see [49].

The physical relevance of causal preserving vector fields is still unclear. Tentative interpretations can be found in [121, 131], and a generalization shedding some light as to their applicability and geometrical properties in [50]. Let us simply remark here that sometimes conserved quantities and constants of motion can be found, see [49, 48] for details. For example, for general affinely parametrized null geodesics whose tangent vector is \( \vec{v} \) it follows that \( g(\vec{\xi}, \vec{v}) \) is monotonically non-decreasing to the future along the geodesic. Moreover, if \( \vec{v} \) is tangent to a canonical null direction for all \( x \) on the curve, then \( g(\vec{\xi}, \vec{v}) \) is constant along this null geodesic. Hence, the null geodesics along the canonical null directions of a causal motion \( \vec{\xi} \) have a constant component along \( \vec{\xi} \). Furthermore, the construction of conserved currents (divergence-free vector fields) is also possible using causal-preserving vector fields as shown in [49, 48], where particular examples for the electromagnetic field and the Bel-Robinson tensor can be found.

Further interesting properties of causal preserving vector fields can be found in a paper by Harris and Low [62] (these authors employ the terminology “causal decreasing
vector fields” for causal preserving vector fields). In the forthcoming results we adapt the language of these authors to the notation followed here.

**Proposition 7.5** If $\bar{\xi}$ is a complete timelike future-directed causal preserving vector field with $\mu_{\bar{\xi}} = \emptyset$ then for any integral curve $\gamma$ of $\bar{\xi}$ we have that $I^{-}(\gamma) = V$.

This allows to prove a splitting theorem.

**Proposition 7.6** Under the assumptions of the previous proposition, the space $Q$ of integral curves of $\bar{\xi}$ is a manifold and we can write $V = Q \times \mathbb{R}$.

### 7.3. Lie subsemigroups

Suppose that we are given a set of causal preserving vector fields with the same canonical null directions. This set cannot form a Lie algebra, and in fact they only have the structure of a cone, that is, we can only form linear combinations with positive coefficients. This is due to the fact that, as explained in the previous subsection, causal symmetries do not form a group. This means that if we are to describe them in terms of actions upon manifolds we cannot use full groups as it is done with ordinary symmetries. Can we nevertheless obtain the corresponding generated subset of causal symmetries? The answer is yes, for it is possible to define actions of submonoids and subsemigroups on Lorentzian manifolds.

Submonoids have received attention in Mathematics in the framework of the theory of Lie subsemigroups. Standard references about this subject are [71, 72] (see [94] for a good review). Here we will limit ourselves to a brief account mainly to bring together pieces of information which ordinarily lie in journals and books only read by mathematicians. We will thus use concepts and notation which are standard in Lie groups theory. A Lie subsemigroup $S$ is a subset of a Lie group $G$ which is a semigroup. For these objects we can define an analog to the Lie algebra as

$$L(S) = \{ x \in g : \exp(\mathbb{R}^{+}x) \in S \},$$

where $g$ is the Lie algebra of $G$. $L(S)$ is a cone in the vector space $g$. Recall that a cone in a vector space $L$ is a subset $C$ such that $\forall v_1, v_2 \in C$, $\lambda_1 v_1 + \lambda_2 v_2 \in C$ where $\lambda_1, \lambda_2$ are any pair of positive (or negative) scalars and $0 \in C$. From the cone $L(S)$ we may define the vector space $H(L(S)) \equiv L(S) \cap -L(S)$ called the edge of the cone. If such edge satisfies the property

$$e^{ad[h]}L(S) = L(S), \quad \forall h \in H(L(S)),$$

then $L(S)$ is called a Lie wedge. The set $L(S)$ can be mapped to the remainder of the Lie group $G$ by means of the differential map of the right action $R_g : G \to G$. In this way the Lie group becomes a conal manifold where the cone $C(g)$ at each $g \in G$ is defined by

$$C(g) = \{ \bar{w} \in T_g(G) : \exists x \in g \text{ with } \bar{w} = R_{g|e}^\prime(x) \},$$

where $e$ is the neutral element of $G$. Note that $C(g)$ can in principle be constructed from any cone on $g$, be it the Lie wedge of a Lie subsemigroup or not. An interesting question now is to find out the conditions that a cone on $g$ (and by extension a cone field on $G$) be the Lie wedge of a Lie subsemigroup $S \subseteq G$. Next result proven in [115] addresses this matter.
Theorem 7.2 Let $G$ be an analytic group, $\mathfrak{g}$ its Lie algebra and $W \subset \mathfrak{g}$ a cone such that $\mathfrak{g}$ is the smallest Lie algebra containing $W$. Then $W$ generates a Lie subsemigroup $S \subseteq G$ if and only if the subgroup of $G$ generated by the subalgebra $H(W)$ is closed and there exists a function $f \in C^\infty(G)$ with the property
$$df|_g(R'_g|e(x)) > 0, \quad \forall x \in W - H(W), \quad \forall g \in G.$$ 

Remark. This theorem takes the form of the stable causality condition for the manifold $G$ with respect to the causality induced by the cone $C(g)$.

If $V$ is a differentiable manifold let us consider the action $\varphi : G \times V \to V$ of a group $G$ complying with the assumptions of theorem 7.2 plus the property
$$Ad(g)[x] \in L(S), \quad \forall x \in L(S), \quad \forall g \in G, \quad (7.2)$$
where $Ad$ is the adjoint representation of the lie group $G$ on its Lie algebra $\mathfrak{g}$. In this case one can show that there is a natural cone field $C(x), x \in V$ which is invariant under this action, namely,
$$\varphi'_g|_x(C(x)) = C(\varphi_g(x)),$$
where $\{\varphi_g\}_{g \in G}$ is the group of transformations of $V$ defined from the action $\varphi$. The explicit relation between the cones $C(g)$ of the Lie group and $C(x)$ of the differentiable manifold is
$$C(x) = \{\tilde{Y} \in T_x(V) : \exists \tilde{w} \in C(g) : \Psi'_g(\tilde{w})|_g = \tilde{Y}\}, \quad \Psi_y(g) \equiv \varphi(g, y) = x,$$
where a simple calculation shows that $C(x)$ does not depend on $y$ (full details can be found in [72]).

If one wishes to study actions in which the invariance of the cone is replaced by $\varphi'_g|_x(C(x)) \subseteq C(\varphi_g(x))$ (this is the case of causal symmetries with $C(x)$ the Lorentzian cone) then this last condition will only be true for elements $g$ belonging to the semigroup $S \subset G$ if we demand that condition (7.2) hold only for $g \in S$. Note however, that causal symmetries are more general than this as they are ruled by actions of infinite-dimensional groups so the theory just presented could be useful to study finite-dimensional submonoids of causal symmetries. As a matter of fact, one can sometimes consider a set of causal preserving vector fields as a cone in a subgroup of biconformal vector fields [50], and the previous construction applies automatically.

Further details can be consulted in [48, 49, 50, 70, 69, 73, 95, 96, 113, 118, 119].

8. Future perspectives

It is time now to recapitulate drawing conclusions from our review and suggesting some possible fruitful lines of research. Lots of topics have been discussed, some of them classical, well-established and known among relativists, some others mainly known in mathematics circles and also others which probably have not yet reached a widespread knowledge due to their novelty. It is precisely this last category which we would like to emphasize in these conclusions.

To start with, the definition of future and past tensors shown in subsection 2.1 has many potential applications. Some have been already studied and commented in this
review, specially in connection with the preservation of the null cone, the existence of causal symmetries, or the setting of causal mappings; others are still to be fully explored, specially the generalized null-cone algebraic structure which induces on the whole bundle $T^*_r(V)$, and the classification and possible decompositions that tensor fields inherit from this. This is a subject characteristic and exclusive of Lorentzian geometry, as it requires essentially the existence of the Lorentzian metric.

Concerning section 3, we believe there is not much to do related to the definition and characterization of abstract causal/etiological spaces, with an important exception: quantum causality and the theory of “causal sets” outlined in subsection 3.3.2. This is a very important line of research, with a strong vitality, and it is worth devoting some efforts in that direction. The main sought but so far unreached result is how to connect the discrete causal set or spin network with the smooth spacetime we believe to see.

One of the main results presented in this review is the possibility of sorting Lorentzian manifolds in abstract causality classes or causal structures in the sense of definition 4.6. In our opinion, this settles the issue of what the concept of “causal structure” of a Lorentzian manifold really means. More importantly, it has allowed to resolve a long-standing question by means of the theorem 4.4, giving a precise meaning to the local causal equivalence of Lorentzian manifolds. This also permits to talk about the causal asymptotic equivalence of asymptotically flat spacetimes, which was only possible if a full conformal completion was previously achieved. Similar local or asymptotic equivalences can be further investigated, and one can easily produce a definition of “asymptotical causal equivalence to $(M,g)$”, where $(M,g)$ is any particular or preferred spacetime, say, de Sitter, or anti de Sitter, or a particular Robertson-Walker geometry, or a plane wave, and so on. This route is yet to be examined. At the same time, causal mappings (definition 4.4) enable us to refine the standard hierarchy of causality conditions, and to construct causal chains of causal structures where, on a fixed manifold, we can compare Lorentzian metrics from the causal point of view, and classify them according to their “goodness” in relation to causality. There are important open problems concerning this line of research which we want to bring to the attention of the mathematical relativity community. For instance, an even more practical procedure to decide if two Lorentzian manifolds are causally related, and then if they are isocausal, would be very convenient. Another question is how many different causal structures can be defined on a given background differential manifold, and whether there exists upper and lower bounds for them in terms of the partial order introduced in subsection 4.3. As an additional remark to the previous comments, let us stress that the definition of causal structure carries over to abstract etiological spaces (ergo also to causal spaces) using the concepts put forward in definition 4.3. Thus the results and questions of the above paragraph can be formulated as well for these more abstract spaces being possible to define and classify causal structures for them, and causal completions and boundaries too.

Another big issue in this review is the definition of the causal boundary of Lorentzian manifolds. This problem has deserved renewed interest recently in the wake of Maldacena's conjecture, and may become a fundamental issue of the subject termed “holography” (see [75]): the probable correspondence between string theory on spacetime backgrounds and a conformal field theory on the boundary at infinity of that background. Prior to this, researchers tried all sort of recipes and methods in order to find a kind of boundary of universal application to all possible spacetimes. Apart from the inspiring, clear and very fruitful ideas of Penrose's concerning the conformal
boundary, subsection 6.1, the main breakthrough here was Geroch, Kronheimer and Penrose’s paper described in subsection 6.2, but further research proved that strong though this procedure undoubtedly is, it has its own limitations that one should bear in mind. All the subsequent amendments suggested over the years and described in subsection 6.3 were found either too theoretical, requiring daunting efforts to be applied in practical cases, or not mending the drawbacks they were supposed to mend. Other several alternative constructions were devised with this aim but it is fair to say that they have remained as a set of rather too complicated definitions with no possible translation into explicit relevant examples. These difficulties to actually construct the causal boundary by whatever procedure have hampered its study in physically relevant cases. There are fresh good news, however: recently new interesting and encouraging results have been obtained, insufflating renewed air to this subject and arousing again the interest of the mathematical and physics communities. We are referring specially to

- the startling result that the GKP boundary of certain classes of pp-wave spacetimes consists of a one dimensional null line, see §6.5.3. It is quite surprising that this result have been obtained only recently and not, say, twenty years ago. In this sense it would be interesting trying to find the full GKP boundary in other pertinent cases instead of paying a deeper attention to the wrong topological properties of the boundary in certain spacetimes with strange or unphysical causal behaviours.
- Harris’ ideas of caring about just the future boundary, and his results (§6.3.6) showing the universality of the future GKP boundary using category theory.
- the generalization of Penrose’s original ideas by using causal mappings instead of conformal ones. As is known the conformal and the GKP boundaries (in any of its improved versions) are very difficult to construct and we believe that the causal boundary based on the concept of isocausality (cf. § 6.4.5) could come in aid. To this end, the technical difficulties concerning causal mappings pointed out above should be addressed and, if these problems were resolved satisfactorily, then the causal boundary in the sense of §6.4.5, as well as the causal diagrams and causal completions (§6.5.2), could be constructed and analyzed explicitly in a large variety of cases very easily. This would allow to recover the simple and very powerful applications of the conformal boundaries of spacetimes by keeping their enormous virtues and intelligibility but avoiding on one hand the traditional problem of the impossibility of finding conformal completions explicitly, and on the other hand the extreme difficulties to build the other types of causal boundaries, which are also not exempt from inconsistencies, as we have just explained.

Causality conditions impose restrictions on the topology and the metric tensor of a spacetime as we explained in section 5. Here we wish to stress theorem 5.6 which may have interesting applications. It is not necessary to stress the importance of globally hyperbolic spacetimes in Physics because they are present in any model where a well-posed formulation of the Cauchy problem for Einstein field equations is required. Therefore this new result could help improving well-posedness results involving globally hyperbolic spacetimes. Moreover we could adapt formulations of the field equations such as the classical ADM to the hypotheses of theorem 5.6 and thereby obtain relevant simplifications. Finally numerical simulations always assume
globally hyperbolic spacetimes so one could set the equations modelling the system with a metric tensor as given by theorem 5.6 with no loss of generality.

Last but not least, yet another issue we feel worth researching by relativists and mathematicians alike is the subject of section 7, namely, causal transformations/symmetries and their infinitesimal generators, causally-preserving vector fields. There is a mathematical sub-branch lying behind these type of transformations (Lie subsemigroups and their actions on manifolds, cf. §7.3) which has its own independent interest. The interpretation we have provided for these transformations and vector fields may help improving or suggesting the mathematical advances apart from bringing to light important physical applications. A physical interpretation for these new transformations or their generators is also an open, seemingly solvable, important question. It is interesting to note that as opposed to other classical symmetries, causal symmetries might be present in virtually all spacetimes with reasonable causality conditions. Therefore they could allow us to formulate rigorously results involving approximate or asymptotic isometries or conformal transformations. Furthermore, they may provide new conserved, or monotonically increasing, quantities. Finally, they have a direct application to important splitting theorems of spacetimes [49, 50], of a more general nature than those presented in subsection 5.2, which are also worth mentioning here as they give characterizations of spacetimes splittable in two orthogonal distributions of any \( p \) and \( q \) dimensions \( (p + q = n) \). Observe in this sense that causal-preserving vector fields may leave a set of, say, \( p \) null directions invariant, and therefore they act as conformal Killing vectors in the distributions spanned by these null directions. One can then try to construct tensors which are invariant under such general splittings, or characterize spacetimes decomposable in two conformally flat pieces, and so on. In summary, causal symmetries and causal-preserving vector fields can always be considered as partly conformal transformations and partly conformal Killing vectors, respectively, and the many implications and possibilities deriving from this fact which immediately spring to mind are certainly worth investigating.

Acknowledgements

We thank M. Sánchez for a careful reading of the manuscript and many improvements and suggestions. Financial support from the Wenner-Gren Foundations (JMMS) and the Applied Mathematics Department at Linköping University (AGP), Sweden, are gratefully acknowledged. We thank the mentioned department, where this work was partly discussed, for hospitality. Financial support under grants BFM2000-0018 and FIS2004-01626 of the Spanish CICyT and no. 9/UPV 00172.310-14456/2002 of the University of the Basque Country is also acknowledged.

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