Multi–Instantons and Exact Results II:
Specific Cases, Higher–Order Effects, and Numerical Calculations

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Abstract: In this second part of the treatment of instantons in quantum mechanics, the focus is on specific calculations related to a number of quantum mechanical potentials with degenerate minima. We calculate the leading multi-instanton contributions to the partition function, using the formalism introduced in the first part of the treatise [J. Zinn-Justin and U. D. Jentschura, e-print quant-ph/0501136]. The following potentials are considered: (i) asymmetric potentials with degenerate minima, (ii) the periodic cosine potential, (iii) anharmonic oscillators with radial symmetry, and (iv) a specific potential which bears an analogy with the Fokker–Planck equation. The latter potential has the peculiar property that the perturbation series for the ground-state energy vanishes to all orders and is thus formally convergent (the ground-state energy, however, is nonzero and positive). For the potentials (ii), (iii), and (iv), we calculate the perturbative $B$-function as well as the instanton $A$-function to fourth order in $g$. We also consider the double-well potential in detail, and present some higher-order analytic as well as numerical calculations to verify explicitly the related conjectures up to the order of three instantons. Strategies analogous to those outlined here could result in new conjectures for problems where our present understanding is more limited.

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We continue here the investigations [1] on multi-instantons and exact results. Sections, equations, tables and figures are numbered consecutively after those of [1], while the numbering of bibliographic items is independent of [1]. For relevant definitions and conventions, the reader is referred to chapters 1 and 2, which are contained in the first part of the treatise [1]. We will often refer to the conjectures summarized in chapter 2.3, and start the discussion here with chapter 5. Many of the instanton calculations rely on generalizations of the ideas and methods introduced in chapter 4 (which is again part of the first paper [1] of this series).
Chapter 5

Instantons in General Potentials with Degenerate Minima

5.1 Orientation

We now consider a general analytic potential possessing two degenerate minima located at the origin and another point $q_0 > 0$:

\[ V(q) = \frac{1}{2} q^2 + O(q^3), \quad (5.1a) \]

\[ V(q) = \frac{1}{2} \omega^2 (q - q_0)^2 + O((q - q_0)^3). \quad (5.1b) \]

For definiteness we assume $\omega > 1$.

In such a situation the classical equations of motion have instanton solutions connecting the two minima of the potential. However, there is no ground state degeneracy beyond the classical limit. Therefore, the one-instanton solution does not contribute anymore to the path integral. Only periodic classical paths are relevant, the leading contribution coming now from the two-instanton configuration [2] in the sense of the discussion in chapter 4.4.1.

Indeed, in the asymmetric case $\omega > 1$, it is useful to redefine the instanton order $n$ in the sense that an instanton configuration of order $n$ describes a trajectory in which the quantal particle returns to the original minimum $n$ times.

To calculate the potential between instantons and the normalization of the path integral, it is convenient to first calculate the contribution at $\beta$ finite of a trajectory described $n$ times and take the large $\beta$ limit of this expression. One finds

\[
\{ \text{Tr } e^{-\beta H} \}_{(n)} = (-1)^n \beta \frac{1}{n \sqrt{\pi g}} \times \sqrt{\frac{\omega C_\omega}{n (1 + \omega)}} \exp \left( -\frac{\omega \beta}{2 n (1 + \omega)} \right) \exp \left( -\frac{n A(\beta)}{g} \right), \quad (5.2)
\]

with the definitions [see also equations (2.73) and (F.15)]

\[
C_\omega = \frac{q_0^2 \omega^{2/(1+\omega)}}{\exp \left( \frac{2 \omega}{1 + \omega} \left[ \int_0^{q_0} dq \left( \frac{1}{\sqrt{2 V(q)}} - \frac{1}{q} - \frac{1}{\omega (q - q_0)} \right) \right] \right)} . \quad (5.3)
\]

Also,

\[
A(\beta) = 2 \int_0^{q_0} \sqrt{2 V(q)} \, dq - 2 C_\omega \frac{(1 + \omega)}{\omega} e^{-(\beta/n) \omega/(1+\omega)} + \ldots . \quad (5.4)
\]

Note that $n$ has not the same meaning here as in chapter 4. Since $\omega$ is different from 1, the partition function may be described, in the path-integral formalism, by trajectories that return to the starting point. Thus, the “one-instanton”—in the sense of chapter 4.4.1—configuration does not contribute to the path integral, and $n$ instead
counts the number of instanton anti-instanton pairs in the language of chapter 4. That is to say, the natural “one-instanton” configuration in the asymmetric potential corresponds to the “two-instanton” trajectory in the conventions of chapter 4. Therefore, \( n \) here actually corresponds to \( 2n \) in the limit \( \omega \to 1 \).

### 5.2 The \( n \)-Instanton Action

We now call \( \theta_i \) the successive amounts of time the classical trajectory spends near \( q_0 \), and \( \varphi_i \) near the origin. The \( n \)-instanton action then takes the form

\[
A(\theta_i, \varphi_j) = n a - 2 \sum_{i=1}^{n} \left( C_1 e^{-\omega \theta_i} + C_2 e^{-\varphi_i} \right),
\]

with \( \sum_{i=1}^{n} (\theta_i + \varphi_i) = \beta \) and [in agreement with (2.60)]

\[
a = 2 \int_{0}^{q_0} \sqrt{2V(q)} \, dq.
\]

By comparing the value of the action at the saddle point,

\[
\theta_i = \frac{\beta}{n(1 + \omega)}, \quad \varphi_i = \frac{\omega \beta}{n(1 + \omega)},
\]

with the expression (5.4), one verifies that one can choose [see also (2.64)]

\[
C_1 = C_\omega, \quad C_2 = C_\omega/\omega;
\]

by adjusting the definitions of \( \theta \) and \( \varphi \).

### 5.3 The \( n \)-Instanton Contribution to the Partition Function

The \( n \)-instanton contribution to the partition function, in the case of asymmetric wells, has the form

\[
\{ \text{Tr} e^{-\beta H} \}_n = \beta e^{-\beta/2} \frac{e^{-na/g}}{(\pi g)^{n/2}} N_n \int_{\theta_i, \varphi_i \geq 0} \delta \left( \sum_{i} \theta_i + \varphi_i - \beta \right)
\times \exp \left[ -\frac{1}{2} (1 - \omega) \theta_i - \frac{1}{g} A(\theta, \varphi) \right].
\]

The additional term \( \sum_{i=1}^{n} \frac{1}{2} (1 - \omega) \theta_i \) in the integrand comes from the determinant generated by the Gaussian integration around the classical path. The normalization can be obtained by performing a steepest descent integration over the variables \( \theta_i \) and \( \varphi_i \) and comparing the result with expression (5.2). The result is

\[
N_n = \frac{(C_\omega \sqrt{\omega})^n}{n}.
\]

The factor \( 1/n \) comes from the symmetry of the action under cyclic permutations of the \( \theta_i \) and \( \varphi_i \).

We now set

\[
\lambda = \frac{e^{-\alpha/g}}{\pi g} C_\omega \sqrt{\omega}, \quad \mu_1 = -\frac{2C_1}{g}, \quad \mu_2 = -\frac{2C_2}{g};
\]

in such a way that

\[
\frac{e^{-\alpha/g}}{\pi g} C_\omega \sqrt{\omega} = \lambda \omega \sqrt{\mu_1 \mu_2}.
\]
As in chapter 4, we introduce the Laplace transform $G^{(n)}(E)$ of $\{\text{Tr} e^{-\beta H}\}_{(n)}$. It involves now two integrals, 

$$\omega\sqrt{\mu_2} \int_0^{+\infty} \exp \left\{ \left[ \frac{1}{2}(1 - \omega) + E - \frac{1}{2} \right] \theta - \mu_2 e^{-\omega \theta} \right\} d\theta \sim \mathcal{I}(E/\omega - 1/2, \mu_2),$$

(5.13)

where $\mathcal{I}(s, \mu)$ has been defined in (4.47) and evaluated for $\mu \to \infty$ [equation (4.53)]:

$$\mathcal{I}(s, \mu) \sim \mu^{s+1/2} \Gamma(-s).$$

(5.14)

Then, $G^{(n)}(E)$ can be written as

$$G^{(n)}(E) = \frac{1}{n\lambda^n} \frac{\partial}{\partial E} \left[ \mathcal{I}(E - \frac{1}{2}, \mu_1) \mathcal{I}(E/\omega - \frac{1}{2}, \mu_2) \right]^n.$$  

(5.15)

From the sum over $n$, one then infers $\Delta(E)$, the sum of the leading multi-instanton contributions to the Fredholm determinant $\mathcal{D}(E)$. One finds

$$\Delta(E) = \frac{1}{\Gamma(\frac{1}{2} - E/\omega) \Gamma(\frac{1}{2} - E)} + \left( -\frac{2C_\omega}{\omega} \right)^E \left( -\frac{2C_\omega}{\omega g} \right)^{E/\omega} \frac{e^{-a/g}}{2\pi},$$

(5.16)

where now we have added the harmonic oscillator contributions corresponding to the two wells. A generalization of this equation, including effects of higher order in $g$, leads to (2.64).

This equation $\Delta(E) = 0$ has two series of energy eigenvalues, close for $g \to 0$ to the poles

$$E_N = N + \frac{1}{2} + \mathcal{O}(\lambda),$$

$$E_N = (N + \frac{1}{2}) \omega + \mathcal{O}(\lambda),$$

(5.17a)

(5.17b)

of the two $\Gamma$-functions. The same expression contains the instanton contributions to the two different sets of eigenvalues. One verifies that multi-instanton contributions are singular for $\omega = 1$. However, if one directly sets $\omega = 1$ in the expression (5.16), one obtains

$$\Delta(E) = \frac{1}{\Gamma^2(\frac{1}{2} - E)} + \left( -\frac{2C_\omega}{\omega} \right)^{2E} \frac{e^{-a/g}}{2\pi},$$

(5.18)

an expression consistent with the product $\Delta_+ (E) \Delta_- (E)$ of the results (4.60) found for the double-well potential, and likewise consistent with (2.21).
Chapter 6

Instantons in the Periodic Cosine Potential

6.1 Orientation

Analytic periodic potentials lead to additional complications, which the example of the cosine potential illustrates. We consider the Hamiltonian

\[ H = \frac{g}{2} \left( \frac{d}{dq} \right)^2 + \frac{1}{16g} (1 - \cos 4q) , \]  

(6.1)

where the peculiar normalization has been chosen to avoid the proliferation of big integer factors.

The unitary operator \( T \) that translates wave functions by one period \( T = \pi/2 \) of the potential, \( T \psi(q) \equiv \psi(q + T) \),

(6.2)

commutes with the Hamiltonian. Because the adjoint of \( T \) fulfills \( T^+ \psi(q) \equiv \psi(q - T) \), the eigenvalues of \( T \) have unit modulus. Indeed, \( T \) can be diagonalized simultaneously with the Hamiltonian,

\[ T \psi(q) = e^{i\varphi} \psi(q) . \]  

(6.3)

This corresponds to decomposing the initial Hilbert space \( \mathcal{H} \) into a sum of spaces \( \mathcal{H}_\varphi \). The coordinate \( q \) can then be restricted to one period or more conveniently considered as an angular variable parameterizing a circle.

In \( \mathcal{H}_\varphi \) the Hamiltonian has a discrete spectrum with eigenvalues \( E_N(g, \varphi) \) and eigenfunctions \( \psi_{N,\varphi}(q) \):

\[ T \psi_{N,\varphi} = e^{i\varphi} \psi_{N,\varphi} , \quad H \psi_{N,\varphi} = E_N(g, \varphi) \psi_{N,\varphi} . \]  

(6.4)

The eigenvalues \( E_N(g, \varphi) \) are periodic functions of the angle \( \varphi \) and \( E_N(g, \varphi) = N + \frac{1}{2} + \mathcal{O}(g) \).

To each state of the harmonic oscillator is thus associated in \( \mathcal{H} \), for \( g \) small, a band when \( \varphi \) varies.

6.2 The Partition Function in the \( \varphi \)-Sector

We now introduce the partition function in \( \mathcal{H}_\varphi \):

\[ Z(\beta, g, \varphi) = \sum_N e^{-\beta E_N(g, \varphi)} . \]  

(6.5)
In the initial Hilbert space $\mathcal{H}$ we also define the quantity
\[ Z_l(\beta, g) = \text{Tr} \left( T^l e^{-\beta H} \right) \]
\[ = \int_0^T dq \langle q | T^l e^{-\beta H} | q \rangle = \int_0^T dq \langle q + l T | e^{-\beta H} | q \rangle \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \sum_N \sum_{l=-\infty}^{+\infty} e^{-\beta E_N(g,\varphi)} \int dq \psi^*_N(q + lT) \psi_N(q) \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} d\varphi \sum_N \sum_{l=-\infty}^{+\infty} e^{-\beta E_N(g,\varphi)} e^{-i l \varphi} \quad (6.6) \]

Inverting this last relation, one finds
\[ Z_l(\beta, g, \varphi) = \sum_{l=-\infty}^{+\infty} e^{i l \varphi} Z_l(\beta, g). \quad (6.7) \]
This is a generalized partition function with twisted boundary conditions, which depends on the rotation angle $\varphi$.

The path integral representation of $Z_l(\beta, g)$ can be written as
\[ Z_l(\beta, g) = \int [dq(t)] \exp \left[ -\frac{S(q)}{g} \right] \quad (6.8) \]
with
\[ S(q) = \int_{-\beta/2}^{+\beta/2} dt \left( \frac{1}{2} \dot{q}(t)^2 + \frac{1}{16} (1 - \cos 4q) \right). \quad (6.9) \]

The integration variable $q$ parameterizes a circle and the boundary condition is periodic with the constraint that $q(t)$ belongs to the topological sector of trajectories turning $l$ times around the circle.

Note that since
\[ i l \varphi = \frac{2i}{\pi} \varphi (q(\beta/2) - q(-\beta/2)) = \frac{2i}{\pi} \varphi \int_{-\beta/2}^{+\beta/2} dt \dot{q}(t), \quad (6.10) \]
the factor $\exp[i l \varphi]$ can be incorporated into the path integral by adding a topological term, the integral of a local density, to the action:
\[ \frac{S(q)}{g} \rightarrow \frac{S(q)}{g} - \frac{2i}{\pi} \varphi \int_{-\beta/2}^{+\beta/2} dt \dot{q}(t). \quad (6.11) \]

The sum (6.7) can thus be written as
\[ Z(\beta, g, \varphi) = \int [dq(t)] \exp \left[ -\frac{S(q)}{g} + \frac{2i}{\pi} \varphi \int_{-\beta/2}^{+\beta/2} dt \dot{q}(t) \right] \quad (6.12) \]
with now unrestricted periodic boundary conditions on the circle: $q(\beta/2) = q(-\beta/2) \mod \pi/2$.

### 6.3 Perturbation Theory and Instantons

In an expansion for $\beta \to \infty$, due to the boundary conditions in the expression (6.9), $Z_0(\beta, g)$ is dominated by the perturbative expansion and $Z_l(\beta, g)$ for $l \neq 0$ by instantons: in a band all eigenvalues are degenerate to all orders in perturbation theory. Only for $l = \pm 1$ do instantons correspond to solutions of the classical equation of motion:
\[ q_c(t) = \arctan e^{\pm t} \Rightarrow S(q_c) = \frac{1}{2}. \quad (6.13) \]
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6.4 Multi–Instantons

For $|l| > 1$, $Z_l(\beta, g)$ is dominated by multi-instantons.

The energy eigenvalue $E_N(g, \varphi)$ can be expanded in a Fourier series:

$$E_N(g, \varphi) = \sum_{l=-\infty}^{+\infty} E_{N,l}(g) e^{i l \varphi}, \quad E_{N,l} = E_{N,-l}. \quad (6.14)$$

For $g$ small, $E_{N,l}(g)$ is dominated by $l$-instanton contributions. In particular, for the ground state energy $E_0(g, \varphi)$ in the $\varphi$ sector, the $l = 1$ term behaves like

$$E_{0,l=1}(g) \sim \frac{1}{\sqrt{\pi g}} e^{-1/2g}. \quad (6.15)$$

6.4 Multi–Instantons

There is one important difference between the double well and the cosine potentials. In the case of the double well potential, each configuration is a succession of alternatively instantons and anti-instantons. Here, by contrast, the paths consist in an arbitrary succession of turns in the positive and the negative direction, that is an arbitrary succession of instantons and anti-instantons. Therefore, we assign a sign $\varepsilon = +1$ to an instanton and a sign $\varepsilon = -1$ to an anti-instanton. A straightforward calculation, similar to the calculation presented above (for details see Appendix C.2) yields the following interaction term between two consecutive instantons of types $\varepsilon_1$ and $\varepsilon_2$ separated by a distance $\theta_{12}$:

$$\frac{2 \varepsilon_1 \varepsilon_2}{g} e^{-\theta_{12}}. \quad (6.16)$$

The interaction between instantons is repulsive, while it is attractive between instantons and anti-instantons.

We redefine the parameters $\lambda$ and $\mu$ previously introduced in equation (4.42) in the context of the double-well problem,

$$\lambda = \frac{1}{\sqrt{2\pi}} e^{-1/2g}, \quad \mu = \frac{2}{g}. \quad (6.17)$$

The one-instanton contribution at leading order can then be written as

$$\frac{1}{\sqrt{\pi g}} e^{-1/2g} = \lambda \sqrt{\mu}. \quad (6.18)$$

With this notation, the $n$-instanton contribution reads

$$Z^{(n)}(\beta, g, \varphi) = \beta e^{-\beta/2} \lambda^n \int_{\theta_i \geq 0} \delta \left( \sum_{i=1}^{n} \theta_i - \beta \right) \frac{J_n(\theta)}{\lambda^{(1/2 - E)}} \quad (6.19)$$

with

$$J_n(\theta) = \sum_{\varepsilon_i = \pm 1} \exp \left( \sum_{i=1}^{n} \frac{2}{g} \varepsilon_i \varepsilon_{i+1} e^{-\theta_i} + i \varepsilon_i \varphi \right). \quad (6.20)$$

The additional term $i \varepsilon_i \varphi$ comes from the topological term in the expression (6.12). We have identified $\varepsilon_{n+1}$ and $\varepsilon_1$.

Since the interaction between instantons contains both attractive and repulsive contributions, we sum series with $g$ imaginary and then perform the analytic continuation of both the Borel sums and the instanton contributions.

Following the same steps as in the case of the double-well potential, we introduce the contribution $G^{(n)}(E, g, \varphi)$ to the resolvent, Laplace transform of $Z^{(n)}(\beta, g, \varphi)$. The integral over the $\theta_i$ again involves only the function (4.47), which we replace by its asymptotic form (4.53). We obtain

$$G^{(n)}(E, g, \varphi) = \frac{\lambda^n}{n} \frac{\partial}{\partial E} \left( \left[ \Gamma \left( \frac{1}{2} - E \right) \right]^{n} \mu^{nE} \right. \times \sum_{\{\varepsilon_i = \pm 1\}} \exp \left[ \sum_{i=1}^{n} i \varepsilon_i \varphi + (E - E_i) \ln (\varepsilon_i \varepsilon_{i+1}) \right], \quad (6.21)$$
We supply here a few more terms in comparison to (2.81), The spectral condition (6.29) is fully compatible with (2.80) at leading order in where the choice in the determination of \( \ln(\varepsilon_i \varepsilon_{i+1}) \) depends on the initial phase of \( g \). We choose
\[
\ln(\varepsilon_i \varepsilon_{i+1}) = -\frac{1}{2} i \pi \left(1 - \varepsilon_i \varepsilon_{i+1}\right).
\] (6.22)
The expression (6.21) can then be rewritten as
\[
G^{(n)}(E, g, \varphi) \sim \frac{\lambda^n}{n} \frac{\partial}{\partial E} \left[ \Gamma\left(\frac{1}{2} - E\right) \mu^E \right]^n \times \sum_{\{\varepsilon_i = \pm 1\}} \exp \left[ \sum_{i=1}^n i \varepsilon_i \varphi - \frac{1}{2} i \pi \left( E - \frac{1}{2} \right) \left(1 - \varepsilon_i \varepsilon_{i+1}\right) \right].
\] (6.23)
The summation over the set \( \{\varepsilon_i\} \) corresponds to calculating the partition function of a one-dimensional Ising model with the transfer matrix
\[
M = \begin{pmatrix}
e^{i\varphi} & e^{-i\pi (E-1/2)} \\
e^{-i\pi (E-1/2)} & e^{i\varphi}
\end{pmatrix}.
\] (6.24)
The sum then is simply \( \text{Tr} M^n \). The expression (6.23) then becomes
\[
G^{(n)}(E, g, \varphi) \sim \frac{\lambda^n}{n} \frac{\partial}{\partial E} \left[ \Gamma\left(\frac{1}{2} - E\right) \mu^E \right]^n \text{Tr} M^n.
\] (6.25)
The sum \( G(E, g, \varphi) \) of all leading order multi-instanton contributions can now be calculated. One finds (using \( \ln \det = \text{Tr} \ln \))
\[
G(E, g, \varphi) = -\frac{\partial}{\partial E} \ln \det \left[ 1 - \lambda \Gamma\left(\frac{1}{2} - E\right) \mu^E M \right].
\] (6.26)
The contribution to the Fredholm determinant \( D(E, \varphi) \) in the \( \varphi \)-sector thus is
\[
\Delta(E, \varphi) = \det \left[ 1 - \lambda \Gamma\left(\frac{1}{2} - E\right) \mu^E - M \right]
= 1 - 2 \lambda \cos \varphi \Gamma\left(\frac{1}{2} - E\right) \mu^E + \lambda^2 \left[ \Gamma\left(\frac{1}{2} - E\right) \right]^2 \mu^{2E} \left[ 1 + e^{-2\pi i E} \right].
\] (6.27)
Again, after addition of the partition function of the harmonic oscillator, the expression becomes
\[
\Delta(E, \varphi) = \frac{1}{\Gamma\left(\frac{1}{2} - E\right)} - \frac{2 \cos \varphi}{\sqrt{2\pi}} \left( \frac{2}{g} \right)^E e^{-1/2g} \left( -\frac{2}{g} \right)^E e^{-1/2g} \frac{\Gamma\left(\frac{1}{2} + E\right)}{\Gamma\left(\frac{1}{2} - E\right)}.
\] (6.28)
Remarkably enough, the equation \( \Delta(E, \varphi) = 0 \) can also be written in a form that is symmetric in the exchange \( g, E \leftrightarrow -g, -E \):
\[
\left( \frac{2}{g} \right)^{-E} \frac{E^{1/2g}}{\Gamma\left(\frac{1}{2} - E\right)} + \left( \frac{2}{g} \right)^E \frac{E^{-1/2g}}{\Gamma\left(\frac{1}{2} + E\right)} = \frac{2 \cos \varphi}{\sqrt{2\pi}},
\] (6.29)
a property that depends explicitly on the normalization of the one-instanton contribution. This symmetry, however, is slightly fictitious because the equation is actually quadratic in \( \Gamma\left(\frac{1}{2} - E\right) \) and only one root is relevant for \( g > 0 \). The spectral condition (6.29) is fully compatible with (2.80) at leading order in \( g \). Indeed, the generalization to higher orders in \( g \) is
\[
\left( \frac{2}{g} \right)^{-B_{pc}(E, g)} \frac{e^{A_{pc}(E, g)/2}}{\Gamma\left(\frac{1}{2} - B_{pc}(E, g)\right)} + \left( \frac{2}{g} \right)^{B_{pc}(E, g)} \frac{e^{-A_{pc}(E, g)/2}}{\Gamma\left(\frac{1}{2} + B_{pc}(E, g)\right)} = \frac{2 \cos \varphi}{\sqrt{2\pi}}.
\] (6.30)
We supply here a few more terms in comparison to (2.81),
\[
B_{pc}(E, g) = E + \left( E^2 + \frac{1}{4} \right) g
+ \left( 3 E^3 + \frac{5}{4} E \right) g^2 + \left( \frac{25}{2} E^4 + \frac{35}{4} E^2 + \frac{17}{32} \right) g^3
+ \left( \frac{245}{4} E^5 + \frac{525}{8} E^3 + \frac{721}{64} E \right) g^4 + O(g^5).
\] (6.31)
The first few terms of the expansion of the instanton $A_{pc}$-function are:

$$
A_{\nu}(E, g, j) = \frac{1}{g} + \left( 3 E^2 + \frac{3}{4} \right) g \\
+ \left( 11 E^3 + \frac{23}{4} E \right) g^2 + \left( -\frac{199}{4} E^4 + \frac{341}{8} E^2 + \frac{215}{64} \right) g^3 \\
+ \left( \frac{1021}{4} E^5 + 326 E^3 + \frac{4487}{64} E \right) g^4 + O(g^5). \tag{6.32}
$$
Chapter 7

Instantons in Radially Symmetric Oscillators

7.1 Orientation

It is interesting to consider a last example, the analytic continuation of the energy eigenvalues of the $O(\nu)$ symmetric anharmonic oscillator corresponding to the Hamiltonian

$$ H = -\frac{1}{2} \nabla^2 + \frac{1}{2} q^2 + g (q^2)^2 $$

from $g > 0$ to $g < 0$. The radial one-dimensional Hamiltonian is given in (2.84),

$$ H_l(g) = -\frac{1}{2} \left( \frac{d}{dr} \right)^2 - \frac{1}{2} \nu - 1 \frac{d}{dr} + \frac{1}{2} \frac{l (l + \nu - 2)}{r^2} + \frac{1}{2} r^2 + g r^4. $$

For $g > 0$, the potential is bound from below, and there are no degenerate minima. Therefore, the quantization condition for $g > 0$, which is given in (2.101), can be considered as rather trivial: only the perturbative $B$-function enters, and since there are no degenerate minima, there are no nontrivial saddle points of the Euclidean action, and thus no instantons to consider.

For $g < 0$, there are two possibilities: one may either endow the Hamiltonian with a self-adjoint extension, as discussed in chapter 2.2.4 and below in chapter 7.2, or one may consider resonances, as in chapter 2.2.5 and below in 7.3.

7.2 Self–Adjoint Extension

7.2.1 Double–Well, Symmetry Breaking and $O(\nu)$–Potentials

One particular aspect facilitates considerably the analysis of the self-adjoint extension of (7.2): There is a general connection [3–6] between the double-well potential with an additional symmetry breaking term (“broken-double-well”) on the one side and anharmonic oscillators with $O(\nu)$-symmetry on the other side (acting in $\mathbb{R}^\nu$). Typically, the sign convention in this investigation is chosen such that the double-well potential is bounded from below, and the anharmonic $O(\nu)$-symmetric oscillator is formulated for $g < 0$. We here briefly explain the relevant method and recall the radial Hamiltonian (2.84) with negative coupling

$$ H_l(-g) = -\frac{1}{2} \left( \frac{d}{dr} \right)^2 - \frac{1}{2} \nu - 1 \frac{d}{dr} + \frac{1}{2} \frac{l (l + \nu - 2)}{r^2} + \frac{1}{2} r^2 - g r^4. $$

For illustrational purposes, we have explicitly replaced $g \rightarrow -g$ in comparison to (2.84). The potential, in this convention, is not bounded from below for a (redefined) positive $g$. 

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7.2 Self–Adjoint Extension

We now try to find a self-adjoint extension of this Hamiltonian; although the coupling is negative, a self-adjoint extension implies that the spectrum will be real. The Hamiltonian (7.3) leads to a radial Schrödinger equation

\[ -\psi''(r) - \frac{\nu - 1}{r} \psi'(r) + \frac{l(l + \nu - 2)}{r^2} \psi(r) + (r^2 - 2gr^4) \psi(r) = 2E \psi(r). \quad (7.4) \]

In order to show the connection between the “broken-double-well” potential and the anharmonic oscillator, we use here a derivation based on differential equations, but the same results can be obtained by path integral methods (for an overview of related issues see [4]). We consider first the more general Hamiltonian (3.78)

\[ H = \frac{g}{2} \left[ -\left( \frac{d}{dr} \right)^2 - \frac{\nu - 1}{r} \frac{d}{dr} + \frac{l(l + \nu - 2)}{r^2} \right] + \frac{1}{g} V(r). \quad (7.5) \]

We have used here the scaling \( x \rightarrow x/\sqrt{g} \). In the case of (2.84), we would have \( V(r) = r^2/2 + r^4 \). Because \( V(r) \) is an even function, we now set

\[ V(r) = W(r^2) = \frac{r^2}{2} + \mathcal{O}(r^4). \quad (7.6) \]

We first eliminate the \( 1/r^2 \) term by the change of variables

\[ \psi(r) = r^l \chi(r), \quad (7.7) \]

and obtain

\[ -\frac{g}{2} \left[ \chi''(r) - \frac{2j + 1}{r} \chi'(r) \right] + \frac{1}{g} W(r^2) \chi(r) = E \chi(r), \quad (7.8) \]

where we have set [see also (2.87)]

\[ j = l + \nu/2 - 1. \quad (7.9) \]

We then take \( r^2 = x \) as a new variable,

\[ \chi(r) = \xi(r^2), \quad (7.10) \]

and find

\[ -g \left[ 2x \xi''(x) + 2(j + 1) \xi'(x) \right] + \frac{1}{g} W(x) \xi(x) = E \xi(x). \quad (7.11) \]

We now write \( \xi \) as a Laplace transform

\[ \xi(x) = \int dp e^{px/\sqrt{g}} \eta(p). \quad (7.12) \]

With proper boundary conditions, the equation becomes

\[ \frac{1}{g} \left[ W \left( -g \frac{d}{dp} \right) + \frac{g}{2} \frac{d}{dp} \right] \eta(p) + \frac{1}{2} \left( 4p^2 - 1 \right) \eta'(p) - 2(j - 1) p \eta(p) = E \eta(p). \quad (7.13) \]

In the special example \( W(x) = x/2 - x^2 \) given in (7.4), to which we restrict the discussion in the sequel, the equation is again a second-order differential equation. After a last transformation to eliminate the \( \eta' \) term,

\[ \eta(p) = a(p) \varphi(p), \quad \text{with} \quad \frac{d}{a} = \frac{1 - 4p^2}{4g}, \quad (7.14) \]

one finally obtains

\[ -\frac{g}{2} \varphi''(p) + \frac{1}{2g} \left( p^2 - \frac{1}{4} \right)^2 \varphi(p) - j p \varphi(p) = \frac{E}{2} \varphi(p). \quad (7.15) \]

This is equivalent to the Schrödinger equation for a particle moving in a double-well potential with minima and \( p \pm 1/2 \), and an additional symmetry-breaking term \( j p \). The equation has been derived by appropriate substitutions in (7.4) and naturally provides a self-adjoint extension of the Hamiltonian (7.3). This concludes the derivation of the correspondence between the anharmonic oscillator at negative coupling and the double-well potential with a linear breaking term, which is expressed by the equations (2.90a) and (2.90b).
For \( j = 0 \), the symmetry-breaking term vanishes. In this case, an comparison of (7.15) with (2.10) leads one automatically to the \( A \) and \( B \) functions of the transformed problem (7.15). These have to be the same as those for the \( O(\nu) \) anharmonic oscillator obtained in (2.89a) and (2.89b) as we set \( j = 0 \). We thus recover the correspondence between the double-well potential on the one side and the anharmonic oscillator with radial symmetry, considered in the case of negative coupling and endowed with a self-adjoint extension, on the other side,

\[
B_{\nu}(E, g, j = 0) = 2B_{dw}(E/2, -g), \quad A_{\nu}(E, g, j = 0) = A_{dw}(E/2, -g).
\] (7.16)

This has already been mentioned in chapter 2.2.4 [see equation (2.91)]. This correspondence is valid on the perturbative level (\( B \) function) as well as on the level of instanton effects (\( A \) function). For \( j \neq 0 \), the reflection symmetry \( p \rightarrow -p \) is broken at the same order as the first quantum correction, so that the degeneracy is lifted.

### 7.2.2 Degenerate Minima and Symmetry Breaking

We now consider a slight generalization of the situation encountered in the Fokker–Planck equation (2.2.4), where the potential has a symmetric structure broken at relative order \( g \), and may allow for a vanishing perturbative expansion. Indeed, in several situations, for instance in the case \( j = \pm 1 \) in (7.15), the potential may have the typical form [we assume the general structure (2.10a) for the Hamiltonian]

\[
V_{\text{tot}}(q) = V(q) + gV_{\text{pert}}(q),
\] (7.17)

where the form \( V \) is a symmetric potential \([V(q) = V(q_0 - q)]\) with degenerate minima and \( V_{\text{pert}}(q) \) breaks the symmetry at order \( g \). When the breaking term is treated at leading order, it is only the difference between the values at the minima of the symmetric potential which is important. Therefore, as a simplifying feature but without loss of generality at leading order, we consider the example of a linear symmetry breaking potential

\[
V_{\text{tot}}(q) = V(q) + g \eta \frac{q}{q_0},
\] (7.18)

where we assume \( V(0) = 0, \eta > 0 \).

The analysis is, in many respects, analogous to the calculations of chapter 5.3 (see also reference [2]). Eigenvalues are only degenerate at leading order, and the degeneracy is lifted at order \( g \), the ground state corresponding to the well at \( q = 0 \). At leading order, the instanton contributions can be derived from chapter 5 with \( \omega = 1 \). Again, we call \( \varphi_i \) the times spent near \( q = 0 \) and \( \theta_i \) near \( q = q_0 \). Then, the \( n \)-instanton action reads

\[
A(\theta_i, \varphi_j) = n a - 2C \sum_{i=1}^{n} \left( e^{-\theta_i} + e^{-\varphi_i} \right),
\] (7.19)

with \( \sum_{i=1}^{n} (\theta_i + \varphi_i) = \beta \) and [see also (2.58) and (2.60)]

\[
a = 2 \int_{0}^{q_0} \sqrt{2V(q)} \, dq, \quad (7.20a)
\]

\[
C = q_0^2 \exp \left( \frac{1}{\sqrt{2V(q_0)}} - \frac{1}{q_0} - \frac{1}{q_0 - q} \right). \quad (7.20b)
\]

The \( n \)-instanton contribution then has the form

\[
\{ \text{Tr} e^{-B_{H}} \}_{(n)} = \frac{\beta}{n} e^{-\beta/2} \left( \frac{C}{\pi g} \right)^n e^{-na/g} \int_{\theta_i, \varphi_i \geq 0} \delta \left( \sum_i \theta_i + \varphi_i - \beta \right) \times \exp \left[ -\eta \sum_{i=1}^{n} \theta_i - \frac{1}{g} A(\theta, \varphi) \right].
\] (7.21)

The additional term \(-\eta \sum_{i} \theta_i\) in the integrand comes from the value of the potential at \( q = q_0 \).
The only required integral [equation (4.47)] has already been evaluated. The sum of leading order multi-instanton contributions reads

\[ \Delta(E) = \frac{1}{\Gamma(-E) \Gamma(1-E)} + \left( -\frac{2C}{g} \right)^{2E-\eta} g^{-\eta/2} \frac{e^{-a/g}}{2\pi}. \]  

(7.22)

Note that when \( \eta \) is a positive integer, the one-instanton contribution to the states \( E_N = N + \frac{1}{2} \) is real, indicating that the behaviour of the perturbative expansion at large order \( k \) is at least smaller by a factor \( 1/k \) than naively expected. For \( N < \eta \), the instanton contribution is of order \( e^{-a/g} \), while for \( N \geq \eta \) it is of order \( e^{-a/2g} \).

### 7.2.3 The Fokker–Planck Hamiltonian

A simple example illustrating the remarks of chapter 7.2.2 is provided by the Fokker–Planck Hamiltonian (see chapter 2.2.4): the stationary solution, as shown below, is not normalizable, and instanton effects determine the energy of the ground state [10].

We recall the Riccati equation (3.14),

\[ g S'(q) - S^2(q) + 2 V(q) - 2 g E = 0. \]  

(7.23)

This equation formally allows for a solution with \( E = 0 \) if the potential \( V(q) \) has the following structure,

\[ V(q) = \frac{1}{2} [U^2(q) - g U'(q)]. \]  

(7.24)

Indeed, this structure automatically leads to a class of potentials for which the perturbative expansion of at least one eigenvalue vanishes identically to all orders in the coupling. We assume \( U(q) \) to be a polynomial that has two zeros at \( q = 0 \) and \( q = q_0 > 0 \), such that

\[ U(q) = q + \mathcal{O}(q^2), \quad U(q_0 - q) = U(q), \]  

(7.25)

and assume that it is an exact solution of the Riccati equation (3.14) with \( E = 0 \).

We now specialize the treatment to the Fokker–Planck potential

\[ V_{FP}(p) = \frac{1}{2} \left( p^2 - \frac{1}{4} \right)^2 + g p, \]  

(7.26)

which is obtained in a natural way by setting \( j = -1 \) in (7.15). In a perturbative expansion around the well \( p = -\frac{1}{2} \), one finds a ground state with \( E = 0 \) to all orders. The equation may alternatively be written as \( (p = q - 1/2) \),

\[ V_{FP}(q = p + \frac{1}{2}) = \frac{1}{2} q^2 (1 - q)^2 + g \left( q - \frac{1}{2} \right). \]  

(7.27)

At leading order in \( g \), \( V(q) \) is symmetric with degenerate minima. The breaking term lifts the degeneracy at order \( g \) and implies that the perturbative ground state, corresponding to the well at \( q = 0 \), has \( E = 0 \) while the lowest state in the other well has \( E = 1 \).

In fact the issue is more complicated because the wave function

\[ \psi(q) = \exp \left[ -\frac{1}{g} \int^q dq' U(q') \right] = \exp \left[ \frac{1}{g} \left( \frac{q^3}{3} - \frac{q^2}{2} \right) \right] \propto \exp \left[ \frac{1}{g} \left( \frac{1}{3} p^3 - \frac{1}{4} p \right) \right] \]  

(7.28)

is not normalizable, and thus is not an eigenfunction. An analogy with the Fokker–Planck equation suggests that the case \( E = 0 \) be identified with an equilibrium probability distribution. Therefore, the non-normalizable wave function (7.28) may naturally be identified with a “pseudo-equilibrium” distribution.

The true ground state has \( E > 0 \) and at leading order for \( g \rightarrow 0 \) is dominated by the one-instanton contribution. Shifting \( E \) in the expression (7.22) as \( E \rightarrow E + 1/2 \) and setting \( \eta = 1 \), one finds

\[ \Delta(E) = \frac{1}{\Gamma(-E) \Gamma(1-E)} + \left( -\frac{2C}{g} \right)^{2E-\eta} g^{-\eta/2} \frac{e^{-a/g}}{2\pi}, \]  

(7.29)
the generalization being (2.96)
\[ \frac{1}{\Gamma(-B_{FP}(E,g)) \Gamma(1 - B_{FP}(E,g))} + \left(\frac{-2}{g}\right)^{2B_{FP}(E,g)} \exp\left(-\frac{A_{FP}(E,g)}{2\pi}\right) = 0. \] (7.30)

At leading order, for the ground state,
\[ E_0(g) \sim \frac{e^{-1/3g}}{2\pi}. \] (7.31)

An imaginary part appears only at two-instanton order and governs the large-order behaviour of the non-Borel summable one-instanton expansion, where for the classification of the instanton order we follow the convention as outlined in chapter 5.1. The arguments presented here can be easily generalized to the non-symmetric situation.

The perturbation series, in the Fokker–Planck potential, vanishes to all orders for the ground state only. For excited states, the leading-order as well as corrections of relative order \( g, g^2, g^3, \ldots \) are nonvanishing. The following statements illustrate this phenomenon. First, we supplement the perturbative \( B_{FP} \)-function given in (2.97) by an expression valid up to the order \( g^4 \),
\[ B_{FP}(E,g) = E + 3E^2g + \left(35E^3 + \frac{5}{2}E\right)g^2 + \left(\frac{1155}{2}E^4 + 105E^2\right)g^3 + \left(\frac{45045}{4}E^5 + \frac{15015}{4}E^3 + \frac{1155}{8}E\right)g^4. \] (7.32)

Inverting the perturbative quantization condition (2.97) for general \( N \), we obtain the following perturbative expansion up to \( O(g^4) \),
\[ E_N(g) \sim N - 3N^2g - \left(17N^3 + \frac{5}{2}N\right)g^2 - \left(\frac{375}{2}N^4 + 165N^2\right)g^3 - \left(\frac{10689}{4}N^5 + \frac{9475}{4}N^3 + \frac{1105}{8}N\right)g^4. \] (7.34)

For the ground state \((N = 0)\), all the terms vanish, whereas for excited states with \( N = 1, 2, \ldots \), the perturbation series is manifestly nonvanishing. For completeness, we also supplement the instanton \( A_{FP} \)-function (2.99b) for the Fokker–Planck potential by a few more terms,
\[ A_{FP}(E,g) = \frac{1}{3g} + \left(17E^2 + \frac{5}{6}E\right)g + \left(\frac{47431}{12}E^4 + \frac{11485}{12}E^2 + \frac{1105}{72}E\right)g^2 + \left(\frac{317629}{4}E^5 + \frac{64535}{2}E^3 + \frac{4109}{2}E\right)g^3 + O(g^5). \] (7.35)

The functions \( B_{FP} \) and \( A_{FP} \) determine the perturbative expansion, and the perturbative expansion about the instantons, in higher order (see also chapter 8 for an application of related ideas to the double-well problem).

(Remark.) When investigating a slight generalization of the potential (7.26), namely [see also (2.90b)]
\[ V_{FP}(p) = \frac{1}{2}p^2 - \frac{1}{4} + gp, \] (7.36)

it may be shown that for negative integer \( j \), the perturbation series of at least one energy level terminates. Assuming that the solution of equation (7.11) is regular at \( x = 0 \), we expand it, as well as the potential, in a Taylor series, after changing variables \( x \mapsto gx \),
\[ W(x) = \sum_{n=1} W_n g^n x^n, \quad \xi(x) = \sum_{n=0} \xi_n g^n x^n. \] (7.37)
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We then obtain the recursion relations

\[-2 g (n + 1) (j + 1 + n) \xi_{n+1} + \sum_{p=0}^{n} g^{n-p-1} W_{n-p} \xi_p = E \xi_n.\]  \hspace{1cm} (7.38)

If \( j \) is a negative integer: \( j = -N - 1 \) (with \( N \geq 0 \)), the set of linear equations with \( 0 \leq n \leq N \) is closed. *Idem est*, the coefficient \( j + 1 + n \) vanishes in this case for \( n = N \), and \( \xi_n = 0 \) for all \( n > N \). Alternatively, one observes that the determinant of the \([N + 1] \times [N + 1]\) system of linear equations defined by equation (7.38) has to vanish, a condition that determines \( E \) as a solution of an algebraic equation of degree \( N - 1 \). This property reflects the equivalence of \( O(\nu) \) models with \( \nu \) even and negative with fermion Hamiltonians. For \( N = 0 \) and thus \( j = -1 \), one recovers the known result

\[ E = 0, \]  \hspace{1cm} (7.39)

which holds in a strict sense only on the level of perturbation theory [see also equation (7.31)]. For \( j = -2 \), one finds \( E = \pm 1 \). The instanton corrections follow directly from equations (7.22) and (7.29). As an illustration, at next order, and thus, for \( j = -3 \), one finds

\[ E^3 - 4 E - 8 g W''(0) = 0, \]  \hspace{1cm} (7.40)

an equation that has three solutions for small \( g \). Also, for \( g = 0 \), the solutions are \( E = 1 - |j| + 2k \) with \( k \) integer, \( 0 \leq k \leq |j| - 1 \). We conclude this chapter by noting that the Hamiltonian (7.15) implies a spectral symmetry with respect to the sign change \( j \rightarrow -j \). However, here this symmetry is broken in the discussion following (7.38), because the potential (7.26) has a unique global minimum at \( j = -1/2 \), not \( j = +1/2 \) [analogous statements hold for the generalization (7.36)]. For this reason, the system of equations is closed for negative integer \( j \) (but the closure does not hold for positive \( j \)).

For a more detailed treatment of the intriguing aspects related to the Fokker–Planck potential, the reader is referred to [11].

7.2.4 \( O(\nu) \)--Symmetric Quartic Potentials

We have related in chapter 7.2.1 perturbative expansions of \( O(\nu) \) symmetric quartic potentials and double-well potentials with linear symmetry breaking. The additional contribution to the potential \(-j\beta H\) has the effect of adding a contribution \( \pm j/2 \) to the action depending whether the instanton is close to \( 1/2 \) or \(-1/2 \), respectively [12]. We now call \( \theta_i \) the successive amounts of time the classical trajectory spends near \(-1/2 \), and \( \varphi_i \) near \(1/2 \). The \( n \)-instanton action then takes the form

\[ A(\theta_i, \varphi_j) = \frac{n}{3} - 2 \sum_{i=1}^{n} (e^{-\theta_i} + e^{-\varphi_i}), \]  \hspace{1cm} (7.41)

with \( \sum_{i=1}^{n} (\theta_i + \varphi_i) = \beta \). We set

\[ \lambda = \frac{e^{-1/\beta g}}{2\pi}, \quad \mu = -\frac{2}{g}. \]  \hspace{1cm} (7.42)

The \( n \)-instanton contribution then takes the form

\[ \{ \text{Tr} \ e^{-\beta H} \}_{(n)} = \frac{\beta}{n} e^{-\beta/2} (-\lambda \mu)^n \int_{0, \varphi \geq 0} \delta \left( \sum_{i} \theta_i + \varphi_i - \beta \right) \]

\[ \times \exp \left[ \sum_{i=1}^{n} \frac{1}{2} j (\theta_i - \varphi_i) - \frac{1}{g} A(\theta, \varphi) \right]. \]  \hspace{1cm} (7.43)

The sum then involves the integrals (taking into account \( E \rightarrow E/2 \))

\[ \sqrt{n} \int_{0}^{+\infty} \exp \left\{ \left( \pm \frac{j}{2} \xi + \frac{1}{2} E - \frac{1}{2} \right) \vartheta - \mu e^{-\vartheta} \right\} d\vartheta = \mathcal{I} \left( \frac{1}{2} (\pm j + E - 1), \mu \right). \]  \hspace{1cm} (7.44)
Using the asymptotic form (4.53),
\[ I(s, \mu) \sim \mu^{s+1/2} \Gamma(-s), \]
we obtain the spectral equation
\[ \Delta_j(E) = \frac{1}{\Gamma\left(\frac{1}{2}(1 + j - E)\right) \Gamma\left(\frac{1}{2}(1 - j - E)\right)} + \left(\frac{2}{g}\right) \frac{e^{-1/3g}}{2\pi} = 0. \]
(7.46)
The generalization to higher orders in \( g \) is given by (2.92) and reads
\[ \Gamma\left[\frac{1}{2}(1 + j - B_\nu(E, -g, j))\right] \Gamma\left[\frac{1}{2}(1 - j - B_\nu(E, -g, j))\right] + \left(\frac{2}{g}\right)^{B_\nu(E, -g, j)} \exp\left(-A_\nu(E, -g)\right) = 0. \]
(7.47)
We supplement here a few terms in comparison to (2.89). The perturbative \( B_\nu \)-function reads [see also equation (B.12)]
\[ B_\nu(E, g, j) = E + \left(\frac{3}{2} E^2 + \frac{j^2}{2} - \frac{1}{3}\right) g \]
\[ + \left(\frac{35}{4} E^3 + \frac{25}{4} E - \frac{15}{4} j^2 E\right) g^2 \]
\[ + \left(-\frac{1155}{16} E^4 - \frac{735}{8} E^2 + \frac{315}{8} j^2 E^2 - \frac{35}{16} j^4 + \frac{105}{8} j^2 - \frac{175}{16}\right) g^3 \]
\[ + \left(\frac{45054}{64} E^5 + \frac{45045}{32} E^3 + \frac{31185}{64} E - \frac{15015}{32} j^2 E^3 \right) \]
\[ + \left(\frac{3465}{64} j^4 E - \frac{12705}{32} j^2 E\right) g^4 + O(g^5). \]
(7.48)
The first few terms of the expansion of the instanton \( A_\nu \)-function are:
\[ A_\nu(E, g, j) = -\frac{1}{3} g^{-1} + \left(\frac{3}{4} j^2 - \frac{19}{12} - \frac{17}{4} E^2\right) g \]
\[ + \left(\frac{227}{8} E^3 - \frac{77}{8} j^2 E + \frac{187}{8} E\right) g^2 \]
\[ + \left(\frac{47431}{192} E^4 - \frac{34121}{96} E^2 + \frac{3717}{32} j^2 E^2 - \frac{341}{64} j^4 \right) \]
\[ + \left(\frac{1281}{32} j^2 - \frac{28829}{576}\right) g^3 \]
\[ + \left(\frac{317629}{128} E^5 + \frac{264725}{48} E^3 + \frac{842909}{384} E + \frac{19215}{128} j^3 E \right) \]
\[ + \left(-\frac{4445}{3} j^2 E^3 - \frac{253045}{192} j^2 E\right) g^4 + O(g^5). \]
(7.49)
In (2.89a) and (2.89b), the functions \( B_\nu \) and \( A_\nu \) are given up to terms of the order \( g^2 \), respectively.

### 7.3 Resonances

#### 7.3.1 The \( \mathcal{O}(2) \)-Anharmonic Oscillator

We first discuss resonances in the example case \( \nu = 2 \) (see also chapter 2.2.5), the generalization then being simple. Note that for \( \nu = 2 \), we have \( j = l + \nu/2 - 1 = l \) [see equations (2.87) and (7.9)]. As in chapter 7.2.2, we
extensively use here the ideas outlined in chapter 5.3. For \( g > 0 \), as discussed in chapter 7.1, there are no saddle points of the Euclidean action beyond the trivial one, and thus no instantons to consider. For \( g < 0 \), the instanton solution exists and has the form

\[
q(t) = u f(t),
\]

(7.50)
in which \( u \) is a fixed unit vector. The leading one-instanton contribution to the ground state energy is

\[
\text{Im} E^{(1)}(g) = \frac{4}{g} e^{1/3g} \left( 1 + O(g) \right) \quad \text{for} \ g \to 0_-.
\]

(7.51)

It is easy to calculate the instanton interaction, and thus \( n \)-instanton action

\[
A(\theta_i) = -\frac{1}{3} n - 4 \sum_i e^{-\theta_i} \cos \varphi_i,
\]

(7.52)
in which \( \theta_i \) is the distance between two successive instantons and \( \varphi_i \) the angle between them:

\[
\cos \varphi_i = u_i \cdot u_{i+1}.
\]

(7.53)

It is convenient to consider the quantity

\[
Z(\beta, \alpha) = \text{Tr} \left[ R(\alpha) e^{-\beta H} \right] = \int [dq(t)] \exp \left[ -S(q)/g \right],
\]

(7.54)

where \( R(\alpha) \) is a rotation matrix which rotates vectors by an angle \( \alpha \). In the path integral, it leads to the boundary condition that \( q(t) \) at initial and final times differ by an angle \( \alpha \):

\[
\hat{q}^{-1/2} \beta \cdot \hat{q}^{1/2} = \cos \alpha.
\]

(7.55)

As before, we introduce the Laplace transform of the twisted partition function \( Z(\beta, \alpha) \) defined in (7.54),

\[
G(E, \alpha) = \int_0^\infty d\beta e^{\beta E} \text{Tr} \left[ R(\alpha) e^{-\beta H} \right] = \sum_{l, N} e^{-i l \alpha - \beta E_{l, N}},
\]

(7.56)

where \( l \) is the angular momentum. The boundary condition in the path integral (7.54) implies for the multi-instanton configuration the constraint

\[
\sum_{i=1}^n \varphi_i = \alpha,
\]

(7.57)

As before, we introduce the Laplace transform of the twisted partition function \( Z(\beta, \alpha) \) defined in (7.54),

\[
G(E, \alpha) = \int_0^\infty d\beta e^{\beta E} \text{Tr} \left[ R(\alpha) e^{-\beta H} \right] = \sum_{l, N} e^{-i l \alpha - \beta E_{l, N} - E}.
\]

(7.58)

The \( n \)-instanton contribution to expression (7.55) then takes the form

\[
G^{(n)}(E, \alpha) \sim \frac{\lambda^n}{2^{1/3n}} \sum_{l=-\infty}^{+\infty} e^{-i l \alpha} [I(E - 1, l, \mu)]^n,
\]

(7.59)

where we have again redefined \( \lambda \) and \( \mu \) [cf. equations (4.42) and (6.17)],

\[
\lambda = -i e^{1/3g}, \quad \mu = -\frac{4}{g},
\]

(7.60)

\[
I(s, l, \mu) = \frac{\mu}{2\pi} \int_0^{2\pi} d\varphi \int_0^{+\infty} d\theta \exp \left( s \theta + i l \varphi - \mu e^{-\theta} \cos \varphi \right).
\]

(7.61)
We introduce the generating function of \( n \)-instanton contributions at fixed angular momentum \( l \):
\[
G_l(E, g) = -\frac{\partial}{\partial E} \ln [1 + \lambda I(E - 1, l, \mu)].
\]
(7.62)

To evaluate \( I(s, l, \mu) \), we first integrate over \( \theta \), where we employ the approximations \( \mu \to \infty \) and thus \( g \to 0_- \), as in going from (4.51) to (4.52). We find
\[
I(s, l, \mu) = \mu^{s+1} \Gamma(-s) \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{i\varphi} (\cos \varphi)^s.
\]
(7.63)

The last integration yields
\[
\int_0^{2\pi} \frac{d\varphi}{2\pi} e^{i\varphi} (\cos \varphi)^s = \frac{2^{-s-1}}{s!} \left(1 + (-1)^l e^{i\pi s}\right) \frac{\Gamma(1+s)}{\Gamma(\frac{1}{2}(s-l)+1) \Gamma(\frac{1}{2}(s+l)+1)},
\]
(7.64)

and thus
\[
I(s, l, \mu) = -\left(\frac{\mu}{2}\right)^{s+1} \frac{\pi}{\sin(\pi s)} \frac{1 + (-1)^l e^{i\pi s}}{\Gamma(\frac{1}{2}(s-l)+1) \Gamma(\frac{1}{2}(s+l)+1)}
\]
\[
= \left(\frac{\mu}{2}\right)^{s+1} e^{i\pi(s+l)/2} \frac{\Gamma(\frac{1}{2}(l-s))}{\Gamma(\frac{1}{2}(s+l)+1)},
\]
(7.65)

the first expression being explicitly even in \( l \).

The sum of leading order contributions to the Fredholm determinant at fixed angular momentum \( l \) thus is
\[
\Delta_l(E) = \frac{1}{\Gamma(\frac{1}{2}(l + 1 + E))} - i \left(-\frac{2}{g}\right)^E \frac{e^{i\pi(E+j+1)/2} e^{1/3g}}{\Gamma(\frac{1}{2}(j + 1 + E))}.
\]
(7.66)

The eigenvalues \( E_{l,N} \) are solutions of the equation \( \Delta_l(E) = 0 \) that satisfy
\[
E_{l,N} = l + 2N + 1 + \mathcal{O}(g), \quad N \geq 0.
\]
(7.67)

7.3.2 The \( \mathcal{O}(\nu) \)--Symmetric Hamiltonian

One can extend this result to the general \( \mathcal{O}(\nu) \) case since, at fixed angular momentum \( l \), the Hamiltonian depends only on the combination \( j = l + \nu/2 - 1 \), and thus the generalization of the discussion in the previous chapter simply involves the replacement \( l \to j \). Hence, making in equation (7.66) the corresponding substitution, one obtains
\[
\Delta_j(E) = \frac{1}{\Gamma(\frac{1}{2}(j + 1 + E))} - i \left(-\frac{2}{g}\right)^E \frac{e^{i\pi(E+j+1)/2} e^{1/3g}}{\Gamma(\frac{1}{2}(j + 1 + E))}.
\]
(7.68)

This equation is consistent with (2.102) at leading order in \( g \). The imaginary parts of the energy eigenvalues
\[
E_{j,N} = j + 2N + 1 + \mathcal{O}(g), \quad N \geq 0,
\]
(7.69)

for \( g \to 0_- \) follow:
\[
\text{Im} E_{j,N} \equiv \frac{2}{N! \Gamma(j + 1 + N)} \left(-\frac{2}{g}\right)^{j+1+2N} e^{1/3g} (1 + \mathcal{O}(g)),
\]
(7.70)

in full agreement with the ground-state result (7.51). Using the Cauchy formula, one can then derive from this expression large-order estimates for perturbative expansions [13]. At next order in \( \lambda \), one obtains the two-instanton contribution, which is related by the same dispersion relation to the large-order behaviour of the perturbative expansion around one instanton.

Finally, note that checks about these expressions are provided by the perturbative relation between the \( \mathcal{O}(\nu) \) anharmonic oscillator with negative coupling and the double-well potential with linear symmetry breaking derived in chapter 7.2.2.
Chapter 8

Instanton Calculations in the Double–Well Problem

8.1 Orientation

The purpose of the current chapter consists in the illustration of the general discussion of instantons by concrete calculational example. We choose the quantum-mechanical double-well oscillator with a potential of the form

\[ V(q) = q^2 (1 - q)^2 / 2 \]

as in equation (2.10). The resurgent expansion for the energy of a level with principal quantum number \( N \) and parity \( \epsilon \) is then given by equation (2.13). We remember the nonperturbative factor (2.27)

\[ \xi(g) = \frac{1}{\sqrt{\pi g}} \exp \left[ -\frac{1}{6 g} \right], \tag{8.1} \]

which characterizes the instanton contributions and the logarithm (2.28)

\[ \chi(g) = \ln \left( -\frac{2}{g} \right), \tag{8.2} \]

which generates, for \( g \) positive, imaginary contributions that cancel among the resummed perturbative expansion and the instanton contributions.

The calculations, and even the results which are discussed in the current chapter, have a somewhat involved structure and are rather lengthy. The formulas should be taken \textit{cum grano salis}, exemplifying the application of general concepts discussed in previous chapters to a case of special interest. The lengthy formulas and numerical results (see also Table F.1 below) are not displayed for their own sake; they receive a meaning and an interpretation in the context of the general conjectures introduced and motivated in chapters 2, 3 and 4.

8.2 Corrections to Asymptotics

According to (2.14), the perturbation series for the level \( N \) is independent of the parity \( \epsilon \) and can be written as

\[ E^{(0)}_N(g) = \sum_{K=0}^{\infty} E^{(0)}_{N,K} g^K. \tag{8.3} \]

We have determined the first 300 terms of the perturbative expansion of the ground state(s) with \( N = 0 \ (\epsilon = \pm) \) in closed analytic form (a complete list is available for internet download at [7]). The coefficients are expressed in terms of rational numbers. An example is given in equation (2.7). The first terms read [see also equation (2.6)]:

\[ E^{(0)}_0(g) = \frac{1}{2} - g - \frac{9}{2} g^2 - \frac{89}{8} g^3 - \frac{5013}{8} g^4 - \frac{88251}{8} g^5 - \frac{3662169}{16} g^6 + O(g^7). \tag{8.4} \]
The magnitude of the terms grows factorially, and the series is nonalternating: all coefficients except the first are negative. We had stressed in chapter 2.1.1 that the imaginary part incurred by analytic continuation from $g$ negative is compensated by the explicit imaginary part of the two-instanton contribution to the energy eigenvalues. For the ground state(s) with $N = 0$, the formal expansion of the two-instanton effect as given by equation (2.15) reads

$$E^{(2)}_{e,0}(g) = \frac{e^{-1/3g}}{\sqrt{\pi} g} \left[ \ln \left( \frac{-2}{g} \right) \sum_{K=0}^{\infty} e_{0,211} K g^K + \sum_{K=0}^{\infty} e_{0,212} K g^K \right].$$

The imaginary part (generated by the logarithm for positive $g$) reads

$$\text{Im}E^{(2)}_{e,0}(g) = \pm \frac{e^{-1/3g}}{g} \left[ e_{0,210} + e_{0,211} g + e_{0,212} g^2 + \mathcal{O}(g^3) \right].$$

The leading factorial growth of the perturbative coefficients $E^{(0)}_{0,K}$ for the ground state is known [2] to be of the form $E^{(0)}_{0,K} \sim -3^{K+1} K!/\pi$. By analytic continuation of the perturbation series for positive $g$, an imaginary part of the order of $\exp(-1/3g)$ is obtained. The large-order behaviour of the perturbative coefficients and the imaginary part of the two-instanton energy shift are connected by the (dispersion-type) relations (2.42)—(2.44). Indeed, as discussed in chapter 2.1.3, the power corrections in $g$ [equation (8.6)] are connected with the corrections of order $K^{-1}$ to the perturbative coefficients $E^{(0)}_{0,K}$ (see also appendix D.1).

It is therefore interesting to numerically determine the corrections to the leading factorial growth of the perturbative coefficients. We start with the available results for the first 300 perturbative coefficients and subtract the known leading asymptotics of the form $E^{(0)}_{0,K} \sim -3^{K+1} K!/\pi$, as well as divide the results of the subtraction by the leading asymptotics. An approximation for the coefficient of the correction term of relative order $1/K$ can be found via multiplication of the results of the previous operation by $K$. This approximation can be improved by elimination of power corrections of higher order in $1/K$ via “extrapolation to $K = \infty$.” For this latter step, various algorithms may be used. One possibility consists in the Neville algorithm [14], which is a variant of Richardson extrapolation [15]. The result is again a numerical estimate for the coefficient of the $1/K$-term. The exact, rational form can then be found easily by routines built into modern-day computer algebra systems, for example the Rationalize function of [16]. These operations may be repeated, and conjectures may be found for the coefficients multiplying the corrections of relative order $1/K$ to the factorial growth of the perturbative coefficients. We find

$$E^{(0)}_{0,K} \sim -3^{K+1} K! \left\{ 1 - \frac{53}{18} \frac{1}{K} - \frac{1277}{648} \frac{1}{K^2} - \frac{405395}{34992} \frac{1}{K^3} - \frac{218793923}{2519424} \frac{1}{K^4} - \frac{35929260709}{45349632} \frac{1}{K^5} + \mathcal{O} \left( \frac{1}{K^6} \right) \right\}.$$  

If we rewrite the corrections to the leading factorial growth of the perturbative coefficients in terms of a factorial series,

$$E^{(0)}_{0,K} \sim -3^{K+1} K! \left\{ 1 - \sum_{j=1}^{\infty} \frac{a_j}{(K - j + 1)j} \right\} =$$

$$= -3^{K+1} K! \left\{ 1 - \frac{a_1}{K} - \frac{a_2}{K (K - 1)} - \frac{a_3}{K (K - 1) (K - 2)} - \frac{a_4}{K (K - 1) (K - 2) (K - 3)} - \cdots \right\},$$

then it is easy to identify the coefficients entering into (8.6) with the $a_j$ coefficients. Based on the discussion in chapter 2.1.3, it is easy to show that

$$3^j (-a_j) = \epsilon_{0,21j}.$$  

The results listed in equation (8.7) therefore lead to the following conjectures for the two-instanton coefficients:

$$\epsilon_{0,211} = -\frac{53}{6}, \quad \epsilon_{0,212} = \frac{1277}{72}, \quad \epsilon_{0,213} = -\frac{336437}{1296},$$

$$\epsilon_{0,214} = -\frac{141158555}{31104}, \quad \epsilon_{0,215} = -\frac{17542610737}{186624}.$$  

8.2 Corrections to Asymptotics
The numbers 53, 1277 and 336 437 are prime.

8.3 Perturbative Expansion and Instanton Theory

We will now use a completely different and more direct route to the calculation of instanton coefficients: It is based on the explicit solution of the quantization condition (2.21) by an ansatz as given by the resurgent expansion (2.13). In order to carry out a calculation, we first have to determine the coefficients in the expansions of \( B_{dw}(E, g) \) and \( A_{dw}(E, g) \) defined in (2.23a) and (2.23b). An accurate calculation requires more terms than those given in (2.24a) and (2.24b).

The calculation of the perturbative function \( B_{dw}(E, g) \) can be carried out easily based on equation (3.31). Indeed, recursive algorithms are known [8]. The generalization of (2.24a) to higher orders is as follows:

\[
B_{dw}(E, g) = E + g \left( 3E^2 + \frac{1}{4} \right) + g^2 \left( \frac{35}{2}E^3 + \frac{25}{4}E \right) + g^3 \left( \frac{115}{2}E^4 + \frac{735}{4}E^2 + \frac{175}{32} \right) + g^4 \left( \frac{45045}{4}E^5 + \frac{45045}{8}E^3 + \frac{31185}{64}E \right) + g^5 \left( \frac{969969}{4}E^6 + \frac{280705}{16}E^4 + \frac{1924923}{64}E^2 + \frac{159159}{256} \right) + g^6 \left( \frac{23090287}{4}E^7 + \frac{88267179}{16}E^5 + \frac{100553453}{64}E^3 + \frac{25746721}{256} \right) + g^7 \left( \frac{2151252675}{16}E^8 + \frac{2788660875}{16}E^6 + \frac{9526065549}{128}E^4 + \frac{692049787}{4096} \right) + g^8 \left( \frac{214886239425}{64}E^9 + \frac{353522522925}{64}E^7 + \frac{1691601686775}{512}E^5 + \frac{663834081625}{16384}E^3 + \frac{25746721}{1024} \right) + O(g^9).
\]

The calculation of the “instanton function” \( A_{dw}(E, g) \) is a little more difficult (see the appendices F.6 and F.7): The calculation is based on the evaluation of successively higher orders of the contour integral of the WKB expansion as given in (3.57). The generalization of (2.24b) to higher orders is given by:

\[
A_{dw}(E, g) = \frac{1}{3}g + g \left( 17E^2 + \frac{19}{2} \right) + g^2 \left( \frac{227}{12}E^3 + \frac{187}{4}E \right) + g^3 \left( \frac{47431}{24}E^4 + \frac{34121}{576}E^2 + \frac{28829}{192} \right) + g^4 \left( \frac{317629}{16}E^5 + \frac{264725}{16}E^3 + \frac{842909}{192}E \right) + g^5 \left( \frac{26145967}{15}E^6 + \frac{16601579}{12}E^4 + \frac{63996919}{240}E^2 + \frac{6167719}{960} \right) + g^6 \left( \frac{812725953}{20}E^7 + \frac{349089111}{80}E^5 \right) + g^7 \left( \frac{4398906487}{320}E^8 + \frac{1280965929}{1280}E^3 + \frac{1280965929}{1280}E \right) + g^8 \left( \frac{443323117271}{448}E^9 + \frac{26522473925}{192}E^6 + \frac{4948336000477}{7680}E^4 \right) + \left( \frac{10166658134543}{107520}E^6 + \frac{10166658134543}{172032}E^3 + \frac{10166658134543}{172032}E \right) + g^9 \left( \frac{22315986340103}{896}E^9 + \frac{4909541135621}{112}E^6 + \frac{2904228665297}{1024}E^3 + \frac{2904228665297}{1024}E \right) + O(g^9).
\]
8.4 Instanton Coefficients

All coefficients, up to eight-instanton order, up to seventh order in $g$, have been calculated (see Table F.1). This requires the WKB expansion up to $g^8 S_8(E, q)$. Due to the prefactor $1/g$, we obtain relevant terms with positive powers of $E$ from $S_10$ of order $g^8$. This is irrelevant if we want to calculate all terms up to the order of $g^8$, and we may therefore neglect $S_10$ (see also appendix F.6).

The perturbation theory function $B_{dw}(E, g)$, together with the instanton function $A_{dw}(E, g)$, entirely determine the perturbative expansion about the $n$-instanton effect ($n$ arbitrary) to order $g^8$, i.e. all coefficients $e_{0,nkl}$ with $l \leq 8$ in the resurgent expansion (2.13), for arbitrary $n$ and $k$. In order to illustrate the effect of the instanton contributions on the energy levels, we give here formulas for the complete instanton effects through the order of $g^8$ rather than lists of coefficients. This is inspired by the notation (2.15). For the states with parity $\epsilon = \pm$ and $N = 0$, the one-instanton effect reads [see also (2.27) and (2.28)]:

$$E^{(1)}_{\epsilon, 0} = -\epsilon \xi(g) \left(1 - \frac{71}{12}g - \frac{6299}{288}g^2 - \frac{2691107}{10368}g^3 - \frac{2125346615}{497664}g^4 - \frac{509978166739}{5971968}g^5 - \frac{84613407444319}{429981696}g^6 - \frac{26252915454007369}{1559780352}g^7 - \frac{717976540715437267525}{495338913792}g^8 + O(g^9) \right).$$

(8.13a)

Note that according to [8], many more terms in the perturbative expansion about one instanton may be calculated. There are even recursive formulae available. The first coefficients in the perturbative expansion about one instanton therefore read as follows:

$$e_{0,100} = 1, \quad e_{0,101} = -\frac{71}{12}, \quad e_{0,102} = -\frac{6299}{288}, \quad e_{0,103} = -\frac{2691107}{10368}, \quad e_{0,104} = -\frac{2125346615}{497664}. \quad (8.13b)$$

The coefficients $e_{0,10l}$ ($l = 0, \ldots, 300$) are available in electronic form [7].

Now we consider the two-instanton energy shift. Equation (8.5) clarifies that the two-instanton shift is given by the sum of two infinite series in $g$, which differ by the presence (or absence) of the logarithmic prefactor $\chi(g)$. The
two-instanton effect is parity independent, and the explicit formula for $N = 0$ reads

$$
\begin{align*}
E_{c,0}^{(2)}(g) &= \xi^2(g) \left[ \chi(g) \left( 1 - \frac{53}{6} g - \frac{1277}{72} g^2 - \frac{336437}{1296} g^3 
\right.ight.
o\ &\quad - \frac{141158555}{31104} g^4 - \frac{17542610737}{186624} g^5 
\left. \right.
o\ &\quad - \frac{14922996684685}{6718464} g^6 - \frac{2359159111315567}{40310784} g^7 
\left. \right. 
\left. - \frac{3279840218627988925}{1934917632} g^8 + O(g^9) \right] 
\right] 
\left( \gamma + \left( \frac{23}{2} - \frac{53}{6} \gamma \right) g + \left( \frac{13}{12} - \frac{1277}{72} \gamma \right) g^2 
\right.
o\ &\quad + \left( \frac{45941}{144} - \frac{336437}{1296} \gamma \right) g^3 + \left( \frac{20772221}{2592} - \frac{141158555}{31104} \gamma \right) g^4 
\left. \right. 
\left. + \left( \frac{12783531515}{62208} - \frac{17542610737}{186624} \gamma \right) g^5 
\right. 
\left. + \left( \frac{2110670726275}{373248} - \frac{14922996684685}{6718464} \gamma \right) g^6 
\right. 
\left. + \left( \frac{22499871554449805}{13436928} - \frac{2359159111315567}{40310784} \gamma \right) g^7 
\right. 
\left. + \left( \frac{429051019455941467}{80621568} - \frac{3279840218627988925}{1934917632} \gamma \right) g^8 + O(g^9) \right) 
\end{align*}
$$

The logarithmic coefficients $\epsilon_{0,2l}$ ($l = 0, \ldots, 8$), determined here by direct analytic calculation, verify the conjectures ($l = 0, \ldots, 5$) which were previously derived based on the corrections to the large-order growth of the perturbative coefficients (8.10) and the dispersion relations discussed in chapter 2.1.3. The three-instanton correction to energy eigenvalues, for states with $N = 0$, involves three powers of the nonperturbative factor $\xi(g)$. The analytic expressions become very involved, as there are three infinite perturbative series in $g$, multiplying the logarithmic terms $\chi^2(g), \chi(g)$, and a series multiplying the terms which lack the logarithm. The coefficients of the instanton expansion can still be calculated in closed analytic form, by direct reference to
the quantization condition (2.22). Up to terms of order $g^8$, the three-instanton shift is given by

$$E^{(3)}_{\text{inst}}(g) = -e^3 g \left[ \chi^2(g) \frac{1}{2} - \frac{141}{8} g - \frac{489}{64} g^2 - \frac{89635}{256} g^3 - \frac{2372519}{4096} g^4 - \frac{2362765483}{16384} g^5 + \mathcal{O}(g^6) \right]$$

$$+ \chi(g) \left\{ 3 \gamma + \left( \frac{63}{4} - \frac{141}{4} \right) g + \left( \frac{825}{8} - \frac{489}{32} \right) g^2 + \left( -\frac{24483}{64} - \frac{89635}{128} \right) g^3 \right\}$$

$$+ \left( \frac{3369081}{256} - \frac{27325159}{2048} \right) g^4 + \left( \frac{-1534609037}{4096} - \frac{2362765483}{1892} \right) g^5$$

$$+ \left( \frac{179089589633}{16384} - \frac{1378627835215}{196608} \right) g^6$$

$$+ \left( \frac{17371037863755515}{131072} - \frac{139916898211047653}{786432} \right) g^7$$

$$+ \left( \frac{\left( \frac{3}{2} \gamma^2 + \frac{3}{2} \zeta(2) \right)}{2} + \left( \frac{\left( \frac{17}{2} \gamma - \frac{63}{4} \right)}{2} \right) \zeta(2) \right) g^8$$

$$+ \left( \frac{297}{2} - \frac{2483}{8} - \frac{9635}{64} \right) \zeta(2) g^9$$

$$+ \left( \frac{21199}{192} - \frac{3369081}{256} \right) \gamma - \frac{89635}{256} \gamma^2 - \frac{89635}{768} \zeta(2) g^{10}$$

$$+ \left( \frac{27147}{128} - \frac{3369081}{256} \right) \gamma - \frac{27325159}{4096} \gamma^2 - \frac{27325159}{12888} \zeta(2) g^{11}$$

$$+ \left( \frac{9180800297}{61440} - \frac{139916898211047653}{1378627835215} \right) \gamma - \frac{2362765483}{16384} \gamma^2 - \frac{2362765483}{49152} \zeta(2) g^{12}$$

$$+ \left( \frac{572797894871}{92160} - \frac{138780589633}{1378627835215} \right) \gamma - \frac{1378627835215}{1179648} \gamma^2 - \frac{1378627835215}{1179648} \zeta(2) g^{13}$$

$$+ \left( \frac{983333533580334823}{41287680} - \frac{131072}{130982} \right) \gamma - \frac{148426925165305}{1572864} \gamma^2 - \frac{148426925165305}{4718592} \zeta(2) g^{14}$$

$$+ \left( \frac{1500070370554927511}{1655072} - \frac{17311037863755515}{1572864} \right) \gamma - \frac{139916898211047653}{150994944} \zeta(2) g^{15} + \mathcal{O}(g^{16}) \right\} . \quad (8.15)$$

### 8.5 Reference Values

All accurate verifications of the theory of instantons, at small coupling $g$, necessitate a very precise determination of energy eigenvalues in the small-$g$ region. A numerically accurate determination of eigenvalues can be a rather hard problem, even in a one-dimensional case. However, calculations are simplified when using multiprecision libraries [17–19]. In the current chapter, we intend to give reference values for specific small-$g$ example cases. The results for $E_{\pm,0}$ in all cases are rather close together, and the tiny energy differences then point toward the instanton-mediated energy shifts. In [9], we indicated two such values, valid up to 180 decimals, for the case $g = 0.001$. Here, we restrict the discussion to slightly fewer decimals while stressing that essentially arbitrary accuracy is available when using appropriate numerical algorithms [9]. For $g = 0.002$, we obtain for the ground state to 100 decimals

$$E_{+,0}(0.002) = 0.49798 16336 05614 52785 33444 97756 93929 30135 47830$$

$$+ 1880 18141 65406 30388 85981 28620 52208 12662 00662 \text{ ,} \quad (8.16)$$

whereas the first excited state has the energy

$$E_{-,0}(0.002) = 0.49798 16336 05614 52785 33444 97756 93930 00653 58949$$

$$+ 74478 49607 06416 37435 43472 00173 52993 86232 45517 \text{ .} \quad (8.17)$$

The two energies differ on the level of the one-instanton contribution which is given by equation (8.13) and has opposite sign for states with opposite parity. Indeed, the energy difference $E_{-,0}(0.002) - E_{+,0}(0.002)$ can be calculated to high accuracy by simply evaluating the partial sums of the perturbative expansion about one instanton, in
analogy to the case $g = 0.001$ considered in [9]. As mentioned previously, the coefficients $c_{0,10l} (l = 0, \ldots, 300)$ are available electronically [7] to the interested reader. For $g = 0.005$, we obtain as reference values, to 45 decimals,

$$E_{+,0}(0.005) = 0.494881507320647627215484956033004028732300532,$$  

and

$$E_{-,0}(0.005) = 0.494881507320699290848098189004435423390722906.$$  

(8.18a)

The mean energy is

$$\frac{E_{+,0}(0.005) + E_{-,0}(0.005)}{2} = 0.494881507320673459031791572518.$$  

(8.18c)

Again, the real part of the Borel sum of the perturbation series (2.50) is slightly different, due to the two-instanton correction,

$$\text{Re} \mathcal{B} \left\{ E_{0}^{(0)}(0.005) \right\} = 0.494881507320673459031791568107.$$  

(8.18d)

The energy difference is

$$E_{-,0}(0.005) - E_{+,0}(0.005) = 5.166363261323297143139465842237 \times 10^{-14}.$$  

(8.18e)

The case $g = 0.007$ yields the following energies

$$E_{+,0}(0.007) = 0.492762513834552888078527435274125157773355043,$$  

and

$$E_{-,0}(0.007) = 0.492762514424291380998937058280417599208041287.$$  

(8.19b)

The mean energy is

$$\frac{E_{+,0}(0.007) + E_{-,0}(0.007)}{2} = 0.4927625141294221345387322.$$  

(8.19c)

The real part of the Borel sum of the perturbation series (2.50) is slightly different, due to the two-instanton correction,

$$\text{Re} \mathcal{B} \left\{ E_{0}^{(0)}(0.007) \right\} = 0.4927625141294221343923438.$$  

(8.19d)

The energy difference is

$$E_{-,0}(0.007) - E_{+,0}(0.007) = 5.897384929204096230062924414346 \times 10^{-10}.$$  

(8.19e)

Finally, for the slightly less problematic case of $g = 0.01$, we only list the energies

$$E_{+,0}(0.01) = 0.489497520976030, \quad E_{-,1}(0.01) = 0.489498132721197.$$  

(8.20)
8.6 The Function $\Delta(g)$ for States with $N = 0$

We recall the definition of the function $\Delta(g)$, given in (2.52),

$$\Delta(g) = 4\left\{\frac{1}{2} (E_{+,0} + E_{-,0}) - B \left\{E_0^{(0)}(g)\right\}\right\} \left(\frac{1}{(E_{+,0} - E_{-,0})^2 \ln(2g^{-1}) + \gamma}\right).$$

We provide here, based in part on the data given in chapter 8.5, some reference values of the for the function $\Delta(g)$, for a somewhat larger coupling $g$ as compared to those listed in Table 2.1:

\[
\begin{align*}
\Delta(0.011) &= 1.01147, \quad \Delta(0.012) = 1.01211, \quad \Delta(0.014) = 1.01322, \quad \Delta(0.015) = 1.01368, \\
\Delta(0.016) &= 1.01408, \quad \Delta(0.017) = 1.01439, \quad \Delta(0.018) = 1.01465, \quad \Delta(0.019) = 1.01482, \\
\Delta(0.020) &= 1.01498, \quad \Delta(0.021) = 1.01486, \quad \Delta(0.022) = 1.01464, \quad \Delta(0.023) = 1.01444, \\
\Delta(0.024) &= 1.01392, \quad \Delta(0.025) = 1.01339, \quad \Delta(0.026) = 1.01233, \quad \Delta(0.027) = 1.01144, \\
\Delta(0.028) &= 1.00991, \quad \Delta(0.029) = 1.00816, \quad \Delta(0.030) = 1.00653.
\end{align*}
\]

The available higher-order corrections (in $g$) to the two-instanton energy shift [see equation (8.14)] allow for an accurate comparison of the numerically determined energy eigenvalues with the asymptotic expansion of the function $\Delta(g)$ (see figure 8.1). We recall that the calculation of the instanton coefficients (8.14) relies on higher-order WKB expansions are described in appendices F.6 and F.7.
8.7 Leading Instanton Effects for States with \( N = 0 \)

Based on the simplified quantization condition (4.60), derived using the path-integral formalism, it is relatively easy to determine the leading (in \( g \)) coefficients of the instanton expansion (2.13) to any order in the nonperturbative factor \( \xi(g) \equiv \exp(-1/6g)/\sqrt{\pi g} \), and zeroth order in \( g \) (this is equivalent to the coefficients \( e_{N,nkl} \) with \( l = 0 \)). In the current chapter, we intend to provide results for the states with \( N = 0 \), up to eight-instanton order, in closed analytic form. The coefficients display an interesting analytic structure; a rapid growth of their absolute magnitude is observed in higher instanton orders for constant (zeroth) order in \( g \) (see also Table F.1). We briefly recall here that the leading two- and three-instanton coefficients have been given in (2.40), (8.14) and (8.15). The four-instanton correction is discussed in appendix F.8. The four-instanton coefficients determine the polynomial \( P_4^0 \) implicitly defined by (2.39):

\[
\begin{align*}
e_{0,430} &= \frac{8}{3}, \quad e_{0,420} = 8 \gamma, \quad e_{0,410} = 8 \gamma^2 + 2 \zeta(2), \\
e_{0,400} &= \frac{8}{3} \gamma^3 + 2 \gamma \zeta(2) + \frac{1}{3} \zeta(3).
\end{align*}
\]

The five-instanton coefficients which enter into \( P_5^0 \), read:

\[
\begin{align*}
e_{0,540} &= \frac{125}{24}, \\
e_{0,530} &= \frac{125}{6} \gamma, \\
e_{0,520} &= \frac{125}{4} \gamma^2 + \frac{25}{4} \zeta(2), \\
e_{0,510} &= \frac{125}{6} \gamma^3 + \frac{25}{2} \gamma \zeta(2) + \frac{5}{3} \zeta(3), \\
e_{0,500} &= \frac{125}{24} \gamma^4 + \frac{25}{4} \gamma^2 \zeta(2) + \frac{5}{3} \gamma \zeta(3) + \frac{29}{16} \zeta(4).
\end{align*}
\]

The result for the six-instanton coefficients are a little more complex:

\[
\begin{align*}
e_{0,650} &= \frac{54}{5}, \\
e_{0,640} &= 54 \gamma, \\
e_{0,630} &= 108 \gamma^2 + 18 \zeta(2), \\
e_{0,620} &= 108 \gamma^3 + 54 \gamma \zeta(2) + 6 \zeta(3), \\
e_{0,610} &= 54 \gamma^4 + 54 \gamma^2 \zeta(2) + 12 \gamma \zeta(3) + \frac{51}{4} \zeta(4), \\
e_{0,600} &= \frac{54}{5} \gamma^5 + 18 \gamma^3 \zeta(2) + 6 \gamma^2 \zeta(3) \\
&\quad + \zeta(2) \zeta(3) + \frac{51}{4} \gamma \zeta(4) + \frac{1}{5} \zeta(5).
\end{align*}
\]
The seven-instanton coefficients, to lowest order in \( g \), read:

\[
\begin{align*}
\epsilon_{0,760} &= \frac{\gamma^5}{720}, \\
\epsilon_{0,750} &= \frac{\gamma^5}{120}, \\
\epsilon_{0,740} &= \frac{\gamma^5}{48} \gamma^2 + \frac{2401}{48} \zeta(2), \\
\epsilon_{0,730} &= \frac{\gamma^5}{36} \gamma^3 + \frac{2401}{12} \gamma \zeta(2) + \frac{343}{18} \zeta(3), \\
\epsilon_{0,720} &= \frac{\gamma^5}{48} \gamma^4 + \frac{2401}{8} \gamma^2 \zeta(2) + \frac{343}{6} \gamma \zeta(3) + \frac{1911}{32} \zeta(4), \\
\epsilon_{0,710} &= \frac{\gamma^5}{120} \gamma^5 + \frac{2401}{8} \gamma^3 \zeta(2) + \frac{343}{6} \gamma^2 \zeta(3) + \frac{49}{6} \gamma \zeta(2) \zeta(3) \\
&+ \frac{1911}{16} \gamma \zeta(4) + \frac{7}{5} \zeta(5), \\
\epsilon_{0,700} &= \frac{\gamma^5}{720} \gamma^6 + \frac{2401}{48} \gamma^4 \zeta(2) + \frac{343}{18} \gamma^3 \zeta(3) + \frac{49}{6} \gamma^2 \zeta(2) \zeta(3) \\
&+ \frac{7}{18} \gamma^2 \zeta(4) + \frac{1911}{32} \gamma \zeta(4) + \frac{7}{5} \zeta(5) + \frac{789}{128} \zeta(6).
\end{align*}
\]

The eight-instanton effect features the following coefficients:

\[
\begin{align*}
\epsilon_{0,870} &= \frac{214}{315}, \\
\epsilon_{0,860} &= \frac{214}{45} \gamma, \\
\epsilon_{0,850} &= \frac{214}{15} \gamma^2 + \frac{211}{15} \zeta(2), \\
\epsilon_{0,840} &= \frac{214}{75} \gamma^3 + \frac{211}{15} \gamma \zeta(2) + \frac{29}{5} \zeta(3), \\
\epsilon_{0,830} &= \frac{214}{9} \gamma^4 + \frac{212}{3} \gamma^2 \zeta(2) + \frac{211}{9} \gamma \zeta(3) + \frac{704}{3} \zeta(4), \\
\epsilon_{0,820} &= \frac{214}{15} \gamma^5 + \frac{212}{3} \gamma^3 \zeta(2) + \frac{210}{3} \gamma^2 \zeta(3) + \frac{27}{3} \gamma \zeta(2) \zeta(3) \\
&+ \frac{704}{15} \gamma \zeta(4) + \frac{29}{5} \zeta(5), \\
\epsilon_{0,810} &= \frac{214}{45} \gamma^6 + \frac{211}{3} \gamma^4 \zeta(2) + \frac{211}{9} \gamma^3 \zeta(3) + \frac{28}{3} \gamma^2 \zeta(2) \zeta(3) \\
&+ \frac{704}{15} \gamma^2 \zeta(4) + \frac{29}{5} \gamma \zeta(5) + 62 \zeta(6), \\
\epsilon_{0,800} &= \frac{214}{315} \gamma^7 + \frac{211}{15} \gamma^5 \zeta(2) + \frac{29}{9} \gamma^4 \zeta(3) + \frac{27}{3} \gamma^3 \zeta(2) \zeta(3) \\
&+ \frac{27}{9} \gamma^2 \zeta(4) + \frac{22}{9} \gamma \zeta(3) \zeta(4) + \frac{25}{9} \gamma \zeta(5), \\
&+ \frac{4}{5} \zeta(2) \zeta(5) + 62 \gamma \zeta(6) + \frac{1}{7} \zeta(7).
\end{align*}
\]

It is intriguing to observe that in all individual terms contributing to a particular coefficient, the sum of the power of \( \gamma \) and of all (integer) arguments of \( \zeta \) functions equals a constant. An analogous pattern may be observed in the context of specific sums that enter into the evaluation of higher-order corrections to the vacuum-polarization charge density around a nucleus, as used in the derivation of equations (66) and (67) of [20] (relevant formulas are given in appendix IV \textit{ibid.}).

### 8.8 The Function \( \Delta_1(g) \) for States with \( N = 1 \)

In this chapter, we will be concerned with the generalization of the function \( \Delta(g) \) [equation (2.52)] to states with \( N = 1 \). The perturbation series for \( N = 1 \), up to order \( g^0 \), has been determined analytically based on the recursive techniques outlined in [8]. Coefficients are made available at [7]. The instanton coefficients for states with \( N = 1 \) are different from those listed in equations (2.18), (2.19), (8.13), (8.13b) and (8.14), and in the Table F.1, where the states with \( N = 0 \) are considered. However, it is relatively easy to calculate the coefficients \( e_{1,nkl} \) with \( n \leq 2 \).
and \( l \leq 6 \) based on the quantization condition (2.22) and the explicit formulas for the function \( B_{dw}(E, g) \) and \( A_{dw}(E, g) \) given in (8.11) and (8.12). For the states with parity \( \epsilon \) and \( N = 1 \), the one-instanton effect is enhanced by a factor \( g^{-1} \) in comparison to the \( N = 0 \)-states [see also (2.27) and (2.28)]:

\[
E_{\epsilon,1}^{(1)} = -\frac{2}{g} \xi(g) \left( 1 - \frac{347}{12} g + \frac{5317}{288} g^2 - \frac{15991559}{10368} g^3 - \frac{28062012119}{497664} g^4 \right)
\]

\[
- \frac{12918255230839}{5971968} g^5 - \frac{38332497252543415}{42981696} g^6 + O(g^7)
\)

(8.28)

The coefficient \( e_{1,100} \) is governed by the general formula (2.35). The two-instanton energy shift, for states with \( N = 1 \), reads

\[
E_{\epsilon,1}^{(2)} = \frac{4}{g^2} \xi^2(g) \left[ \chi(g) \left( 1 - \frac{293}{6} g + \frac{39823}{72} g^2 - \frac{2767481}{1296} g^3 - \frac{902381531}{31104} g^4 \right) + \frac{233318679457}{186624} g^5 - \frac{36747818955893}{6718464} g^6 + O(g^7) \right]
\]

\[
\left( \gamma - 1 + \frac{61}{3} - \frac{293}{6} \gamma \right) g^2 + \left( \frac{21179}{72} - \frac{39823}{72} \gamma \right) g^3 - \frac{1592699}{648} g^4 + \frac{2767481}{1296} g^5
\]

\[
+ \left( \frac{5487367691}{31104} - \frac{902381531}{31104} \gamma \right) g^6 + \frac{33161353697}{93312} - \frac{233318679457}{186624} \gamma \right) g^7 + O(g^8)
\]

(8.29)

Again, the coefficients \( e_{1,210} \) and \( e_{1,200} \) are governed by the general formulas (2.37) and (2.38).

### Figure 8.2: Double-well potential

Comparison of numerical data obtained for the function \( \Delta_1(g) \) defined in (8.35) with the sum of the terms up to the order of \( g^{10} \) of its asymptotic expansion for \( g \) small, where we express both the numerator as well as the denominator of (8.35) as a power series in \( g \). There is good agreement between numerically determined (“exact”) values (data points) and the smooth curve given by the analytic asymptotics. See also Table 8.1.

We consider the example \( g = 0.005 \), both in order to illustrate the calculation and in order to provide reference values for independent verification. The energy of the state \((\pm, 1)\) as determined numerically reads:

\[
E_{\pm,1}(0.005) = 1.46321 33515 77109 24646 17593 40427 75209 25202 .
\]

(8.30)
The energy of the state $(-1, 1)$ is slightly higher:

$$E_{-1}(0.005) = 1.46321\,33515\,95341\,45149\,69757\,27023\,63549\,18578.$$  \hfill (8.31)

The energy difference

$$\Delta E(0.005) = E_{-1}(0.005) - E_{+1}(0.005) \approx 1.823 \times 10^{-11}. \hfill (8.32)$$

The mean energy is

$$\frac{E_{+1}(0.005) + E_{-1}(0.005)}{2} = 1.46321\,33515\,86225\,34897\,93675\,37523\,63549\,18578. \hfill (8.33)$$

The real part of the Borel sum of the perturbation series is

$$B\left\{E_1^{(0)}(0.005)\right\} = 1.46321\,33515\,86225\,34897\,88950\,50((1)). \hfill (8.34)$$

Those decimal figures which agree are underlined. The difference is due to the two-instanton energy shift. The definition (2.52) of the function $\Delta$ which relates the one- and two-instanton effects to the Borel sum of the perturbation series, should be modified in order to accommodate for the changed structure of the leading two-instanton coefficients. We define the function $\Delta_1(g)$ as

$$\Delta_1(g) = 4\left\{\frac{1}{2} (E_{+1} + E_{-1}) - B\left\{E_1^{(0)}(g)\right\}\right\}, \hfill (8.35)$$

In order to determine the leading asymptotics of the function $\Delta_1(g)$, we have used the perturbative expansions about one- and two-instantons given in equations (8.28) and (8.29). If we additionally perform an expansion in inverse powers of the logarithm $\ln(g)$, we arrive at the result

$$\Delta_1(g) = 1 + 9g - \frac{57}{2} \frac{g}{\ln(2/g)} \left(1 + \frac{1 - \gamma}{\ln^2(2/g)} + O\left(\frac{1}{\ln^3(2/g)}\right)\right) + O(g^2), \hfill (8.36)$$

which generalizes the formula (2.53) to the case $N = 1$. Of course, many more terms in the asymptotic expansion of $\Delta_1(g)$ can be determined based on equations (8.28) and (8.29). Sample values for the function $\Delta_1(g)$ are given in Table 8.1, and the function is plotted against its small-$g$ asymptotic expansion in figure 8.2.

<table>
<thead>
<tr>
<th>coupling $g$</th>
<th>0.005</th>
<th>0.006</th>
<th>0.007</th>
<th>0.008</th>
<th>0.009</th>
<th>0.010</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta_1(g)$ num.</td>
<td>1.02097</td>
<td>1.02434</td>
<td>1.02748</td>
<td>1.03039</td>
<td>1.03304</td>
<td>1.03540</td>
</tr>
<tr>
<td>$\Delta_1(g)$ asymp.</td>
<td>1.02098</td>
<td>1.02434</td>
<td>1.02748</td>
<td>1.03039</td>
<td>1.03305</td>
<td>1.03542</td>
</tr>
</tbody>
</table>

**8.9 Three–Instanton Effects for $N = 0$**

According to the equations (2.27), (2.39) and (2.40), the three-instanton shift of the states with $N = 0$, at leading order in $g$, is given by

$$E^{(3)}_0(g) = -\varepsilon \xi^3(g) \left\{P_3^0[\ln(-2/g)] + O(g \ln^2(g))\right\}, \hfill (8.37)$$
CHAPTER 8: INSTANTON CALCULATIONS IN THE DOUBLE–WELL PROBLEM

8.9 Three–Instanton Effects for \( N = 0 \)

Figure 8.3: Double-well potential: Comparison of numerical data obtained for the function \( D(g) \) defined in (8.42) with the sum of the terms up to the order of \( g^{10} \) of its asymptotic expansion for \( g \) small. Again, we express both the numerator as well as the denominator of the expression defining \( D(g) \) [see equation (8.42)] as a power series in \( g \).

where the expression \( P^0_3[\ln(-2/g)] \) reads

\[
P^0_3(g) = \frac{3}{2} \left[ \ln\left(\frac{2}{g}\right) + \gamma \right]^2 + \frac{\pi^2}{12}.
\]  

(8.38)

In order to evaluate the physically relevant part of the three-instanton shift, we have to give an interpretation to the squared imaginary part generated by the logarithm, for positive \( g \), under the analytic continuation of the logarithm to negative argument,

\[
\ln\left(\frac{-2}{g}\right) \rightarrow \ln\left(\frac{2}{g}\right) + i\pi.
\]  

(8.39)

There is a potential problem. The explicit imaginary part of the three-instanton shift, generated by the analytic continuation of the logarithm, has to cancel the imaginary part of the (generalized) Borel sum of the one-instanton shift. In chapter 2.1.3, this fact has been used in order to derive the leading large-order asymptotics of the perturbative expansion about one instanton. The imaginary part therefore cancels when the resurgent expansion (2.13) is summed, and we need not consider it when evaluating real energy shifts. On squaring the right-hand side of (8.39), we isolate the physically relevant part of \( P^0_3(g) \) which for \( g > 0 \) reads:

\[
g > 0 : \quad \text{Re } P^0_3(g) = \frac{3}{2} \left[ \ln\left(\frac{2}{g}\right) + \gamma \right]^2 - \frac{17}{12} \pi^2.
\]  

(8.40)

According to the conjecture (2.13), the energy difference \( E_{-,0}(g) - E_{+,0}(g) \), for \( g > 0 \), is approximately equal to twice the value of the (positive) one-instanton energy shift of the state \((-+,0)\). The difference of the quantity \( E_{-,0}(g) - E_{+,0}(g) \) and the real part of the Borel sum of the perturbative expansion about one instanton is approximately given by the three-instanton shift,

\[
E_{-,0}(g) - E_{+,0}(g) - 2B\left\{ E_{1}^{(0)}(g) \right\} \approx 2E_{-0}^{(3)}(g),
\]  

(8.41)
Table 8.2: The ratio $D(g)$ as a function of $g$. The values obtained from the asymptotic expansion for $g$ small (up to the order $g^8$, denoted “asymp.”) are compared to numerically determined values (see chapter 8.5, values are denoted as “num.”). The interested reader may verify the numerical values based on the data given in equations (8.18) and (8.19).

<table>
<thead>
<tr>
<th>coupling $g$</th>
<th>0.005</th>
<th>0.006</th>
<th>0.007</th>
<th>0.008</th>
<th>0.009</th>
<th>0.010</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D(g)$ num.</td>
<td>1.00883</td>
<td>1.00931</td>
<td>1.00944</td>
<td>1.00919</td>
<td>1.00856</td>
<td>1.00754</td>
</tr>
<tr>
<td>$D(g)$ asymp.</td>
<td>1.00877</td>
<td>1.00920</td>
<td>1.00925</td>
<td>1.00891</td>
<td>1.00816</td>
<td>1.00698</td>
</tr>
</tbody>
</table>

where we neglect the five-instanton contribution and recall that $E^{(3)}_{-0}(g) > 0$ for $g > 0$. In order to define a function $D(g)$ in distant analogy to (2.52) and (8.35), we should normalize to unity in the limit $g \to 0$. This can be achieved as follows: According to (2.35), the quantity $E_{-0}(g) - E_{+0}(g)$ is approximately equal to $2\xi(g)$ in the small-$g$ limit, where $\xi(g)$ is defined in (2.27). The third power of $E_{-0}(g) - E_{+0}(g)$ then compensates the three powers of $\xi(g)$ comprised by the three-instanton effect (8.15). In order to normalize to unity, the only remaining missing terms are a prefactor 4 and an inverse normalization factor $\text{Re}P_3^0(g)$ which is given in (8.40). We then define the function $D(g)$ as

$$D(g) = \frac{4}{3 \left[ \ln \left( \frac{2}{g} \right) + \gamma \right]^2} - \frac{17}{12} \pi^2 \left( \frac{E_{-0} - E_{+0}}{(E_{-0} - E_{+0})^3} \right)$$

(8.42)

The leading asymptotics for small $g$ read [we also expand in inverse powers of the logarithm, as in (2.53) and (8.36)]:

$$D(g) = 1 + 6g + \frac{g}{\ln(2/g)} \left( -\frac{21}{2} + \frac{63}{3} \ln(2/g) + \mathcal{O} \left( \frac{1}{\ln^2(2/g)} \right) \right) + \mathcal{O}(g^2).$$

(8.43)

Of course, more terms in the asymptotic expansion of $D(g)$ for small $g$ can be calculated in a straightforward manner on the basis of equations (8.13) and (8.15). A table of numerically calculated values of $D(g)$, and of the values suggested by the expansion up to the order $g^8$, is given in table 8.2, and a graphical representation is obtained in figure 8.3. In the obtaining the table 8.2 and figure 8.3, the expansion of $D(g)$ is carried out through the order of $g^8$, and no additional expansion in inverse powers of the logarithm $\ln(2/g)$ is performed.
Chapter 9

Conclusions

The central theme of the current article is the discussion of (multi-)instanton effects, which manifest themselves in nonperturbative, nonanalytic (in the coupling) contributions to the path integral which in turn defines the partition function (see chapters 1, 2 and 3). The expansion about nontrivial saddle points of the Euclidean action leads naturally to energy shifts which involve nonanalytic factors of the form \( \exp(-a/g) \). The instanton interaction and multi-instanton effects find a natural interpretation in terms of classical trajectories along which the particle may oscillate between degenerate minima. A particular detailed discussion of instantons in the (symmetric) double-well problem is provided in chapter 4, and the considerations are generalized to asymmetric wells in chapter 5, to the periodic cosine potential in chapter 6, to radially symmetric oscillators (self-adjoint extension and resonances), and potentials of the Fokker–Planck type (chapter 7). In chapter 8, we complement the preceding chapters by higher-order analytic as well as numerical calculations related to the double-well problem, up to the order of eight instantons (see also appendix F).

The conjectured quantization conditions are presented in chapter 2.3. For completeness, we give a list of the classes of potentials for which conjectures on quantization conditions are discussed in the current article,

- the double-well potential (section 2.1 and chapter 4),
- more general symmetric potentials with degenerate minima (section 2.2.1 and appendix E),
- a potential with two equal minima but asymmetric wells (section 2.2.2 and chapter 5),
- a periodic-cosine potential (section 2.2.3 and chapter 6),
- resonances of the \( \mathcal{O}(\nu) \)-symmetric anharmonic oscillator, for negative coupling (sections 2.2.4 and 7.2),
- a special potential which has the property that the perturbative expansion of the ground-state energy vanishes to all orders of the coupling constant (see chapters 2.2.4 and 7.2.3),
- and eigenvalues of the \( \mathcal{O}(\nu) \)-symmetric anharmonic oscillator, for negative coupling but with the Hamiltonian endowed with nonstandard boundary conditions (section 2.2.5 and 7.3).

Indeed, exact results may be derived for large classes of analytic potentials, as discussed in chapter 2. The considerations presented in chapters 3.2 and 3.3 are important for more general investigations in later chapters of the current work, and indeed might be useful for any further conceivable generalizations. Specifically, the rather general quantization condition summarized in equations (3.55) and (3.57) may be adapted to large classes of analytic potentials. The perturbative \( B \)-function and the instanton \( A \)-function are known to higher orders in the coupling \( g \), for a many of the potentials discussed here. For the periodic cosine potential [Eqs. (6.31) and (6.32)] and the Fokker–Planck potential [Eqs. (7.32) and (7.35)], as well as the \( \mathcal{O}(\nu) \) anharmonic oscillator [Eqs. (7.48) and (7.49)], we perform higher-order calculations of the \( B \)- and \( A \)-functions up to the order \( g^4 \). For the double-well potential, we perform calculations up to \( \mathcal{O}(g^8) \) [see Eqs. (8.11) and (8.12)], i.e. up to eight-instanton order.

The investigation of nontrivial saddle points of the Euclidean action allows for the identification of finite-action contributions to the path integral, the instantons, which may be calculated exactly in the limit of a large separation.
of the instanton tunneling configurations (see the discussion in chapter 4.4.2). Approximate quantization conditions may be derived on the basis of related considerations that evaluate the sum of the leading instanton effects, with an arbitrary number of oscillations on the instanton trajectory but considering only the leading term in the instanton interaction [see equations (4.60), (5.16), (6.29), (7.22), (7.29), (7.66), and (7.68)].

We now briefly recall general consequences of the resurgent expansion for the energy levels of the double-well potential (2.13), which reads

\[ E_{ε,N}(g) = \sum_{l=0}^{∞} E_{N,l}^{(0)} g^l + \sum_{n=1}^{∞} \left( \frac{2}{g} \right)^N n \left( -\varepsilon \frac{1}{6g} \right)^{n-1} \sum_{k=0}^{n-1} \sum_{l=0}^{∞} e_{N,nkl} g^l. \]  

(9.1)

It is thus clear that ordinary perturbation theory, which relies on an expansion in powers of the coupling constant, is not even qualitatively sufficient for a description of the energy levels as a function of the coupling constant \( g \).

In other words, whenever there are nontrivial saddle points of the classical action, it is not sufficient to consider perturbation theory about the trivial saddle point of the action. Generalized expansions are required which typically include nonanalytic factors of the form \( \exp(-a/g) \) where [see equation (2.60)]

\[ a = 2 \int_0^{q_0} dq \sqrt{2V(q)}, \]

(9.2)

\( q \) and \( q_0 \) are the positions of the degenerate minima, and \( V \) is the potential.

Rather important differences prevail among cases where there is parity symmetry, and asymmetric cases. Indeed, for the (symmetric) double-well potential, the splitting of opposite-parity states is given by a one-instanton effect of the order of \( 2 \exp[-a/(2g)] \). By contrast, there is no such degeneracy of states in the case of asymmetric wells, and for the ground-state energy (7.31) of the Fokker–Planck Hamiltonian, we have an energy of the order of \( \exp(-a/g) \). For the asymmetric case, the classical trajectories have to return to the original minimum (this corresponds to an even number of “tunnelings” between the degenerate minima), or else the trajectories do not contribute to the path integral. This is discussed in chapters 4.2.2 and 5.1.

Let us outline a few unsolved questions related to the discussed problems, which may be investigated in the future.

- First and foremost, the exact WKB- and instanton-inspired methods as well as the numerical investigations discussed here may be extended to more general potentials [see the general equation (3.57)]. One example is provided by potentials with three or more degenerate minima which interpolate between the cases of two degenerate minima on the one hand and the periodic cosine potential on the other hand. A further, particularly intriguing case is the Fokker–Planck potential (see chapter 7.2.3) for which the perturbation series vanishes to all orders (the manifestly positive ground-state energy is determined in this case by instanton effects).

- Second, there is a certain unsolved issue related to the complete (re-)summation of the instanton expansion. Indeed, the nonperturbative factor \( \exp(-a/g) \) may assume rather large values for moderate values of \( g \). In the specific case (2.13), while the power series

\[ \sum_{l=0}^{∞} e_{N,nkl} g^l \]

(9.3)

find a natural summation procedure in terms of the Borel method in complex directions, the large-order properties of the expansion in powers of

\[ \frac{e^{-1/6g}}{√πg} \]

(9.4)

are largely unknown. The explicit calculations in chapter 8 might be helpful for a verification of large-order estimates [regarding the expansion in \( \exp(-1/6g) \)]. The same applies to the entries of Table F.1.

- Third, as outlined in chapter 3.2.5, there is an interesting connection to fundamental properties of Borel summability. Note that even in situations where the perturbative expansion of eigenvalues is Borel summable, it is not clear whether the functions \( B_i(E, g) \) and \( A(E, g) \) [see equation (3.57)] are Borel summable in \( g \) at \( E \) fixed. Indeed, the wave function \( ψ(q) \) is unambiguously defined only when the quantization condition (3.55)
is satisfied with $N$ a non-negative integer. When $E$ is not an eigenvalue, the solution of the Schrödinger equation is an undefined linear combination of two particular solutions. Thus, it may be interesting to investigate the behaviour of the functions $B_i(E, g)$ and $A(E, g)$ when $E$ approaches an eigenvalue, in closer detail. It is conceivable that this approach might reveal connections between the $A$- and the $B$-functions and eventually lead to a unified understanding of the structure of these functions. Again, the explicit calculations in chapter 8 might be useful in the context of a verification of such relations, if they indeed exist.

- Fourth, there might be some possibilities for an extension of the exact methods discussed in the current article to higher-dimensional scenarios where our current understanding is more limited. There is a well-known analogy between a one-dimensional field theory and one-dimensional quantum mechanics, the one-dimensional field configurations being associated with the classical trajectory of the particle. Therefore, model problems derived from field-theoretic models via a reduction of the dimensionality may find a natural and exact treatment via the semi-classical methods discussed in the current article. Indeed, the loop expansion in field theory corresponds to the semi-classical expansion [21, chapter 6].

We conclude with a few remarks on conceivable applications to field theories, and in particular non-Abelian gauge field theories. These questions follow naturally after the famous examples found by Lipatov [22–24] for scalar theories. For a study of the non-Abelian case, one could be tempted to assume that the periodic potential discussed in chapter 6 might be a good starting point. However, one of the pre-eminent problems is related to the scale invariance of the classical equations of motion (classical chromodynamics) of the fields which are invariant under the transformation $A_{\mu}(x) \rightarrow \lambda A_{\mu}(\lambda x)$. In a multi-dimensional field theory, instantons are relevant at all possible length scales (small and large separations), and it is not a priori clear how to perform the integration of the small- and large-scale instantons. An explicit infrared cutoff destroys the gauge invariance of the theory. For a survey of field-theoretic aspects related to instantons, one may consult the following review articles [25–29] (the list is necessarily incomplete). The nonlinear $\sigma$ model (see e.g. [30, chapter 14]), or the $\phi^4$ theory [30, chapter 34], or even manifestly nonrelativistic field theories might provide better defined scenarios for a systematic, desirably exact study of instanton effects. In general, a better and more systematic understanding of situations which have by now precluded a thorough analysis, might be gained by suitable generalizations of the methods discussed here. As demonstrated, these have been successfully applied to solvable model problems, in one-dimensional quantum mechanics, which is equivalent to a field theory (point-like in space, with one time dimension).

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Appendix F

WKB Calculation and Instanton Contributions

F.1 An Important Asymptotic Expansion

To obtain the perturbative expansion of the function $A(E, g)$ (equation (3.57)) we need the WKB expansion (B.13) of $S(q)$. To extract $A$ from equation (3.57) we have first to expand the term $-\ln \Gamma\left[\frac{1}{2} - B(E, g)\right] + B(E, g) \ln(-g/2)$ for $B$ large. This necessity arises because $B(E, g) \sim E = O(g^{-1})$ in the context of the WKB expansion.

For the derivative $\psi'$ of the function $\psi(z) = \partial/\partial z \ln \Gamma(z)$, the following integral representation is known:

$$\psi'(\frac{1}{2} + z) = \frac{1}{2} \int_0^{\infty} \frac{t}{\sinh(t/2)} e^{-tz} \, dt = \frac{\partial^2}{\partial z^2} \Gamma\left(\frac{1}{2} + z\right). \quad (F.1)$$

For large $z$, the integral is dominated by the region of small $t$, and one may therefore derive an asymptotic expansion of $\psi'(\frac{1}{2} + z)$ by expanding the integrand [excluding the exponential factor $\exp(-t z)$] in $t$, and subsequent integration of the terms resulting from this expansion. Since $\psi'(z) = \partial^2/\partial z^2 \ln \Gamma(z)$, the integration constants have to be adjusted properly. One obtains the following asymptotic expansion of the $\Gamma$-function,

$$\ln \Gamma\left(\frac{1}{2} + z\right) = z \ln z - z + \frac{1}{2} \ln(2\pi) - \frac{1}{24} z + \frac{7}{2880} z^3 - \frac{31}{40320} z^5 + \frac{127}{215040} z^7 - \frac{511}{608256} z^9 + O(z^{-9}). \quad (F.2)$$

F.2 Properties of the Mellin Transform

The Mellin transform of a function $F$ is commonly defined as

$$M(s) = \int_0^L dE \, F(E) \, E^{-s-1}. \quad (F.3)$$

A rather useful property of the Mellin transform is this: If the function $F$ can be expanded into a power series in its argument, then the transform $M$ develops poles at positive integer argument, and the corresponding residues yield the coefficients of the power series for $F$. Moreover, for functions $F(E)$ which contain logarithmic terms of the form $E^n \ln E$, the Mellin transform has double poles whose residues, again, give the coefficients of the logarithmic terms.

Let us consider a slight generalization of the Mellin transform with an arbitrary upper limit of integration $L$. Then,

$$\int_0^L dE \, E^n \, E^{-s-1} = \frac{L^{n-s}}{n-s} = \frac{1}{n-s} + \ln L + O(n-s). \quad (F.4)$$
The coefficient of the leading term in the Laurent series about $S$ is $-a$. For a term of the form
\[ \int_0^L dE \ E^n \ln E \ E^{-s-1} = \frac{L^{n-s}}{(n-s)^2} + \frac{L^{n-s} \ln L}{n-s} \]
\[ = - \frac{1}{(n-s)^2} + \frac{\ln^2 L}{2} + O(n-s). \quad (F.5) \]

The coefficient of the leading term in the Laurent series about $n = s$ is independent of $L$. This is important for the considerations presented in the following chapters.

### F.3 Mellin Transform of the WKB Expansion

To calculate the successive terms of $A(E, g)$ in an expansion in powers of $g$ we have then to replace in equation (3.57) $S_\pm$ by its WKB expansion, and expand each term for $Eg$, that is $E$ small. A standard method to obtain this expansion which contains only powers of $E$ and powers of $E$ multiplied by $\ln E$ is to calculate the Mellin transform of integral (3.57). We thus consider the function
\[ M(s) = \frac{1}{g} \int dE dq \ S_+(q, E, g) \ E^{-s-1}. \quad (F.6) \]

One verifies, replacing $S_\pm$ by its WKB expansion, that the function $M(s)$ has double poles at integer values of $s$. The residue of the double pole at $s = n$ yields the coefficient of $-E^n \ln E$ in the expansion of the integral (3.57) for $g$ small and the residue of the simple pole the coefficient of $-E^n$:
\[ \frac{1}{n-s} \quad \rightarrow \quad E^n, \quad (F.7a) \]
\[ - \left( \frac{1}{n-s} \right)^2 \quad \rightarrow \quad E^n \ln E. \quad (F.7b) \]

The formulas lead to a proper identification of the terms in the Mellin transform of the contour integrals of successive orders of the WKB expansion, with the expansion in $g$ of the contour integrals themselves of the successive orders of the WKB expansion. Therefore, the equation (F.7), for each order in the expansion in $g$, is in fact a Mellin backtransformation.

We now investigate the contour integrals of successive orders of the perturbative expansion of the WKB expansion (sic!). We recall that the WKB expansion is an expansion in powers of $g$ at $gE$ fixed [equation (B.13)], and that a general result for $S_0$ has been given in (3.40). [A general second-order result for $g^2 S_2$ has been given in (B.16).] Expansions in $g$ of $S_0$ and $S_2$ are given in (B.18) and (B.20), respectively. Formulas relevant for the contour integrals are given in (B.19) and (B.22).

We call $I_0(s)$ the leading contribution to $M(s)$ coming from $S_0$, and $I_2(s)$ the contribution coming from $g^2 S_2$:
\[ M(s) = I_0(s) + I_2(s) + O(g^3). \quad (F.8) \]

The function $I_0(s)$ is then given by [see equations (B.18) and (B.19)]:
\[ I_0(s) = - \frac{1}{ig} \int dE dq \ \left[ 2gE - U^2(q) \right]^{1/2} E^{-s-1} \]
\[ = - \frac{g^s}{i} \int dE dq \ \left[ 2gE - U^2(q) \right]^{1/2} \ (gE)^{-s-1} \]
\[ = 2^{s+1} g^{s-1} \frac{\Gamma(-s) \Gamma(3/2)}{\Gamma(3/2 - s)} \int_0^{q_0} dq \ U^{1-2s}(q), \quad (F.9) \]

where $q = 0$ and $q = q_0$ are the two minima of the potential. The function $I_0(s)$ has a simple pole for $s = 0$ and double poles for $s$ positive integer, indicating the presence of logarithms in higher powers of $g$.

The contribution from $g^2 S_2$ is [see equation (B.21)]:
\[ I_2(s) = \frac{1}{8} (2g)^{s+1} \frac{\Gamma(-s) \Gamma(-3/2)}{\Gamma(-3/2 - s)} \int_0^{q_0} dq \ U^{-3-2s} U'^2. \quad (F.10) \]

The function $I_2(s)$ has a simple pole for $s = -1$ and double poles for $s$ integer, $s \geq 0.$
F.4 General Potentials at Leading WKB Order

The residue at \( s = 0 \) of \( I_0(s) \) as defined in (F.9) is proportional to the instanton action (5.6):

\[
\frac{2}{g} \int_0^{q_0} dq U(q) = \frac{a}{g}. \tag{F.11}
\]

The residue and the double pole at \( s = 1 \) are also of crucial importance for the quantization condition (2.64). Indeed, by calculating the residues of the double and simple poles of \( I_0(s) \) at \( s = 1 \), we obtain the coefficients of \( E \ln E \) and \( E \) and thus terms which are generated only by the expansion of the \( \Gamma \)-functions and the factor \((-2C_1/g)^B\) of equation (3.57). We now combine (B.18), (B.19), and (F.9) in order to evaluate \( I_0(s) \). If we denote by \( 1 \) and \( \omega^2 \) the values of the second derivatives of the potential at the minima \( q = 0 \) and \( q = q_0 \), respectively [see (2.70)], then

\[
2 \int_0^{q_0} dq U^{1-2s}(q) = 2 \int_0^{q_0} dq \left[ U^{1-2s}(q) - q^{1-2s} - (\omega(q_0 - q))^{1-2s} \right]
+ q_0^{2-2s} \frac{1 + \omega^{1-2s}}{(1 - s)}
+ \frac{1}{(1 - s)} \left( 1 + \frac{1}{\omega} \right) + 2 \left( 1 + \frac{1}{\omega} \right) \ln q_0 + 2 \frac{\ln \omega}{\omega}. \tag{F.12}
\]

It follows, according to the correspondence (F.7), that the contributions to the integral on the left-hand side of (3.57) are

\[
E \left( \ln(-E/\omega) - 1 \right) + E \left( \ln(-E) - 1 \right)
+ E \left[ \left( 1 + \frac{1}{\omega} \right) \ln(-g/2) - 2 \frac{\ln \omega}{\omega} - 2 \ln \tilde{C} - 2 \left( 1 + \frac{1}{\omega} \right) \ln q_0 \right], \tag{F.13}
\]

with

\[
\ln \tilde{C} = \int_0^{q_0} dq \left[ \frac{1}{U(q)} - \frac{1}{q} - \frac{1}{\omega(q_0 - q)} \right]. \tag{F.14}
\]

This result is fully consistent with equations (2.73) and (5.3) as well as (C.10) below,

\[
B_1(E, g) = E + \mathcal{O}(g), \quad B_2(E, g) = E/\omega + \mathcal{O}(g), \quad C_\omega = q_0^2 \omega^{2(1+\omega)} \tilde{C}^{1+\omega}. \tag{F.15}
\]

We have used here the relation \( B(E, g) = E(1 + 1/\omega) \) which is valid for asymmetric wells in the convention (2.70). The result implied by this calculation, if combined with equations (3.55) and (3.57), is fully consistent with the quantization condition (5.16), which in turn is the expansion of the quantization condition (2.64) at leading order in \( g \).

F.5 Symmetric Potentials: Next–to–Leading WKB Order

We now specialize the treatment of the previous chapter to symmetric potentials. In the case of symmetric potentials with degenerate minima, which have \( \omega_1 = \omega_2 = 1 \) in the sense of (2.63) and (2.70), one can introduce the parameterization (B.6). Then,

\[
\int_0^{q_0} dq U^{1-2s}(q) = 2 \int_0^{u_0} du \frac{u^{1-2s}}{\sqrt{\rho(u)}}, \tag{F.16a}
\]

\[
\int_0^{q_0} dq U^{-3-2s}(q) U'^2(q) = 2 \int_0^{u_0} du u^{-3-2s} \sqrt{\rho(u)}, \tag{F.16b}
\]

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where \( u_0 \) is the zero of the function \( \rho \): \[
\rho(u_0) = 0. 
\] (F.17)

We now consider the residues of \( I_0(s) \).

- The residue at \( s = 0 \) of \( I_0(s) \) yields the leading contribution to \( A(E, g) \) [equation (F.9)]:
  \[
  \frac{a}{g} = \frac{2}{g} \int_0^{u_0} dq U(q) = \frac{4}{g} \int_0^{u_0} du \frac{u}{\sqrt{\rho(u)}}, 
  \] (F.18)

- The residues of the poles at \( s = 1 \) of \( I_0(s) \) require slightly more work:
  \[
  2 \int_0^{u_0} du \frac{u^{1-2s}}{\sqrt{\rho(u)}} = \frac{1}{1-s} + 2 \int_0^{u_0} du \left( \frac{1}{\sqrt{\rho(u)}} - 1 \right) + 2 \ln u_0 + \mathcal{O}(s-1). 
  \] (F.19)

We set, in accordance with (2.58),
\[
\ln C = 2 \int_0^{u_0} du \left( \frac{1}{\sqrt{\rho(u)}} - 1 \right) + 2 \ln u_0. 
\] (F.20)

Combining with the pole of the factor \( \Gamma(-s) \), one obtains the coefficients of \( E \ln(E) \) and \( E \), respectively,
\[
2E \ln E - 2E + 2E \ln(g/2C). 
\] (F.21)

- We consider the residue at \( s = 2 \) of \( I_0(s) \). For the term of order \( g \) we need the small-\( u \) expansion (B.6) of the function \( \rho \). Then,
  \[
  2 \int_0^{u_0} du \frac{u^{1-2s}}{\sqrt{\rho(u)}} = 2 \int_0^{u_0} du \left[ u^{1-2s} - \frac{1}{2} \alpha_1 u^{2-2s} + \left( -\frac{1}{2} \alpha_2 + 3 \frac{3}{8} \alpha_1^2 \right) u^{3-2s} \right] 
  + 2 \int_0^{u_0} du \left[ \frac{1}{\sqrt{\rho(u)}} - 1 + \frac{1}{2} \alpha_1 u - \left( -\frac{1}{2} \alpha_2 + 3 \frac{3}{8} \alpha_1^2 \right) u^2 \right] 
  = \left( \frac{1}{2} \alpha_2 + 3 \frac{3}{8} \alpha_1^2 \right) \frac{1}{2-s} + \tilde{a}_{2,2} + \mathcal{O}(s-2) 
  \] (F.22)

with
\[
\tilde{a}_{2,2} = 2 \int_0^{u_0} du \left[ \frac{1}{\sqrt{\rho(u)}} - 1 + \frac{1}{2} \alpha_1 u - \left( -\frac{1}{2} \alpha_2 + 3 \frac{3}{8} \alpha_1^2 \right) u^2 \right] 
  - \frac{1}{u_0^2} \alpha_1 + \frac{\alpha_1}{u_0} + 2 \left( -\frac{1}{2} \alpha_2 + 3 \frac{3}{8} \alpha_1^2 \right) \ln u_0. 
\] (F.23)

The factor then yields
\[
2^{s+1} g^{s-1} \frac{\Gamma(-s)\Gamma(3/2)}{\Gamma(3/2-s)} = -\frac{g}{2-s} + \frac{g}{2} + g \ln(g/2) + \mathcal{O}(s-2). 
\] (F.24)

Combining factors, one obtains a contribution to \( gE^2 \ln E \):
\[
\left( -\frac{1}{2} \alpha_2 + 3 \frac{3}{8} \alpha_1^2 \right) gE^2 \ln E 
\] (F.25)

and to \( gE^2 \):
\[
\left[ \frac{1}{2} + \ln \left( \frac{g}{2} \right) \left( -\frac{1}{2} \alpha_2 + 3 \frac{3}{8} \alpha_1^2 \right) - \tilde{a}_{2,2} \right] gE^2. 
\] (F.26)
APPENDIX F: WKB CALCULATION AND INSTANTON CONTRIBUTIONS

F.5 Symmetric Potentials: Next–to–Leading WKB Order

We now consider the residues of \( I_2(s) \). We first rewrite the expression (F.10) in terms of the function (B.6) as

\[
I_2(s) = \frac{1}{4} (2g)^{s+1} \Gamma(-s) \frac{\Gamma(-3/2)}{\Gamma(-3/2-s)} \int_{0}^{u_0} du \, u^{-3-2s} \sqrt{\rho(u)}. \tag{F.27}
\]

The first residues of \( I_2(s) \) may now be evaluated as follows:

- From \( s \to -1 \), one obtains the contribution \(-1/12E\).
- For \( s \to 0 \), the factor in front of the integral in (F.10) has the expansion

\[
-\frac{g}{2s} - \frac{g}{2} \ln(g/2) - \frac{4g}{3}. \tag{F.28}
\]

In combining (F.16b) with the parameterization (B.6), we conclude that the integral itself yields

\[
\int_{0}^{u_0} du \, u^{-3-2s} \sqrt{\rho(u)}
= \int_{0}^{u_0} du \, u^{-3-2s} \left[ 1 + \frac{1}{2} \alpha_1 u + \left( \frac{1}{2} \alpha_2 - \frac{1}{8} \alpha_1^2 \right) u^2 \right]
+ \int_{0}^{u_0} \frac{du}{u^3} \left[ \sqrt{\rho(u)} - 1 - \frac{1}{2} \alpha_1 u - \left( \frac{1}{2} \alpha_2 - \frac{1}{8} \alpha_1^2 \right) u^2 \right]
= \left( -\frac{1}{2s} + \ln u_0 \right) \left( \frac{1}{2} \alpha_2 - \frac{1}{8} \alpha_1^2 \right) - \frac{\alpha_1}{2 u_0^2} - \frac{\alpha_1}{2 u_0}
+ \int_{0}^{u_0} \frac{du}{u^3} \left[ \sqrt{\rho(u)} - 1 - \frac{1}{2} \alpha_1 u - \left( \frac{1}{2} \alpha_2 - \frac{1}{8} \alpha_1^2 \right) u^2 \right]. \tag{F.29}
\]

Using (F.28) and (F.7b), we infer that the contribution proportional to \( g \ln E \) thus is

\[
-g \ln E \left( \frac{1}{8} \alpha_2 - \frac{1}{32} \alpha_1^2 \right). \tag{F.30}
\]

The coefficient of \( g \) (without logarithms) may be deduced from (F.28) and (F.7a),

\[
\frac{1}{2} \int_{0}^{u_0} \frac{du}{u^3} \left[ \sqrt{\rho(u)} - 1 - \frac{1}{2} \alpha_1 u - \left( \frac{1}{2} \alpha_2 - \frac{1}{8} \alpha_1^2 \right) u^2 \right]
- \frac{1}{4 u_0^2} \frac{\alpha_1}{4 u_0} - \left( \frac{1}{8} \alpha_2 - \frac{1}{32} \alpha_1^2 \right) \left[ \ln \left( \frac{g}{2} \right) + \frac{8}{3} - 2 \ln u_0 \right]. \tag{F.31}
\]

Meanwhile, we have evaluated certain contributions to the Mellin transform of the contour integrals of the leading orders of the WKB expansion. These Mellin transforms were denoted by \( I_0 \) and \( I_2 \), and after a Mellin back-transformation (F.7), these terms enter on the left-hand side of equation (3.57). We would now like to regroup these terms into a form which is amenable to the identification of the functions \( A(E,g) \) and \( B(E,g) \). These latter functions are found on the right-hand side of (3.57). Using the expansion of the \( \Gamma \) function and setting \( B(E,g) = E + g b_2(E) + O(g^2) \), one finds

\[
\frac{1}{2} \ln(2\pi) - \ln \Gamma \left( \frac{1}{2} - B(E,g) \right) + B(E,g) \ln \left( -\frac{g}{2C} \right)
\sim \left[ E + g b_2(E) \right] \ln \left( \frac{gE}{2C} \right) - E - \frac{1}{24E} + \cdots. \tag{F.32}
\]

Here, advantage has been taken of the fact that the WKB-expansion is an expansion in \( g \) at fixed \( Eg \), wherefore \( E = O(1/g) \). In order to obtain \( A(E,g) \), one has to subtract this contribution twice, because we have a symmetric potential, with two equal contributions from each of the two wells. One sees immediately that the terms of order \( g^0 \) cancel and as well as the term \( 2b_2(E) \ln(gE/2) \) [see equation (B.11)], as expected from general arguments.
We are now in the position to write down explicit expressions for the function \( A(E, g) \) at order \( g \), for symmetric potentials which follow the parameterization (B.6). The contribution at order \( g \) is then
\[
A(E, g) = \frac{a}{g} + g \left( a_{2,2} E^2 + a_{2,0} \right) + \mathcal{O}(g^2),
\]
where \( a_{2,2} \) and \( a_{2,0} \) are given by
\[
a_{2,2} = \left( \frac{1}{2} + \ln C \right) \left( -\frac{1}{2} \alpha_2 + \frac{3}{8} \alpha_1^2 \right) - \tilde{a}_{2,2},
\]
\[
a_{2,0} = \frac{1}{2} \int_0^{u_0} \frac{d u}{u^3} \left[ \sqrt{\rho(u)} - 1 - \frac{1}{2} \alpha_1 u - \left( \frac{1}{2} \alpha_2 - \frac{1}{8} \alpha_1^2 \right) u^2 \right] u^2
- \frac{1}{4 u_0^2} - \frac{\alpha_1}{4 u_0} + \left( \frac{1}{8} \alpha_2 - \frac{1}{32} \alpha_1^2 \right) \left( \ln C - \frac{8}{3} + 2 \ln u_0 \right).
\]
The explicit result for \( \tilde{a}_{2,2} \) can be found in (F.23).

\section*{F.6 Explicit evaluation for a Special Family of Potentials}

Let us remember at this stage that the basic paradigm for the evaluation of the “instanton function” \( A(E, g) \) that enters into (2.24b) is the following: start from the conjectured structure of the WKB expansion (3.57),
\[
\frac{1}{g} \oint_{C'} dz S_+(z) = A(E, g) + \ln(2\pi)
- \sum_{i=1}^{2} \left\{ \ln \Gamma \left( \frac{1}{2} - B_i(E, g) \right) + B_i(E, g) \ln(-g/2C_i) \right\}.
\]
from which using (3.43)
\[
\exp \left[ -\frac{1}{g} \oint_{C'} dz S_+(z) \right] + 1 = 0,
\]
the quantization condition (2.64) follows immediately. The perturbative expansions \( B_i(E, g) \) can in general be evaluated easily using the techniques described in chapter 3.2.5. The goal is the calculation of \( A(E, g) \).

One may calculate successive orders in the WKB expansion using the algorithm (3.41), resulting in an approximation for the left-hand side of (F.35). Because the \( B_i(E, g) \) on the right-hand side of (F.35) may easily be calculated and expanded according to (F.2), it is then possible to calculate \( A(E, g) \) by subtracting those terms that are generated by the \( B_i(E, g) \) on the right-hand side of (F.35), from the result obtained for the WKB expansion on the left-hand side, and obtain a result for \( A(E, g) \).

For classes of potentials for which the function \( \rho \) has a simple form, the expressions simplify. We thus consider now the class (B.23) of potentials which satisfy \( \rho(u) = 1 - 4u^m \). Then, the two integrals (F.16a) and (F.16b) can be calculated explicitly. One finds
\[
\int_0^{u_0} d u \frac{u^{1-2s}}{\sqrt{1-4u^m}} = \frac{1}{m} 4^{2(s-1)/m} \frac{\Gamma \left( \frac{2}{m} (1-s) \right) \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} + \frac{2}{m} (1-s) \right)},
\]
\[
\int_0^{u_0} d u \frac{u^{-3-2s}}{\sqrt{1-4u^m}} = \frac{1}{m} 4^{2(s+1)/m} \frac{\Gamma \left( \frac{1}{2} + \frac{2}{m} (1+s) \right)}{\Gamma \left( \frac{1}{2} - \frac{2}{m} (1+s) \right)}.
\]
The leading WKB order then yields
\[
I_0(s) = \frac{1}{m} g^{s-1} 2^{(1+4/m)+2-4/m} \frac{\Gamma(-s) \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{2}{m} (1-s) \right)}{\Gamma \left( \frac{3}{2} - s \right) \Gamma \left( \frac{1}{2} + \frac{2}{m} (1-s) \right)}.
\]
At next order one finds
\[
I_2(s) = \frac{1}{m} g^{s+1} 2^{(1+4/m)+4-4/m-1} \frac{\Gamma(-s) \Gamma(-\frac{3}{2}) \Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{2}{m} (1+s) \right)}{\Gamma \left( \frac{3}{2} - s \right) \Gamma \left( \frac{3}{2} - \frac{2}{m} (1+s) \right)}.
\]
One verifies that the residue of the double pole in $s$ coincide, as expected, with the terms appearing in the expansion of $B(E, g)$.

(The double-well potential.) This is a symmetric case in which $B_1 = B_2 = B$ and $C = 1$ [see equations (2.21), (2.22), (2.24a) and (2.58)]. Note that according to (F.17), $u_0 = 1/2$ for the double-well potential. We recall the relations $U(q) = q(1-q)$, as well as $\rho = \sqrt{U''}$ and $\rho(u) = \sqrt{1-4u}$ valid for the double-well. At leading order, one finds (the duplication formula of the $\Gamma$ function has been used)

$$I_0(s) = -g^{s-1}2^{3s-1} \frac{\Gamma^2(1-s)\Gamma(3/2)}{s\Gamma(5/2-2s)}.$$  \hfill (F.40)

For $s \to 0$, one recovers the contribution $1/3g$.

Expanding for $s \to 1$ and again using the correspondence (F.7), one obtains the contributions to (3.57)

$$2E\ln(-E) - 2E + 2E\ln(-g/2) \sim -2\ln\Gamma\left(\frac{1}{2} - E\right) - 2E\ln(-2/g).$$ \hfill (F.41)

Similarly, for $s \to 2$,

$$6gE^2\ln\left(\frac{Eg}{2}\right) + 17gE^2$$ \hfill (F.42)

and for $s \to 3$:

$$70g^2E^3\ln\left(\frac{Eg}{2}\right) + 236g^2E^3.$$ \hfill (F.43)

To obtain the contributions to $A(E, g)$, one has to subtract various contributions of order $g^2E^3$. One of these originates from the expansion of the term

$$B(E, g) \ln B(E, g) \sim gb_2(E) \ln[E(1+(g/E)b_2(E))]
\sim g^2b_2^2(E)/E \sim 3^2g^2E^3/E = 9g^2E^3$$ \hfill (F.44)

[see equations (2.23a) and (2.24a)]. The term $B(E, g) \ln B(E, g)$, in turn, comes from the expansion (F.2) of the $\Gamma$-function in (F.32). The other terms of order $g^2E^3$ cancel against each other. The result for the term of order $g^2E^3$, which is $236 - 9 = 227$, is in agreement with (2.24b). In order to obtain the results presented in chapter 8.4, the contributions of order $g^3$ and $g^4$ are useful. We have from the contribution of $s \to 4$:

$$1155g^3E^4\ln\left(\frac{Eg}{2}\right) + \frac{49843}{12}g^3E^4$$ \hfill (F.45)

and from $s \to 5$:

$$\frac{45045}{2}g^4E^5\ln\left(\frac{Eg}{2}\right) + \frac{335183}{4}g^4E^5.$$ \hfill (F.46)
APPENDIX F: WKB CALCULATION AND INSTANTON CONTRIBUTIONS

F.6 Explicit evaluation for a Special Family of Potentials

The complete result for the contour integral of the leading WKB term, through the order of $g^6$, is

$$\frac{1}{g} \oint dz S_0(q, g, E) = \frac{1}{3g} + \left\{ 2E \ln \left( \frac{Eg}{2} \right) - 2E \right\}$$

$$+ g \left( 6E^2 \ln \left( \frac{Eg}{2} \right) + 17E^2 \right)$$

$$+ g^2 \left( 70E^3 \ln \left( \frac{Eg}{2} \right) + 236E^3 \right)$$

$$+ g^3 \left( 1155E^4 \ln \left( \frac{Eg}{2} \right) + \frac{49843}{12}E^4 \right)$$

$$+ g^4 \left( \frac{45045}{2} E^5 \ln \left( \frac{Eg}{2} \right) + \frac{335183}{4}E^5 \right) .$$

$$+ g^5 \left( \frac{11056741}{6} E^6 \ln \left( \frac{Eg}{2} \right) + \frac{969969}{2}E^6 \right)$$

$$+ g^6 \left( \frac{515954137}{12} E^7 \ln \left( \frac{Eg}{2} \right) + \frac{22309287}{2}E^7 \right) +$$

$$+ g^7 \left( \frac{2151252675}{8} E^8 \ln \left( \frac{Eg}{2} \right) + \frac{469212586743}{448}E^8 \right)$$

$$+ g^8 \left( \frac{214886239425}{32} E^9 \ln \left( \frac{Eg}{2} \right) + \frac{70860581490397}{2688}E^9 \right),$$

where we neglect terms of order $g^8$ and higher. At next WKB order

$$I_2(s) = g^{-s+1} 2^{5s+1} \frac{\Gamma(-1-s) \Gamma\left(-s - \frac{3}{2}\right) \Gamma\left(s + \frac{3}{2}\right) \Gamma\left(-1 - 2s\right)}{\Gamma\left(-s - 2\right)} .$$

(F.47)

$I_2(s)$ can be rewritten as

$$I_2(s) = -g^{-s+1} 2^{3s-1} \frac{\Gamma^2(-s)\Gamma\left(-s - \frac{3}{2}\right) 1 + \frac{2}{3}s}{\Gamma\left(-s - 2\right)} .$$

(F.48)

For $s \to -1$, one obtains a singular term contributing to the asymptotic expansion of the $\Gamma$-function:

$$I_2 \sim \frac{1}{12(s + 1)} \Rightarrow -\frac{1}{12E} .$$

(F.50)

From the expansion for $s \to 0$, one infers

$$-\frac{g}{2s^2} \left( 1 - s \ln \left( \frac{2}{g} \right) + \frac{11}{3}s \right) .$$

(F.51)

This yields a contribution to (3.57):

$$\frac{1}{2} g \ln \left( \frac{Eg}{2} \right) + \frac{11}{6}g .$$

(F.52)

To obtain a contribution to $A$, one must now take into account the correction coming from replacing $E$ by $B$ in the expansion (F.32) of $\Gamma\left(\frac{1}{2} - B\right)$. At this order only $-1/(12B)$ contributes:

$$-\frac{1}{12B} = -\frac{1}{12E} + \frac{g}{4} + \cdots .$$

(F.53)

It follows that the contribution to $A(E, g)$ is $(11/6) - 1/4 = 19/12$. Thus, the expansion (2.24b) has been verified.

For $s \to 1$, one infers the contribution to (3.57):

$$\frac{25}{2} g^2 E \ln \left( \frac{Eg}{2} \right) + \frac{605}{12} gE .$$

(F.54)
Now three terms in (F.32) contribute, involving $B$ up to order $g^2$ and (equation (2.24a)) one finds $605/12 - 3/2 - 35/12 + 9/12 = 187/4$, in agreement with (2.24b).

The complete result for the contour integral of the WKB term $S_2$, up to the order $g^8$, is

$$\frac{1}{g} \int dz \left[ g^2 S_2(q, g, E) \right] = -\frac{1}{12E} + g \left( \frac{1}{2} \ln \left( \frac{Eg}{2} \right) + \frac{11}{6} \right)$$

$$+ g^2 \left( \frac{25}{2} E \ln \left( \frac{Eg}{2} \right) + \frac{605}{12} E \right)$$

$$+ g^3 \left( \frac{735}{2} E^2 \ln \left( \frac{Eg}{2} \right) + \frac{4522}{3} E^2 \right)$$

$$+ g^4 \left( \frac{45045}{4} E^3 \ln \left( \frac{Eg}{2} \right) + \frac{743439}{16} E^3 \right)$$

$$+ g^5 \left( \frac{2807805}{8} E^4 \ln \left( \frac{Eg}{2} \right) + \frac{69706241}{48} E^4 \right)$$

$$+ g^6 \left( \frac{88267179}{8} E^5 \ln \left( \frac{Eg}{2} \right) + \frac{1097349517}{24} E^5 \right)$$

$$+ g^7 \left( \frac{2788660875}{8} E^6 \ln \left( \frac{Eg}{2} \right) + \frac{69402310265}{48} E^6 \right)$$

$$+ g^8 \left( \frac{353522522925}{32} E^7 \ln \left( \frac{Eg}{2} \right) + \frac{82167014713033}{1792} E^7 \right),$$

where again terms of order $g^9$ and higher are neglected. This result is required for the calculations presented in chapter 8.4.

Without further calculational details, we also give here the contour integral of $g^4 S_4$ up to the order $g^8$:

$$\frac{1}{g} \int dz \left[ g^4 S_4(q, g, E) \right] = \frac{7}{1440E^3} - \frac{11g}{480E^2} + \frac{101g^2}{480E}$$

$$+ g^3 \left( \frac{175}{16} \ln \left( \frac{Eg}{2} \right) + \frac{17473}{288} \right)$$

$$+ g^4 \left( \frac{31185}{32} E \ln \left( \frac{Eg}{2} \right) + \frac{616601}{128} E \right)$$

$$+ g^5 \left( \frac{1924923}{32} E^2 \ln \left( \frac{Eg}{2} \right) + \frac{544644431}{1920} E^2 \right)$$

$$+ g^6 \left( \frac{100553453}{32} E^3 \ln \left( \frac{Eg}{2} \right) + \frac{83125560313}{5760} E^3 \right)$$

$$+ g^7 \left( \frac{9526065549}{64} E^4 \ln \left( \frac{Eg}{2} \right) + \frac{645115861327}{960} E^4 \right)$$

$$+ g^8 \left( \frac{1691601686775}{256} E^5 \ln \left( \frac{Eg}{2} \right) + \frac{60366482211337}{2048} E^5 \right).$$
APPENDIX F: WKB CALCULATION AND INSTANTON CONTRIBUTIONS

F.7 Alternative Methods for the Contour Integrals

The contour integral of \( g^6 S_6 \), up to \( O(g^8) \), reads:

\[
\frac{1}{g} \oint dz \left[ g^6 S_6(q, g, E) \right] = \frac{31}{20160 E^6} + \frac{87 g}{4480 E^4} + \frac{359 g^2}{40320 E^3} - \frac{15 g^3}{128 E^2} + \frac{2515 g^4}{480 E} + g^5 \left( \frac{159 159}{128} \ln \left( \frac{Eg}{2} \right) + \frac{59665 801}{7680} \right) + g^6 \left( \frac{25746 721}{128} E \ln \left( \frac{Eg}{2} \right) + \frac{25285 094 891}{23040} \right) + g^7 \left( \frac{2 506 538 463}{128} E^2 \ln \left( \frac{Eg}{2} \right) + \frac{53760}{20480} \right) + g^8 \left( \frac{758 382 964 625}{512} E^3 \ln \left( \frac{Eg}{2} \right) + \frac{1 884 561 008 165 335}{258048} \right).
\] (F.57)

The term \( g^8 S_8 \), upon contour integration, yields to the order \( g^8 \),

\[
\frac{1}{g} \oint dq S_8(q, g, E) = \frac{127}{107520 E^7} - \frac{7381 g}{322560 E^6} + \frac{217 g^2}{9216 E^5} - \frac{3377 g^3}{215040 E^4} + \frac{2195 g^4}{12288 E^3} - \frac{593 329 g^5}{12288 E^2} + \frac{3344883 g^6}{20480 E} + g^7 \left( \frac{692 049 787}{2048} E \ln \left( \frac{Eg}{2} \right) + \frac{5 392 814 329 807}{258048} \right) + g^8 \left( \frac{663 834 081 625}{8192} E^2 \ln \left( \frac{Eg}{2} \right) + \frac{9 724 807 577 177 167}{20643840} \right).
\] (F.58)

The leading terms in \( 1/E \) reproduce the coefficients of the asymptotic expansion of the \( \Gamma \)-function in equation (F.2).

A similar calculation can immediately be repeated for the cosine potential. Note that for the radial Schrödinger equation, similar calculations are possible with slight technical modifications, because the WKB expansion is valid for both \( E \) and \( \nu + 2l \) large.

F.7 Alternative Methods for the Contour Integrals

In chapter F.6, we have considered the evaluation of contour integrals of successive terms in the WKB expansion by the method of the Mellin transform. In the current chapter, two alternative methods will be discussed: (i) a “subtraction procedure” and (ii) an explicit direct evaluation of the contour integral which is inspired by Lamb-shift calculations [31].

(Method (i).) We consider the double-well problem and use the convention \( U(q) = q (1 - q) \). Starting from the expansion (B.18),

\[
S_0(q, g, E) = q (1 - q) \sum_{n=0}^{\infty} \left( \frac{2 g E}{q^2 (1 - q)^2} \right)^n \frac{\Gamma(n - \frac{1}{2})}{\Gamma(n + 1) \Gamma(-\frac{1}{2})},
\] (F.59)

the contour integral of \( S_0(q, g, E) \) can be written as

\[
\frac{1}{g} \int dq S_0(q, g, E) = \frac{2}{g} \int_0^1 dq S_0(q, g, E) = \frac{2}{g} \int_0^1 dq \sqrt{U^2(q) - 2gE} = \frac{2}{g} \sum_{n=0}^{\infty} \frac{\Gamma(n - \frac{1}{2})}{\Gamma(n + 1) \Gamma(-\frac{1}{2})} (2 g E)^n \int_0^1 dq q^{1-2n} (1 - q)^{1-2n}.
\] (F.60)
Here, we have interchanged the infinite sum with the integration, which is not a mathematically rigorous procedure in the current situation. Moreover, formally, the resulting integrals do not converge for general positive integer \( n \), because of the asymptotic behaviour of the integrand near \( q \to 0 \) and \( q \to 1 \). However, it is easy to evaluate the integral in terms of the Beta function for arbitrary \( n \), and to perform an expansion about integer \( n \), by setting \( n \to n + \eta \). The following series results,

\[
\frac{1}{g} \int dz \, S_0(q, g, E) = \frac{1}{3g} + \left\{ 2E \ln \left( \frac{Eg}{2} \right) - 2E \right\} \\
+ g \left( 6E^2 \left\{ \ln \left( \frac{Eg}{2} \right) + \frac{1}{\eta} + \mathcal{O}(\eta) \right\} + 17E^2 \right) \\
+ g^2 \left( 70E^3 \left\{ \ln \left( \frac{Eg}{2} \right) + \frac{1}{\eta} + \mathcal{O}(\eta) \right\} + 236E^3 \right) \\
+ g^3 \left( 1155E^4 \left\{ \ln \left( \frac{Eg}{2} \right) + \frac{1}{\eta} + \mathcal{O}(\eta) \right\} + \frac{49843}{12}E^4 \right) \\
+ g^4 \left( \frac{45045}{2} E^5 \left\{ \ln \left( \frac{Eg}{2} \right) + \frac{1}{\eta} + \mathcal{O}(\eta) \right\} + \frac{335183}{4}E^5 \right) + \mathcal{O}(g^5). \tag{F.61} \]

When employing the prescription of neglecting the divergent terms in \( \eta \), the result (F.47) is recovered. This strategy is in formal analogy to the “minimal subtraction” procedure—however, we here do not change the dimensionality of the integration.

(Method (ii).) We now discuss the applicability of the so-called \( \epsilon \)-method for the evaluation of the contour integrals. This method has been used for the evaluation of integrals which give rise to logarithmic terms in addition to the usual power terms in an asymptotic expansion (for a discussion, including a number of illustrative examples, and further references, the reader may consider [31, 32]). We observe that the logarithms originate because of the asymptotic behaviour of the integrand near \( q \approx gE \), but actual divergences result when the expansion in powers of \( gE \) is performed. We may therefore consider to introduce an arbitrary separation parameter \( \epsilon \) which separates the regions \((0, \epsilon)\) and \((1 - \epsilon, 1)\) on the one side from the region \((\epsilon, 1 - \epsilon)\) on the other side. In the latter region, we may safely expand in powers of \( gE \), whereas in the first region, we may expand in \( q \) (or \( 1 - q \)) which is effectively an expansion in \( gE \) (this is in analogy to the example considered in appendix A of [31]). The arbitrary parameter \( \epsilon \) cancels if the results of the integrations are expanded first in \( gE \), then in \( \epsilon \).

The integration regions \((\epsilon, 1 - \epsilon)\) gives the following contribution:

\[
\frac{2}{g} \int_{\epsilon}^{1 - \epsilon} dq \sqrt{q^2 (1 - q)^2 - 2gE} \\
= \frac{1}{3g} + 4E \ln \epsilon \\
+ gE^2 \left[ 7 + 12 \ln \epsilon - \frac{6}{\epsilon} - \frac{1}{\epsilon^2} \right] \\
+ g^2 E^3 \left[ \frac{533}{6} + 140 \ln \epsilon - \frac{70}{\epsilon} - \frac{15}{\epsilon^2} - \frac{10}{3\epsilon^3} - \frac{1}{2\epsilon^4} \right] \\
+ g^3 E^4 \left[ \frac{18107}{12} - 2310 \ln \epsilon - \frac{1155}{\epsilon} - \frac{525}{2\epsilon^2} - \frac{70}{\epsilon^3} - \frac{35}{2\epsilon^4} - \frac{7}{2\epsilon^5} - \frac{5}{12\epsilon^6} \right] \\
+ g^4 E^5 \left[ \frac{477745}{16} + 45045 \ln \epsilon - \frac{45045}{2\epsilon} - \frac{21021}{4\epsilon^2} - \frac{3003}{2\epsilon^3} - \frac{3465}{8\epsilon^4} - \frac{231}{2\epsilon^5} - \frac{105}{4\epsilon^6} - \frac{9}{2\epsilon^7} - \frac{7}{16\epsilon^8} \right] \tag{F.62} \]

We should add the result from the regions $(0, \epsilon)$ and $(1 - \epsilon, 1)$:

\[
\frac{2}{g} \left( \int_0^\epsilon \int_{1-\epsilon}^1 dq \sqrt{q^2 (1-q)^2 - 2gE} \right)
= 2E \ln \left( \frac{Eg}{2} \right) - 1 - 2 \ln \epsilon
+ gE \left[ 6 \ln \left( \frac{Eg}{2} \right) + 10 - 12 \ln \epsilon + \frac{6}{\epsilon^2} + \frac{1}{\epsilon^2} \right]
+ g^2 E^2 \left[ 70 \ln \left( \frac{Eg}{2} \right) + \frac{883}{6} - 140 \ln \epsilon + \frac{70}{3 \epsilon^2} + \frac{10}{2 \epsilon^2} \right]
+ g^3 E^3 \left[ 1155 \ln \left( \frac{Eg}{2} \right) + \frac{7934}{3} - 2310 \ln \epsilon + \frac{1155}{\epsilon} + \frac{525}{2 \epsilon^2} + \frac{70}{\epsilon^2} + \frac{7}{2 \epsilon^2} + \frac{5}{12 \epsilon^2} \right]
+ g^4 E^4 \left[ 45045 \ln \left( \frac{Eg}{2} \right) + \frac{862987}{16} - 45045 \ln \epsilon + \frac{45045}{2 \epsilon} + \frac{21021}{4 \epsilon^2} \right.
+ \left. \frac{3903}{2 \epsilon^3} + \frac{3465}{8 \epsilon^4} + \frac{231}{2 \epsilon^5} + \frac{105}{4 \epsilon^6} + \frac{9}{2 \epsilon^7} + \frac{7}{16 \epsilon^8} \right].
\] (F.63)

Adding the contributions, the dependence on $\epsilon$ cancels, and the result (F.47) is recovered.

**F.8 Four–Instanton Coefficients**

We give here the analytic expression for the four-instanton shift of the energy eigenvalues of the double-well oscillator up to eighth order in $g$. There are four infinite series in $g$, each multiplying a specific power of the logarithmic factor $\lambda(g) \equiv \ln(-2/g)$. The shift $E_0^{(4)}(g)$ is the sum

\[
E_0^{(4)}(g) = \sum_{k=0}^{3} L_k(g),
\] (F.64)

with the $L_k(g)$ given below. The leading term, for small $g$, is of the order $\xi^4(g)\lambda^3(g)$.

\[
L_3(g) = \xi^4(g)\lambda^3(g) \left\{ \frac{8}{27} g^2 - \frac{1156}{243} g^3 - \frac{16276129}{1458} g^4 - 550613176 \frac{g^5}{2187} - \frac{123969635693}{2834352} g^6 + \mathcal{O}(g^7) \right\}.
\] (F.65)

$L_2(g)$ is given by

\[
L_2(g) = \xi^4(g)\lambda^2(g) \left\{ \frac{8}{3} \gamma + \frac{352}{9} \gamma^2 + \frac{13394}{243} g^3 - \frac{16276129}{1458} g^4 - \frac{550613176}{2187} g^5 - \frac{123969635693}{2834352} g^6 + \mathcal{O}(g^7) \right\}.
\] (F.66)
The coefficients entering into $L_1(g)$ are more complex,

$$L_1(g) = \xi^4(g) \lambda(g) \left\{ \frac{8}{3} \gamma^3 + 2 \gamma \zeta(2) + \frac{352}{9} \gamma^3 - \frac{88}{3} \zeta(2) \right\} g^2 \right.$$

$$+ \left( \frac{2564}{3} + \frac{3044}{3} \gamma + \frac{1156}{9} \gamma^2 + \frac{289}{9} \gamma \zeta(2) \right) g^3$$

$$+ \left( \frac{9781}{18} - \frac{10700}{9} \gamma - \frac{133394}{81} \gamma^2 - \frac{66679}{162} \zeta(2) \right) g^4$$

$$+ \left( \frac{153739}{648} - \frac{3332681}{81} \gamma^2 - \frac{16276129}{486} \gamma^3 - \frac{16276129}{1944} \zeta(2) \right) g^5$$

$$+ \left( \frac{263237449}{9720} - \frac{316478342}{243} \gamma - \frac{550613176}{729} \gamma^2 - \frac{17653294}{729} \zeta(2) \right) g^6$$

$$+ \left( \frac{82487435753}{58320} - \frac{117850396597}{2916} \gamma - \frac{123969635693}{6561} \zeta(2) - \frac{123969635693}{262448} \zeta(3) \right) g^7$$

$$+ \left( \frac{221545831304111}{3674160} - \frac{40994182753031}{314928} \gamma^2 - \frac{1079940312204816344}{1717917660985273} \zeta(2) \right) g^8$$

$$+ \left( \frac{404089920}{39366} - \frac{14705773688939005}{3779136} \gamma^2 - \frac{14705773688939005}{3779136} \zeta(2) + \mathcal{O}(g^9) \right) .$$

The series $L_0(g)$ is free of any logarithms,

$$L_0(g) = \xi^4(g) \left\{ \frac{8}{3} \gamma^3 + 2 \gamma \zeta(2) + \frac{1}{3} \zeta(3) \right\} g^2$$

$$+ \left( \frac{555}{2} + \frac{2564}{3} \gamma + \frac{1522}{9} \gamma^2 + \frac{1156}{27} \gamma^3 + \frac{761}{6} \zeta(2) + \frac{289}{9} \gamma \zeta(2) + \frac{289}{54} \zeta(3) \right) g^3$$

$$+ \left( \frac{4931}{4} - \frac{9781}{18} \gamma - \frac{5350}{9} \gamma^2 - \frac{133394}{243} \gamma^3 + \frac{2675}{18} \gamma \zeta(2) - \frac{66679}{162} \gamma \zeta(2) - \frac{66679}{972} \zeta(3) \right) g^4$$

$$+ \left( \frac{98395}{48} - \frac{3332681}{648} \gamma - \frac{153739}{162} \gamma^2 - \frac{16276129}{1458} \gamma^3 - \frac{16276129}{1164} \zeta(2) - \frac{16276129}{1164} \zeta(3) \right) g^5$$

$$+ \left( \frac{5322143}{108} - \frac{2623137449}{9720} \gamma - \frac{550613176}{243} \gamma^2 - \frac{550613176}{2187} \gamma^3 \right) g^6$$

$$+ \left( \frac{1803744497}{2592} - \frac{82487435753}{58320} \gamma - \frac{177850396597}{5832} \gamma^2 - \frac{173969635693}{19683} \gamma^3 \right) g^7$$

$$+ \left( \frac{400142382153}{38880} - \frac{221545831304111}{3647160} \gamma - \frac{42495585076178}{6561} \gamma^2 - \frac{40994182753031}{236196} \zeta(2) \right) g^8$$

$$+ \left( \frac{212478258089}{15122} - \frac{1079940312204816344}{314928} \gamma - \frac{1717917660985273}{78732} \zeta(2) \right) g^9$$

$$+ \left( \frac{14705773688939005}{284352} - \frac{14705773688939005}{314928} \gamma^2 - \frac{14705773688939005}{22674816} \zeta(3) \right) g^8 + \mathcal{O}(g^9) .$$

**F.9 Higher–Order Coefficients**

Some instanton coefficients have already been presented in chapters 8.4, 8.7 as well as in appendix F.8. Here, we present numerical data for all coefficients up to eight-instanton order, and up to seventh order in $g$. The analytic expressions become rather involved; numerical data in table F.1 exhibits the rapid (factorial) growth of the coefficients in higher orders in $g$, as well as the rapid growth for fixed order in $g$ in higher instanton-order.
APPENDIX F: WKB CALCULATION AND INSTANTON CONTRIBUTIONS

F9 Higher–Order Coefficients

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<th>n</th>
<th>k</th>
<th>(c_{0,n,k0})</th>
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<th>(c_{0,n,k4})</th>
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Table F.1: The \(c\)-coefficients (instanton coefficients) determine the resurgent expansion (2.13) for the energy eigenvalues of the quantum mechanical double-well oscillator. Here, all coefficients up to eighth order in the instanton interaction (up to the “eight-instanton order”) and up to seventh order in the coupling constant \(g\) are considered. Calculations are carried out for the ground state. We recall that \(c_{0,n,k0}\) is the coefficient multiplying the term \(\xi(g)\lambda(g)g^{k}\). The coefficient \(c_{0,n,(n-1)0}\) multiplies leading term in the \(n\)-instanton order. There is a rapid growth of the absolute magnitude of the coefficients in higher orders in \(g\) as well as in higher orders in \(n\).
Bibliography


[10] I. W. Herbst and B. Simon, Phys. Lett. B 78, 304 (1978). The special examples discussed in this work, together with the vanishing perturbation series, find a natural explanation in terms of the concepts discussed here in chapters 2.2.4 and 7.2.3.


