Chiral Symmetry Breaking and the Dirac Spectrum at Nonzero Chemical Potential

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The relation between the spectral density of the QCD Dirac operator at nonzero baryon chemical potential and the chiral condensate is investigated. We use the analytical result for the eigenvalue density in the microscopic regime which shows oscillations with a period that scales as $1/V$ and an amplitude that diverges exponentially with the volume $V = L^4$. We find that the discontinuity of the chiral condensate is due to the whole oscillating region rather than to an accumulation of eigenvalues at the origin. These results also extend beyond the microscopic regime to chemical potentials $\mu \sim 1/L$.

Introduction. One of the salient features of QCD at low energy is the spontaneous breaking of chiral symmetry characterized by a discontinuity of the chiral condensate. More than two decades ago it was realized by Banks and Casher \[1\] that the discontinuity of the chiral condensate at zero quark mass is proportional to the eigenvalue density of full QCD includes the fermion determinant in the average

$$\rho_{N_f}(x, y, m; \mu) = \frac{\langle \sum_k \delta^2(x + iy - z_k) \det^{N_f}(D + m) \rangle}{\langle \det^{N_f}(D + m) \rangle} \quad (4)$$

Strictly speaking, since $\rho_{N_f}$ is in general complex, this is not a density. Due to chiral symmetry the eigenvalues occur in pairs $\pm z_k$ so that $\rho_{N_f}(x, y, m; \mu) = \rho_{N_f}(-x, -y, m; \mu)$. A second reflection symmetry which holds only after averaging over the gauge field configurations is that $\rho_{N_f}^*(x, y, m; \mu) = \rho_{N_f}(-x, -y, m; \mu)$. Because the fermion determinant vanishes for $z_k = \pm m$ we expect that $\rho_{N_f}(x = \pm m, y = 0, m; \mu) = 0$. The chiral condensate in the chiral limit can be expressed as

$$\Sigma = \lim_{m \to 0} \lim_{V \to \infty} \frac{1}{V} \int dx dy \frac{\rho_{N_f}(x, y, m; \mu)}{x + iy + m} \quad (5)$$

At zero chemical potential the above quantity is known to be proportional to the density of the imaginary eigenvalues at zero $\[1\]$. This happens because then $\rho_{N_f}(x, y, m; \mu = 0) \propto \delta(x)$. When $\mu \neq 0$ the eigenvalues spread into the complex plane and this argument no longer holds. In this case we will show that there is an extended region of the eigenvalue density that contributes to the chiral condensate.

For simplicity, here we will show how the condensate arises from the microscopic limit of the spectral density which is believed to be universal. We have also verified \[2\] that a similar mechanism occurs even for larger $\mu$ provided that the contributing eigenvalues are still in the universal region. The microscopic limit of the spectral density is defined as \[3\]

$$\hat{\rho}_{N_f}(\hat{x}, \hat{y}, \hat{m}; \hat{\mu}) = \lim_{V \to \infty} \frac{1}{(\Sigma V)^2} \rho_{N_f}(\hat{x} \Sigma V, \hat{y} \Sigma V, \hat{m} \Sigma V) \quad (6)$$

In this limit, $\hat{x} = x \Sigma V, \hat{y} = y \Sigma V, \hat{m} = m \Sigma V$ and $\hat{\mu} = \mu F_\pi \sqrt{V}$ are kept fixed with $\Sigma$ given by \[1\] and $F_\pi$ the
pion decay constant. The expression for the condensate in this limit becomes

$$\Sigma = \lim_{m, \hat{\mu} \to \infty} \Sigma \int d\hat{x} d\hat{y} \frac{\hat{\rho}_{N_f}(\hat{x}, \hat{y}, \hat{m}; \hat{\mu})}{\hat{x} + i\hat{y} + \hat{m}}.$$  \hspace{1cm} (7)

The microscopic limit of the spectral density at nonzero chemical potential was recently calculated both for the quenched case and the unquenched case. For a nonzero number of flavors it was found that the eigenvalue density for \(m\Sigma < 2\mu^2 F_\pi^2\) is a strongly oscillating complex function. The oscillations cover a region of the complex eigenvalue plane and, as we will see below, the entire region contributes to the integral in (7). This constitutes a new mechanism where a discontinuity of the chiral condensate in the complex mass plane is obtained from an oscillating eigenvalue density in the complex plane. This mechanism does not rely on the specific form of the eigenvalue density as is demonstrated in the simple example below.

The lack of Hermiticity properties of the Dirac operator at nonzero chemical potential is a direct consequence of the imbalance between quarks and anti-quarks imposed in order to induce a nonzero baryon density. Because of this, it is our opinion that because of its physical origin, a paradigm shift will be necessary to develop viable probabilistic algorithms for this problem. Because of this, it is our opinion that it is particularly important to improve our analytical understanding of chiral symmetry breaking for QCD at nonzero baryon density.

Euclidean QCD at finite baryon density is not the only system without Hermiticity properties that has received much attention recently. We mention the distribution of the poles of \(S\)-matrices which are given by the eigenvalues of a non-Hermitian operator, the Hatano-Nelson model (a random potential together with a nonzero imaginary vector potential), and the description of Laplacian growth in terms of the spectrum of non-Hermitian random matrices. The essential difference from QCD is that in these problems the determinant of the operator only enters in the generating function of the resolvent. We will see that the additional determinant in QCD completely changes the character of the theory.

Example. As an example to illustrate our point, let us consider the eigenvalue density (sg is the sign function)

$$\hat{\rho}_{E_S}(\hat{x}, \hat{y}, \hat{m}; \hat{\mu}) = \frac{\theta(2\hat{\mu}^2 - |\hat{x}|)}{4\pi\hat{\mu}^2} \times \left[ 1 - e^{i\pi(\hat{x}|\hat{x}|+2\hat{\mu}^2)/4\hat{\mu}^2 + (|\hat{x}|^2 - |\hat{m}|^2)/(|\hat{x}|+2\hat{\mu}^2)/4\hat{\mu}^2) \right].$$  \hspace{1cm} (8)

This eigenvalue density has the same reflection symmetries as the eigenvalue density of the QCD Dirac operator and has the property that it vanishes at the point where the fermion determinant is zero. The integral in (7) can be evaluated analytically by means of a complex contour integral in \(\hat{y}\) resulting in

$$\Sigma_{Ex} = \text{sg}(\hat{m})\Sigma + \frac{\Sigma}{\hat{m}} e^{-|\hat{m}|}(e^{-|\hat{m}|} - 1),$$  \hspace{1cm} (9)

which, for large \(\hat{m}\), approaches \(\text{sg}(\hat{m})\Sigma\). What we have learned from this example is that a discontinuity in the chiral condensate can be obtained from an oscillating spectral density rather than from eigenvalues localized on the imaginary axis. We will show next that the same mechanism is at work for QCD at \(\mu \neq 0\).

The microscopic spectral density. The input for our calculation of the chiral condensate is the microscopic spectral density derived for any number of flavors in the \(N_f = 1\) and topological charge equal to zero for which the equations are less extensive. The microscopic eigenvalue density can be decomposed as

$$\hat{\rho}_{N_f=1}(\hat{x}, \hat{y}, \hat{m}; \hat{\mu}) = \hat{\rho}_Q(\hat{x}, \hat{y}; \hat{\mu}) - \hat{\rho}_U(\hat{x}, \hat{y}, \hat{m}; \hat{\mu}),$$  \hspace{1cm} (10)

with \((\hat{z} = \hat{x} + i\hat{y})

$$\hat{\rho}_U(\hat{x}, \hat{y}, \hat{m}; \hat{\mu}) = \frac{|\hat{z}|^2}{2\pi\hat{\mu}^2} e^{-(\hat{z}^2 + z^*^2)/(8\hat{\mu}^2)} \times K_0\left(\frac{|\hat{z}|^2}{4\hat{\mu}^2}\right) I_0(\hat{m}) \int_0^1 dt e^{-2t\hat{\mu}^2} I_0(\hat{z}^2 t) I_0(\hat{m}t).$$  \hspace{1cm} (11)

The first term in (10) is the quenched eigenvalue density given by \(\hat{\rho}_Q(\hat{x}, \hat{y}; \hat{\mu}) = \hat{\rho}_Q(\hat{x}, \hat{y}, \hat{x} + \hat{y}; \hat{\mu})\). As expected, the microscopic spectral density vanishes at \(\hat{z} = \pm \hat{m}\). A plot of the real part of the eigenvalue density for \(\hat{m} = 60\) and \(\hat{\mu} = 8\) is shown in figure. Notice that the oscillatory
region extends from the mass pole at \( \hat{z} = \hat{m} \) and toward the boundary of the support of the spectrum.

The oscillations appear as the microscopic variables become large, i.e. as the thermodynamic limit is approached. In this region an asymptotic formula for the eigenvalue density is accurate and will be used in order to analyze the role of the oscillations for chiral symmetry breaking. We first derive the asymptotic formula for the eigenvalue density and then evaluate the chiral condensate from (7). For \( \hat{\mu}^2 \gg 1 \) and \( (\hat{x} + \hat{m})/(4\hat{\mu}^2) < 1 \), the integral in \([11]\) over \( t \) is very well approximated by \([11]\)

\[
\int_0^1 dt \, t e^{-2\hat{\mu}^2 t^2} I_0(\hat{z}^* t) I_0(\hat{m} t) \\
\approx \frac{1}{4\hat{\mu}^2} \exp\left(\frac{\hat{z}^*^2 + \hat{m}^2}{8\hat{\mu}^2}\right) I_0\left(\frac{\hat{m} \hat{z}^*}{4\hat{\mu}^2}\right).
\]

Furthermore, we are interested in the approach to the thermodynamic limit where \( |\hat{m}| \gg 1, |\hat{z}| \gg 1, |\hat{z}|^2/(4\hat{\mu}^2) \gg 1 \) and \( |\hat{m}\hat{z}|/(4\hat{\mu}^2) \gg 1 \). This justifies the replacement of the Bessel functions by their leading order asymptotic expansion including the Stokes terms. We obtain the following asymptotic result for the integral over \( t \) of the unquenched eigenvalue density

\[
\hat{\rho}_U(\hat{x}, \hat{y}; \hat{m}; \hat{\mu}) \\
\sim \frac{1}{4\pi \hat{\mu}^2} e^{-[\hat{y} + \hat{\mu} \hat{x}]/(8\hat{\mu}^2)} (\hat{x}^2 - (\hat{x})^2)^2/(2\hat{\mu}^2).
\]

This expression has the reflection symmetries discussed below \([11]\). The asymptotic expansion of the quenched part of the spectral density is simply given by

\[
\hat{\rho}_Q(\hat{x}, \hat{y}; \hat{\mu}) \sim \frac{1}{4\pi \hat{\mu}^2} \theta(2\hat{\mu}^2 - |\hat{x}|).
\]

For the argument presented below it is important that also the asymptotic expansion of the spectral density vanishes at \( \hat{x} + \hat{y} = \pm \hat{m} \).

The chiral condensate. As explained in the introduction, the chiral condensate does not depend on the baryon chemical potential. Hence, in the microscopic limit it is known that \([12, 13]\) (momentarily we use the original variables to emphasize the volume dependence)

\[
\Sigma_{N_f=1}(m) = \frac{I_1(mV \Sigma)}{I_0(mV \Sigma)}
\]

which is discontinuous, \( \Sigma_{N_f=1}(m) = \text{sg}(m) \Sigma \), in the thermodynamic limit at fixed quark mass. The question we wish to answer is how oscillations of the spectral density conspires into a \( \mu \) independent condensate. We stress that this is not just a challenging mathematical problem; understanding which parts of the eigenvalue density contributes to the chiral condensate will give direct insight in the physical consequences of the sign problem.

We now derive the chiral condensate from \([7]\) using the asymptotic form of the microscopic eigenvalue density. We first consider the integral over \( \hat{y} \). The contribution from the quenched part of the spectral density \([14]\) is given by

\[
\frac{1}{4\pi \hat{\mu}^2} \int_{-\infty}^{\infty} d\hat{y} \frac{1}{x + i\hat{y} + \hat{m}} = \text{sg}(\hat{x} + \hat{m}) \frac{1}{4\pi \hat{\mu}^2}.
\]

The contribution from \( \hat{\rho}_U \) in \([7]\) is evaluated by a saddle point approximation. The contour in the complex \( \hat{y} \) plane is deformed into a contour from \( -\infty \) to \( \infty \) over the saddle point at \( \hat{y} = i \text{sg}(\hat{x})(4\hat{\mu}^2 - |\hat{x}| - |\hat{m}|) \) and, if the contour has crossed the pole, an integral around the pole at \( \hat{y} = i(\hat{x} + \hat{m}) \). The saddle point contribution is exponentially suppressed for \( |\hat{x}| < 2\hat{\mu}^2 \) leaving only the integral around the pole. For \( \hat{m} > 0 \) (\( \hat{m} < 0 \)) the pole contribution for \( \hat{x} > 0 \) (\( \hat{x} < 0 \)) is exponentially suppressed. We obtain

\[
\frac{1}{4\pi \hat{\mu}^2} \int_{-\infty}^{\infty} d\hat{y} e^{-[\hat{y} + \text{sg}(\hat{x})(|\hat{x}| + |\hat{m}| - 4\hat{\mu}^2)]^2/(8\hat{\mu}^2) - (|\hat{x}| - 2\hat{\mu}^2)^2/(2\hat{\mu}^2)} \\
\times \frac{1}{x + i\hat{y} + \hat{m}} \\
\sim -\frac{1}{2\hat{\mu}^2} \theta(\hat{m}) \theta(-\hat{x} - \hat{m}) - \theta(-\hat{m}) \theta(\hat{x} + \hat{m})
\]

where we have used that the exponent vanishes at the pole. For \( \hat{x} > 2\hat{\mu}^2 \) the eigenvalue density is zero so it is now trivial to do the integration over \( \hat{x} \) to get

\[
\Sigma_{N_f=1} = \frac{\sum N_f=1}{2\hat{\mu}^2} \int_{-2\hat{\mu}^2}^{2\hat{\mu}^2} d\hat{x} \left[\frac{1}{2} \text{sg}(\hat{x} + \hat{m}) + \theta(\hat{m}) \theta(-\hat{x} - \hat{m}) - \theta(-\hat{m}) \theta(\hat{x} + \hat{m})\right] \\
= \text{sg}(\hat{m}) \Sigma.
\]

This result agrees with \([13]\) for \( |\hat{m}| \gg 1 \) where the asymptotic expansion of \( \Sigma_{N_f=1}(m) \) is valid. Using the exact microscopic spectral density we would have recovered the mass dependence of \([13]\).

We also emphasize that a finite result for the chiral condensate is not obtained due to a cancellation of the pole and a zero of the fermion determinant. The pole term gives a finite contribution for each of the two terms in \([14]\) which do not vanish at \( \hat{z} = \hat{m} \). We have checked numerically that the same mechanism results in a discontinuity of the chiral condensate for more than one flavor.

The contribution from the unquenched part of the eigenvalue density to the chiral condensate is dominated by the pole term because the exponential in \([17]\) suppresses the integrand at the saddle point in the complex \( \hat{y} \)-plane. The simple result \([17]\) which implies a nonzero chiral condensate in the chiral limit is thus directly related to the complex phase of the spectral density. The oscillating exponential has to suppress terms that diverge exponentially with the volume which is achieved by oscillations in the \( \hat{y} \)-direction with a period that scales as the
Conclusions. For QCD at nonzero chemical potential the chiral condensate is not dominated by the contribution from the smallest eigenvalues. On the contrary, we have found that the contributions from strips parallel to the imaginary eigenvalue axis do not depend on the real part of the eigenvalue as long the eigenvalue is inside the support of the spectrum. This result arises from integrating a spectral density that oscillates with a period of $1/(\Sigma V)$ and an amplitude that diverges exponentially with the volume. Although we have shown only results using the microscopic eigenvalue density, we have checked that our arguments apply up to $\mu \sim 1/L$ (for $V = L^4$). In conclusion, we have uncovered a novel mechanism of chiral symmetry breaking at nonzero chemical potential where an oscillatory spectral density results in a discontinuity of the chiral condensate in the complex mass plane.

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