Number of Spin \( j \) States of Identical Particles

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In this paper we study the enumeration of number (denoted as \( D_I \)) of spin \( j \) states for fermions in a single-\( j \) shell and bosons with spin \( l \). We show that \( D_I \) can be enumerated by the reduction from \( SU(n+1) \) to \( SO(3) \). New regularities of \( D_I \) are discerned.

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The enumeration of number of spin \( I \) states (denoted as \( D_I \)) for fermions in a single-\( j \) shell or bosons with spin \( l \) (We use a convention that \( j \) is a half integer and \( I \) is an integer) is a very common practice in nuclear structure theory. One usually obtains this number by subtracting the combinatorial number of angular momentum projection \( M = I + 1 \) from that with \( M = I \). More specifically, \( D_I \) equals to the combinatorial number of angular momentum projection \( M = I + 1 \), where \( M = m_1 + m_2 + \cdots + m_n \), with the requirement that \( m_1 \geq m_2 \geq \cdots \geq m_n \) for bosons and \( m_1 > m_2 > \cdots > m_n \) for fermions, where \( n \) is the number of particles (This procedure is called Process A in this paper.). The combinatorial numbers of different \( M \)'s look irregular, and such an enumeration would be prohibitively tedious when \( j \) and \( l \) are very large. The number of states of a few nucleons in a single-\( j \) shell is usually tabulated in textbooks, for sake of convenience.

Another well-known solution was given by Racah\(^2\) in terms of the seniority scheme, where one has to introduce (usually by computer choice) additional quantum numbers. More than one decade ago, a third route was studied by Katriel et al.\(^6\) and Sunko et al.\(^4\), who constructed generating functions of the number of states for fermions in a single-\( j \) shell or bosons with spin \( l \).

There were two efforts in constructing analytical formulas of \( D_I \). In Ref.\(^6\), \( D_0 \) for \( n = 4 \) was obtained analytically. In Ref.\(^4\), \( D_I \) was constructed empirically for \( n = 3 \) and 4, and some \( D_0 \)'s for \( n = 5 \). It is therefore desirable to obtain a deeper insight into this difficult problem.

Equivalent to Process A, we propose here another procedure, called process B and explained as follows. Let \( \mathcal{P}(I_0) \) be the number of partitions of \( I_0 = i_1 + i_2 + \cdots + i_n \), with \( 0 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq 2l + 1 - n \) for fermions or \( 0 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq 2l \) for bosons. Here \( I_{\text{max}} = nj - \frac{n(n-1)}{2} \) for fermions in a single-\( j \) shell, and \( I_{\text{max}} = nl \) for bosons with spin \( l \). One defines \( \mathcal{P}(n, I_0) = D_I - I_{\text{max}} = 1 \) for \( I_0 = 0 \). Then one has \( \mathcal{P}(n, I_0) = \mathcal{P}(n, I_0 - 1) \).

Now we look at \( D_I \) for \( \bar{n} \) “bosons” of spin \( L = \frac{3}{2} \), with \( \bar{n} = 2l \) for bosons or \( \bar{n} = 2j + 1 - n \) for fermions. \( I_{\text{max}} \) of these \( \bar{n} \) “bosons” with spin \( L \) equals that of \( n \) bosons with spin \( l \) or that of \( n \) fermions in a single-\( j \) shell. Furthermore, \( \mathcal{P}(n, I_0) \) of \( I_0 = i_1 + i_2 + \cdots + i_n \) with the requirement \( 0 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq 2L = n \) always equals that of \( I_0 = i_1 + i_2 + \cdots + i_n \) with the requirement \( 0 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq 2j + 1 - n \) for fermions or \( 0 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq 2l \) for bosons.

This result can be explained from the fact as follows. The \( \mathcal{P}(n, I_0) \) of \( \bar{n} \) “bosons” with spin \( L \) corresponds to Young diagrams up to \( n \) rows, and \( 2l \) columns for bosons or \( 2j + 1 - n \) columns for fermions. The conjugates of these Young diagrams are those up to \( 2l \) rows for bosons or \( 2j + 1 - n \) rows for fermions, and up to \( n \) columns, which correspond to partitions in Process B for \( n \) fermions in a single-\( j \) shell or bosons with spin \( l \). Therefore, Process B for \( \bar{n} \) bosons with spin \( L = n/2 \) provides us with an alternative to construct \( D_I \) for \( n \) bosons with spin \( l \) or \( n \) fermions in a single-\( j \) shell.

This alternative (Process B for \( \bar{n} \) bosons with spin \( L \)) suggests the following identity. If \( l = (2j + 1 - n)/2 \) (\( n \) is even), i.e., \( I_{\text{max}} \) of bosons equals that of fermions, then \( D_I \) for bosons equals that of fermions. This identity is easily confirmed. It means that one can obtain \( D_I \) of \( n \) fermions in a single-\( j \) shell by using that of \( n \) bosons with spin \( l \) or \( 2j + 1 - n \) or \( n \) fermions in a single-\( j \) shell or bosons with spin \( l \).

Process B for \( \bar{n} \) bosons with spin \( L = n/2 \) is also useful in constructing formulas of \( D_I \). One can see this point from the fact that Process B involves \( SU(n+1) \) symmetry, which is independent of \( j \) and \( l \), while in Process A different \( j \) shell for fermions and spin \( l \) for bosons involve different symmetries (\( SU(2j+1) \) and \( SU(2l+1) \)).

Below we exemplify our idea by \( n = 4 \). The relevant symmetry for Process B of \( \bar{n} \) bosons with spin \( L \) is \( SU(5) \) (i.e., \( L = n/2 = 2 \), \( d \) bosons). \( \bar{n} \) equals \( 2l \) and \( 2j - 3 \), for four bosons and four fermions, respectively.

Our first result is that \( D_I \) of four bosons with spin...
\( l \) always equals that of four fermions in a single \( j \) shell when \( l = (2j - 3)/2 \). Our second result is that we can derive \( D_l \) of four bosons with spin \( l \) by this new method. Here one needs \( D_l \) of \( \tilde{n} = 2l \). This problem was studied in the interacting boson model, suggested by Arima and Iachello \( ^{[3]} \) in seventies. Below we revisit the enumeration of \( D_l \) for \( d \) bosons with particle number \( \tilde{n} = 2l \).

Let us follow the notation of Ref. \( ^{[3]} \) and define \( \tilde{n} = 2l = 2n + v = 2n + 3n_\lambda + \lambda \). \( D_l \) of \( \tilde{n} \) \( d \) bosons is enumerated via the procedure as follows. (1) \( v \) takes value \( 2l, 2l - 2, 2l - 4, \ldots, 0 \), which corresponds to \( n = 0, 1, 2, \ldots, n/2 = l \), respectively. (2) For each value of \( v \), \( n_\Delta \) takes value from 0 to \( \left[ \frac{\lambda}{6} \right] \). (3) For each set of \( v \) and \( n_\Delta \), \( \lambda \) is determined by \( v - 3n_\Delta \). (4) For each \( \lambda \) obtained in step (3), the allowed spin is given by \( \lambda, \lambda + 1, \lambda + 2, \ldots, 2\lambda - 3, 2\lambda - 2, 2\lambda \). Note that there is no state with \( 2\lambda - 1 \). One easily sees that there is no \( I = 1 \) states for \( d \) bosons, because \( \lambda = 1 \) presents \( I = 2 \) state (\( 2\lambda - 1 \) is missing).

In order to obtain \( D_l \), it is necessary to know the number of \( \lambda \) appearing in the above process for each \( I \). Let us call this number \( f_\lambda \) and define \( \tilde{n} = 2l = 6k + \kappa, \kappa = 0, 2, 4, \) and \( k \geq 1 \). Below we exemplify how we obtain \( f_\lambda \) by the case of \( \kappa = 0 \). We have the following hierarchy:

\[
\begin{array}{ccc}
\lambda & f_\lambda & v \\
0 & k + 1 & 0, 6, 12, \ldots, 6k \\
1 & k & 4, 10, 16, \ldots, 6k - 2 \\
2 & k & 2, 8, 14, \ldots, 6k - 4 \\
3 & k & 6, 12, 18, \ldots, 6k \\
4 & k & 4, 10, 16, \ldots, 6k - 2 \\
5 & k - 1 & 8, 14, 20, \ldots, 6k - 4 \\
6 & k & 6, 12, 18, \ldots, 6k \\
7 & k - 1 & 10, 16, 22, \ldots, 6k - 2 \\
8 & k - 1 & 8, 14, 20, \ldots, 6k - 4 \\
9 & k - 1 & 12, 18, 24, \ldots, 6k \\
10 & k - 1 & 10, 16, 22, \ldots, 6k - 2 \\
11 & k - 2 & 14, 20, 26, \ldots, 6k - 4 \\
12 & k - 1 & 12, 18, 24, \ldots, 6k \\
13 & k - 2 & 16, 22, 28, \ldots, 6k - 2 \\
14 & k - 2 & 14, 20, 26, \ldots, 6k - 4 \\
15 & k - 2 & 18, 24, 30, \ldots, 6k \\
16 & k - 2 & 16, 22, 28, \ldots, 6k - 2 \\
17 & k - 3 & 20, 26, 32, \ldots, 6k - 4 \\
\vdots & \vdots & \vdots \\
\end{array}
\]

From this tabulation we have that \( f_\lambda = k + \delta_{m0} - \delta_{m5} - \left[ \frac{\lambda}{6} \right] \), where \( m \) is equal to \( \lambda \) mod 6 when \( \kappa = 0 \), and \( \left[ \cdot \right] \) means to take the largest integer not exceeding the value inside.

For the sake of simplicity we define \( I = 2I_0 \) for even values of \( I \) and \( I = 2I_0 + 3 \) for odd values of \( I \). For \( I_0 \leq l \),

\[
D_{I=2I_0} = \sum_{\lambda=I_0}^{2I_0} f_\lambda. \tag{1}
\]

For \( \kappa = 0 \) and \( I_0 \leq l \) (\( I = 2I_0 \leq 2l \)),

\[
\begin{align*}
D_{I=2I_0} &= (I_0 + 1)k - (9K^2 - K + 3KK + (2K - 5)(2K - 5)) + \delta_{K0} + \delta_{K3} \\
&= \left[ \frac{I_0 + 3}{6} \right] + \left[ \frac{I_0 + 5}{6} \right] + \delta_{K0} + \delta_{K3}. \tag{2}
\end{align*}
\]

For \( \kappa = 2 \) and \( I_0 \leq l \),

\[
\begin{align*}
D_{I=2I_0} &= (I_0 + 1)(k + 1) - (9K^2 - K + 3KK + (2K - 5)(2K - 5)) \\
&= \left[ \frac{I_0 + 3}{6} \right] - \left[ \frac{I_0 + 4}{6} \right] + \delta_{K4}. \tag{3}
\end{align*}
\]

For \( I \) is odd and \( I \leq 2l \), we use a relation \( D_{I=2I_0} = D_{I=2I_0+3} = \left[ \frac{I_0}{6} \right] + 1 \). This relation was obtained empirically in Ref. \( ^{[3]} \) and can be obtained mathematically by calculating

\[
D_{I=2I_0+3} = \sum_{\lambda=I_0+3}^{2I_0+3} f_\lambda
\]

and compare with \( D_{I=2I_0} \).

For the case with \( I \geq 2l \), we define \( I = I_{\text{max}} - 2I_0 \) for even \( I \) and \( I = I_{\text{max}} - 2I_0 - 3 \) for odd \( I \). \( f_{\lambda=I_0} = \left[ \frac{I_0}{6} \right] - \delta_{(I_0 \text{ mod } 6),0} \). We obtain that

\[
\begin{align*}
D_{I_{\text{max}}-2I_0} &= D_{I_{\text{max}}-2I_0-3} = 3 \left[ \frac{I_0}{6} \right] \left( \left[ \frac{I_0}{6} \right] + 1 \right) - \left[ \frac{I_0}{6} \right] \\
&\quad + \left( \left[ \frac{I_0}{6} \right] + 1 \right) ((I_0 \text{ mod } 6) + 1) + \delta_{(I_0 \text{ mod } 6),0} - 1. \tag{5}
\end{align*}
\]

Thus we solve the problem of enumeration of \( D_l \) for four bosons with spin \( l \) or four fermions in a single-\( j \) shell by using the new enumeration procedure. One may obtain \( D_l \) of other \( n \) (\( n \) is even) cases by applying this method similarly, if the reduction rule of \( SU(n+1) \rightarrow SO(3) \) is available.

A question arises when we apply this method to odd \( n \) cases, for which spin \( L \) of \( n \) bosons involved in Process B is not an integer (\( L = n/2 \)). These bosons are therefore not “realistic”. For such cases \( I \) of \( n \) bosons with
spin $l$ cannot equal that of $n$ fermions in a single $j$ shell. Namely, there is no similar correspondence of $D_I$ between bosons and fermions when $n$ is odd \cite{8}. However, $D_I$ of $\bar{n}$ fictitious bosons with spin $n/2$ ($n$ is odd) obtained by Process A equals that of $n$ bosons with spin $l$ or that of $n$ fermions in a single-$j$ shell, where $\bar{n} = 2l$ (even value) and $2j + 1 - n$ (odd value) for bosons and fermions, respectively. In other words, $D_I$ of $\bar{n}$ fictitious bosons with spin $n/2$ equals that of $n$ bosons with spin $l$ if $\bar{n} = 2l$ or that of $n$ fermions in a single-$j$ shell if $\bar{n} = 2j + 1 - n$, here $n$ is odd. Further discussion is warranted on this problem.

To summarize, We have presented in this paper an alternative to enumerate the number of spin $I$ states, $D_I$, for $n$ fermions in a single $j$ shell or $n$ bosons with spin $l$. We proved that $D_I$ of $n$ bosons with spin $l$ equals that of $n$ fermions in a single-$j$ shell when $2l = 2j + 1 - n$, where $n$ is even. We have also exemplified the usefulness of this new method in constructing analytical formulas of $D_I$ by $n = 4$.

For odd $n$, the procedure of our new method involves half integer spin $L$ for “bosons”. Further consideration of this fictitious situation is necessary.

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\[\text{References}\]

\[\text{1} \] For example, R. D. Lawson, Theory of Nuclear Shell Model (Clarendon, Oxford, 1980), P. 8-20.


\[\text{8} \] A correspondence of $D_I$ was noted in Sec. II of Ref. \cite{8} for large $I$ cases.