Signatures of Primordial Non-Gaussianity in the Large-Scale Structure of the Universe

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Abstract. We discuss how primordial (e.g. inflationary) non-Gaussianity in the cosmological perturbations is left imprinted in the Large-Scale Structure of the universe. Our findings show that the information on the primordial non-Gaussianity set on super-Hubble scales flows into Post-Newtonian terms, leaving an observable imprint in the Large-Scale Structure. Future high-precision measurements of the statistics of the dark matter density and peculiar velocity fields will allow to pin down the primordial non-Gaussianity, thus representing a tool complementary to studies of the Cosmic Microwave Background anisotropies.

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The most compelling feature of inflation \[1\] is to provide the seeds for the Large-Scale Structure (LSS) of the universe and the anisotropies in the Cosmic Microwave Background. Cosmological perturbations are created during inflation from quantum fluctuations and “redshifted” to sizes larger than the Hubble radius. The deviation from a Gaussian statistics characterizing the cosmological perturbations, \textit{i.e.} the presence of higher-order connected correlation functions, emerges as a key observable to discriminate among the various scenarios for the generation of cosmological perturbations produced during inflation. Non-Gaussianity (NG) is one of the primary targets of present and future Cosmic Microwave Background (CMB) satellite missions \[2\].

The main goals of this Letter are to show how NG – set at primordial epochs on large (super-Hubble) scales – propagates to subhorizon scales once cosmological perturbations reenter the Hubble radius during the matter- and dark energy-dominated epochs and to analyze which signatures primordial NG leaves in the Large-Scale Structure (LSS) of the universe through the dark matter density and velocity fields. We start with considering the case of flat universe filled by a perfect non-relativistic matter fluid. For the purposes of this paper this is a good approximation, although for the scales crossing the Hubble radius during the radiation dominated era one should include the radiation in the evolution equations, thus using a complete transfer function also for those scales. We find it convenient to perform our calculations in the comoving-synchronous gauge and then switch to the Poisson gauge \[3\], which allows a more direct comparison with the standard Newtonian and Eulerian approach adopted in LSS studies. We make use of the formalism developed in Refs. \[4, 5\] which the reader is referred to for more details. The synchronous-gauge is defined by the line-element 
\[
d s^2 = a^2(\tau)[-d\tau^2 + \gamma_{ij}(x, \tau)dx^i dx^j],
\]
where \(a(\tau)\) is the scale factor, \(\tau\) is the conformal time and \(x\) represent comoving Lagrangian coordinates for the fluid element. The energy-momentum tensor is 
\[
T^{\mu\nu} = \rho u^\mu u^\nu,
\]
where \(\rho\) is the mass density and, in our comoving coordinates, the fluid four-velocity is given by 
\[
u^\mu = (1/a, 0, 0, 0).
\]
The homogeneous background (Einstein-de Sitter universe) is described by a scale factor \(a(\tau) \propto \tau^2\). A very efficient way to write down Einstein and continuity equations is to introduce the peculiar velocity-gradient 
\[
\psi^i_j \equiv u^i_\tau - \frac{a'}{a} \delta^i_j = \frac{1}{2} \gamma_{ia} \gamma^{aj},
\]
where we have subtracted the isotropic Hubble flow. Primes stand for differentiation w.r.t. the conformal time; semicolons denote covariant differentiation. From the continuity equation \(T^{\mu\nu}_{;\nu} = 0\), we infer the exact solution for the density contrast \(\delta = \delta \rho/\rho\) 
\[
\delta(x, \tau) = (1 + \delta_{in}(x))[\gamma(x, \tau)/\gamma_{in}(x)]^{-1/2} - 1,
\]
where \(\gamma = \det \gamma_{ij}\). The subscript “in” denotes the value of quantities evaluated at some initial time. Such initial conditions will play a key role once they have been properly determined to account for the primordial non-linearities in the cosmological perturbations.

From Eq. \[1\] it is evident that in the comoving-synchronous gauge the real independent degree of freedom is the spatial metric tensor \(\gamma_{ij}\). The energy constraint
Signatures of Primordial Non-Gaussianity in the Large-Scale Structure of the Universe

The spatial metric tensor can be written as

$$\gamma_{ij} = \frac{1}{2} \delta_{ij} + \chi^{(1)}_{ij} + \frac{1}{2} \chi^{(2)}_{ij},$$

where $\chi^{(1)}_{ij}$ and $\chi^{(2)}_{ij}$ are traceless and include scalar, vector and tensor (gravitational waves) perturbations. Similarly, we split the density contrast into a linear and a second-order part as $\delta(x, \tau) = \delta^{(1)}(x, \tau) + \frac{1}{2} \delta^{(2)}(x, \tau)$. At linear order the growing-mode solutions for a matter-dominated epoch in the comoving-synchronous gauge are given by

$$\delta^{(1)}(x, \tau) = \frac{3}{2} \phi(x) + \frac{2}{2\xi} \nabla^2 \phi(x),$$

and

$$\chi^{(1)}_{ij} = -\frac{1}{2} \left( \dot{\chi}_{ij} - \frac{1}{3} \delta_{ij} \nabla^2 \phi \right)$$

for the metric perturbations and

$$\delta^{(1)} = \frac{2}{6} \nabla^2 \phi$$

for the linear density contrast. Here $\phi(x)$ is the so-called peculiar gravitational potential. In writing $\chi^{(1)}_{ij}$ we have eliminated the residual gauge ambiguity of the synchronous gauge as in Ref. [5] and we have neglected linear vector modes since they are not produced in standard mechanisms for the generation of cosmological perturbations (as inflation). We have also neglected linear tensor modes, since they play a negligible role in LSS formation.

Let us now discuss the key issue of the initial conditions which we conveniently fix at the time when the cosmological perturbations relevant today for LSS are well outside the Hubble radius, i.e. when the (comoving) wavelength $\lambda$ of a given perturbation mode is such that $\lambda \gg H^{-1}$, $H = a'/a$ being the conformal Hubble constant.

In the standard single-field inflationary model, the first seeds of density fluctuations are laid down on super-horizon scales from the fluctuations of a scalar field, the inflaton $\Phi$. Recently many other scenarios have been proposed as alternative mechanisms to generate such primordial seeds. These include, for example, the curvaton $\chi$ and the inhomogeneous reheating scenarios $[7]$, where the first density fluctuations are produced through the fluctuations of a scalar field other than the inflaton. In order to follow the evolution on super-Hubble scales of the density fluctuations coming from the various mechanisms, we use the curvature perturbation $\zeta = \zeta^{(1)} + (1/2)\zeta^{(2)} + \cdots$ of uniform density hypersurfaces, where $\zeta^{(1)} = -\psi^{(1)} - \chi^{(1)} / \rho$ and the expression for $\zeta^{(2)}$ can be found in Ref. [8]. The crucial point is that the gauge-invariant curvature perturbation $\zeta$ remains constant on super-Hubble scales after it has been generated during a primordial epoch and possible isocurvature perturbations are no longer present. Therefore, we may...
Different scenarios are characterized by different values of $\zeta^{(2)}$. For example, in the standard single-field inflationary model $\zeta^{(2)} = 2 \left( \zeta^{(1)} \right)^2 + O \left( n_\zeta - 1 \right)$ \[9, 10\], where $n_\zeta$ is the spectral index for the scalar perturbations. In general, we may parametrize the primordial NG level in terms of the conserved curvature perturbation as in Ref. \[11\], $\zeta = 2a_{nl} \left( \zeta_{in}^{(1)} \right)^2$. In the standard scenario $a_{nl} \simeq 1$, in the curvaton case $a_{nl} = (3/4r) - r/2$, where $r \approx (\rho_f/\rho_D)$ is the relative curvaton contribution to the total energy density at curvaton decay \[2\]. In the minimal picture for the inhomogeneous reheating scenario, $a_{nl} = 1/4$. For other scenarios we refer the reader to Ref. \[2\].

The curvature perturbation $\zeta = \zeta_{in}$ allows to set the initial conditions for the metric and matter perturbations accounting for the primordial contributions. Indeed, in the synchronous gauge and on super-Hubble scales the energy density contrast vanishes (see, e.g., Ref. \[12\]) and therefore $\delta_{in} = 0$. From the definiton of $\zeta^{(1)}$ it then follows that $\zeta^{(1)} \simeq -\psi^{(1)} = -5\varphi/3$ and at second-order

$$\zeta^{(2)} \simeq -\psi_{in}^{(2)} = 2a_{nl} \left( \zeta_{in}^{(1)} \right)^2 = \frac{50}{9} a_{nl} \varphi^2,$$

where $a_{nl}$ will always signal the presence of primordial NG according to our parametrization. One of the best tools to detect or constrain the primordial large-scale NG is through the analysis of the CMB anisotropies, for example by studying the bispectrum \[2\]. In that case the standard procedure is to introduce the non-linearity parameter $f_{nl}$ characterizing NG in the large-scale temperature anisotropies \[13, 14, 2\]. To give the feeling of the resulting size of $f_{nl}$ when $|a_{nl}| \gg 1$, $f_{nl} \simeq -5a_{nl}/3$ (see Refs. \[2, 11\], where the sign is opposite for a different convention relating the CMB anisotropies to the gravitational potential).

After having learned how to set initial conditions, our next step is to determine how a primordial NG evolves onto subhorizon scales. By perturbing Eq. (1) up to second order, we get

$$\delta = 3(\psi^{(1)}_{in} - \psi^{(1)}_{in}) + \delta^{(1)}_{in} + \frac{3}{2}(\psi^{(2)}_{in} - \psi^{(2)}_{in}) + \frac{1}{2}\delta^{(2)}_{in} + \frac{1}{8}(\gamma^{(1)}_{ij} - \gamma^{(1)}_{in})^2$$

$$+ \frac{1}{4}(\gamma^{(1)}_{ij} - \gamma^{(1)}_{in})^2 - \frac{1}{2}\delta^{(1)}(\gamma^{(1)}_{ij} - \gamma^{(1)}_{in})^2,$$

where $\gamma_{ij} \equiv \delta_{ij} + \gamma^{(1)}_{ij} + \gamma^{(1)}_{ij}/2$. To compute the metric perturbation $\psi^{(2)}$, we need only the Raychaudhury equation (4) and the energy constraint (2). Perturbing these equations up to second order and inserting the solution for the density contrast (5) in terms of the spatial metric perturbations, one obtains an expression for $\psi^{(2)}$ in terms of the peculiar gravitational potential $\varphi(x)$

$$\psi^{(2)} = -\frac{50}{9} a_{nl} \varphi^2 + \frac{10}{27} \left( \frac{3}{4} - a_{nl} \right) \tau^2 \varphi^{ik} \varphi_{,k} + \frac{10}{27} (1 - a_{nl}) \tau^2 \varphi \nabla^2 \varphi$$

$$- \tau^4 \frac{10}{252} \left( \frac{3}{10} \varphi^{jk} \varphi_{,jk} - (\nabla^2 \varphi)^2 \right).$$

(6)
The energy constraint also provides the solution for the scalar perturbation of the traceless part of the second-order spatial metric $\chi^{(2)}_{ij}$, while the vector and tensor parts follow by solving the momentum constraint and the remaining evolution equation, according to the same procedure outlined in Ref. [5]. We find $\chi^{(2)}_{ij} = -\frac{20}{9}(1-a_{nl})\tau^2\varphi_{,ij} - \frac{20}{9}(\frac{3}{2} - a_{nl})\tau^2\varphi_{,i}\varphi_{,j} + \frac{20}{97}\tau^2 \left[(1-a_{nl})\varphi\nabla^2\varphi + \left(\frac{3}{2} - a_{nl}\right)\varphi^k\varphi_{,k}\right]\delta_{ij} + \frac{19}{126}\tau^4\varphi^{,i}\varphi_{,ij} + \frac{\tau^4}{126} \left(-12\varphi_{,ij}\nabla^2\varphi + 4(\nabla^2\varphi)^2 \delta_{ij} - \frac{19}{3}\varphi^{,kl}\varphi_{,kl}\delta_{ij}\right) + \Delta^{(2)}_{ij}$, where $\Delta^{(2)}_{ij}$ describing second-order tensor modes generated by linear scalar perturbations and possible time-independent terms arising from the initial conditions, will not be necessary for our purposes. Finally, for the density contrast in the synchronous gauge we plug the solution (6) into Eq. (5) to obtain

$$
\delta^{(2)} = \frac{\tau^4}{126} \left[5(\nabla^2\varphi)^2 + 2\varphi^{,ij}\varphi_{,ij}\right] + \frac{10}{9}\tau^2 \left[(\frac{3}{4} - a_{nl})\varphi_{,k}\varphi^k\right]
+ (2-a_{nl})\varphi\nabla^2\varphi .
$$

The expression obtained for the metric perturbation $\psi^{(2)}$ is characterized by a large-scale part ($\tau \to 0$) which is dominated by the primordial NG contribution. It is such primordial NG which propagates onto smaller scales, once the mode reenters the horizon during the matter-dominated epoch. This is evident from the dependence on $a_{nl}$ of the term proportional to $\tau^2$ in Eq. (6).

The last step of our computation consists in expressing the relevant quantities in the Poisson gauge [3], i.e. the generalization beyond linear order of the longitudinal gauge, by which a more direct comparison with the standard Newtonian approximation adopted in LSS observations and N-body simulations is possible. Second-order coordinate transformations may be written as $x^\mu = x^\mu - \xi^\mu_{(1)} - (\xi^\mu_{(2)} - \xi^\mu_{(2)}_{,k}\xi^k_{(1)})/2$, with $\xi^\mu_{(r)} = \alpha^\mu_{(r)}$ and the space shift splits into the scalar and vector parts as $\xi^i_{(r)} = \beta^i_{(r)} + d^i_{(r)}$. The density contrast and the velocity perturbations transform correspondingly as in Ref. [5].

The first-order transformation parameters to go from the synchronous to the Poisson gauge are given by $[5]$ $\alpha^{(1)} = \tau\varphi/3$, $\beta^{(1)} = \tau^2\varphi/6$ and $d^{(1)i} = 0$. It follows that at linear order the metric perturbations are given in terms of the peculiar gravitational potential $\varphi$ [5] as $\psi^{(1)} = \phi^{(1)} = \varphi$ where $\phi = \phi^{(1)} + \phi^{(2)}/2$ and $\delta g_{00} = -a^2(1 + 2\phi)$ defines the lapse function in the Poisson gauge. Linear tensor perturbations are gauge-invariant while the density contrast and the velocity perturbations read $[5]$ $\delta^{(1)} = -2\varphi + \tau^2\nabla^2\varphi/6$ and $v^{(1)i} = -\tau\varphi^i/3$ respectively. For simplicity we do not report here the explicit expressions we find for the second-order transformation parameters and we give the results for the relevant perturbations in the Poisson gauge. We refer the reader to Eqs. (5.12), (5.13) and (5.14) of Ref. [5], which give the general formulae for $\beta^{(2)}$, $d^{(2)i}$ and $\alpha^{(2)}$. We just mention here that the initial conditions for the second-order perturbations will show up in such a computation. For the density contrast and the peculiar velocity in the Poisson gauge we find

$$
\delta^{(2)}_{\varphi} = \frac{\tau^4}{126} \left[5(\nabla^2\varphi)^2 + 2\varphi^{,ij}\varphi_{,ij} + 7\varphi^{,i}\nabla^2\varphi_{,i}\right] + \frac{\tau^2}{9} \left[10 \left(\frac{9}{20} - a_{nl}\right)\varphi_{,k}\varphi^k\right].
$$
where \( \nabla^2 \Theta \equiv \Psi - \frac{1}{3} \varphi^i \varphi_i \) with \( \nabla^2 \Psi \equiv -\frac{1}{2}[\nabla^2 \varphi]^2 - \varphi,ik \varphi^{ik} \). Eqs. (8) and (9) are the main results of this Letter. In particular they clearly show how the primordial NG which is initially generated on large scales is transferred to the density contrast and peculiar velocity on subhorizon scales. The expression for the density contrast is made of three contributions: the standard second-order Newtonian piece (proportional to \( \tau^4 \) which is insensitive to the non-linearities in the initial conditions; a Post-Newtonian (PN) piece (proportional to \( \tau^2 \) which carries the most relevant information on primordial NG; the super-horizon terms (independent of \( \tau \)). Our findings show in a clear way that the information on the primordial NG set on super-Hubble scales flows into the PN terms, leaving an observable imprint in the LSS.

It might also be useful to write the density contrast in Fourier space in terms of the linear density contrast, by defining a kernel \( \mathcal{K}(k_1, k_2; \tau) \), depending on the wavevector of the perturbation modes, as

\[
\delta_k(\tau) = \delta^{(1)}_k(\tau) + \int \frac{dk_1 dk_2}{(2\pi)^3} \mathcal{K}(k_1, k_2; \tau) \delta^{(1)}_{k_1} \delta^{(1)}_{k_2} \delta(k_1 + k_2 - k). 
\tag{10}
\]

The kernel \( \mathcal{K} \) reads

\[
\mathcal{K}(k_1, k_2; \tau) = \frac{5}{7} + \frac{2}{7} \frac{(k_1 \cdot k_2)^2}{k_1^2 k_2^2} + \frac{1}{2} \frac{(k_1 \cdot k_2)}{k_1^2 k_2^2} \left( k_1^2 + k_2^2 \right) \\
- 6 f_{nl}(k_1, k_2; \tau) \frac{k_1^2}{k_1^2 k_2^2 \tau^2}.
\tag{11}
\]

where

\[
f_{nl}(k_1, k_2; \tau) = \frac{5}{3} a_{nl} + \frac{63 (k_1 \cdot k_2)}{42 k^2} + \frac{172 (k_1^2 + k_2^2)}{42 k^2}
+ \frac{(k_1^2 + k_2^2)(k_1 \cdot k_2)}{k_1^2 k_2^2 k^2} \times \left( \frac{4}{7} (k_1 \cdot k_2) + k_1^2 + k_2^2 \right)
- \frac{6 k_1^2 k_2^2}{7 k^4} + \frac{6 (k_1 \cdot k_2)^2}{7 k^4}.
\tag{12}
\]

In order to obtain the expression in Eq. (11) we have performed an expansion in \((k_1, k_2, \tau)^{-1} \ll 1 \) up to terms \((k_1, \tau)^{-2}\) starting from Eq. (8). The non-linearity parameter \( f_{nl} \) defined in this way generalizes the standard definition of Refs. [15] inferred from the Newtonian gravitational potential \( \Phi = \varphi + f_{nl} (\varphi^2 - \langle \varphi^2 \rangle) \). Our findings indicate that the non-Gaussianity in the density contrast shows a non-trivial shape dependence which might help in detecting the PN nonlinearities. In Fig. 1 we plot the \( k \)-dependence of the non-linearity parameter \( f_{nl} \) which can be useful in distinguishing the different contributions in the measured bispectrum. The mass-density bispectrum is given by

\[
\langle \delta(k_1) \delta(k_2) \delta(k_3) \rangle = (2\pi)^3 \delta^{(3)}(k_1 + k_2 + k_3) [2 \mathcal{K}(k_1, k_2; \tau) P_\delta(k_1) P_\delta(k_2) + \text{cycl}.],
\]

for \( \tau \neq 0 \). The non-Gaussianity in the density contrast shows a non-trivial dependence which might help in detecting the PN nonlinearities. In Fig. 1 we plot the \( k \)-dependence of the non-linearity parameter \( f_{nl} \) which can be useful in distinguishing the different contributions in the measured bispectrum. The mass-density bispectrum is given by
Signatures of Primordial Non-Gaussianity in the Large-Scale Structure of the Universe

Figure 1. Wavenumber dependence of the non-linearity parameter \( f_{nl} \). The continuous and dashed-dotted lines correspond to the k-dependent part \( \Delta f_{nl} \) of \( f_{nl} \) (\( f_{nl} \) for \( a_{nl} = 0 \)), with an angle between \( k_1 \) and \( k_2 \) of \( \pi/3 \) and \( 2\pi/3 \) respectively. The horizontal lines show the k-independent part \( |5a_{nl}/3| \) for \( a_{nl} = 1, 10, 40, 100 \).

where \( P_\delta(k) \) is the linear mass-density power spectrum. By inspecting Eq. (12) we can roughly estimate the relevance of the primordial NG contribution to the mass-density bispectrum (proportional to \( a_{nl}(1/\tau k)^2 \)) with respect to the Newtonian one. For the two to be comparable on a given scale \( \lambda \sim k^{-1} \), reminding the reader that \( f_{nl} \sim -\frac{5}{3}a_{nl} \) (\( |a_{nl}| \gg 1 \)), we find that the non-linearity parameter \( f_{nl} \) must be as large as 
\[
 f_{nl} \sim (10^5/1+z) \text{(Mpc/\lambda)}^2
\]
Thus, we confirm that, on a comoving scale \( \lambda \sim \) few Mpc, a value of the non-linearity parameter \( f_{nl} \sim 10^3 \) is required for the primordial NG to leave observable effects in the clustering of galaxies at low redshift [15]. Going to higher redshift would allow to pin down lower values of \( f_{nl} \) [16], provided the galaxy-to-dark matter bias evolution is accurately handled [17], or that observables directly related to the dark matter density contrast are considered (see e.g. Ref. [18]). As far as the velocity divergence is concerned, in the limit of large NG, we also recover the standard Newtonian formula relating the NG in the peculiar velocity to the one present in the gravitational potential \( \Phi \), \( \nabla \cdot v_P = -(2/3\mathcal{H})\nabla^2\Phi \). Finally, one can easily extend our findings for large NG to the case in which dark energy is present. For the density contrast, it suffices to replace, in the term proportional to \( a_{nl} \), \( \tau^2 \) with \( 4D_+^2\mathcal{H}_0^{-2}\Omega_m^{-1} \), where \( D_+ \) is the standard Newtonian linear growth factor of density perturbations [19] and \( \Omega_{0m} \) is the present-day matter density parameter. As for the peculiar velocity, one has to replace, in the term proportional to \( a_{nl} \), \( \tau \) with \( 2D_+\mathcal{H}_0^{-2}\mathcal{H}f(\Omega_m) \), where \( f(\Omega_m) = d\ln D_+/d\ln a \) [19].

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References

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