There is no generalization of known formulas for mutually unbiased bases

Claude Archer
Université Libre de Bruxelles,
C.P.165/11 -Physique et Mathématique
Faculté des Sciences Appliquées
avenue F.D. Roosevelt 50,
1050 Bruxelles, Belgium
carcher@ulb.ac.be

Abstract

In a quantum system having a finite number $N$ of orthogonal states, two orthonormal bases $\{a_i\}$ and $\{b_j\}$ are called mutually unbiased if all inner products $\langle a_i | b_j \rangle$ have the same modulus $1/\sqrt{N}$. This concept appears in several quantum information problems. The number of pairwise mutually unbiased bases is at most $N + 1$ and various constructions of $N + 1$ such bases have been found when $N$ is a power of a prime number. We study families of formulas that generalize these constructions to arbitrary dimensions using finite rings. We then prove that there exists a set of $N + 1$ mutually unbiased bases described by such formulas, if and only if $N$ is a power of a prime number.
1 Introduction

1.1 Definitions and previous results

In the $N$-dimensional Hilbert space $\mathbb{C}^N$, two orthonormal bases $\{a_i\}_{1 \leq i \leq N}$ and $\{b_j\}_{1 \leq j \leq N}$ are called mutually unbiased if all inner products $\langle a_i | b_j \rangle$ have the same modulus $|\langle a_i | b_j \rangle| = 1/\sqrt{N}$. A set of mutually unbiased bases is a set of orthonormal bases which are pairwise mutually unbiased. In various physical situations (see subsection 1.2), the problem is to find the maximal number of mutually unbiased bases. The following result is due to W.K. Wootters and B.D. Fields but it has been obtained independently by A. R. Calderbank, P. J. Cameron, W. M. Kantor and J. J. Seidel.

**Theorem 1.1** ([11], [4])

- In dimension $N \geq 2$, the number of mutually unbiased bases is at most $N + 1$.
- If $N$ is a power of a prime number then there exist $N + 1$ mutually unbiased bases.

In dimension $N$, a set of mutually unbiased bases is called complete if it contains $N + 1$ bases. If $N$ is not a prime power, it is not known whether such a complete set exists, even for $N = 6$. Originally, constructions of $N + 1$ mutually unbiased bases in dimension $N$, were based on the arithmetic of a field, where addition and multiplication are invertible ([11]). There exists a field with $N$ elements if and only if $N$ is a power of a prime number. Nevertheless, new constructions have been recently obtained (see [8]) using the arithmetic of rings, where multiplication is not invertible. Since there exist rings of $N$ elements for any $N$, is it possible to use finite rings to construct $N + 1$ mutually unbiased bases for arbitrary dimensions? We will address this issue here. First we will generalize known constructions from [11] and [8] to any finite ring. Then we will prove that for dimensions $N$ that are not prime powers, there does not exist a complete set of mutually unbiased bases described by this generalization.

1.2 Applications to Quantum Information

Mutually unbiased bases (MUB for short) have recently been considered with an increasing interest because of the central role they play in specific quantum information tasks. MUB are related, among others, to state estimation and to protocols of quantum cryptography.
**State estimation.** Mutually unbiased bases play an important role in state estimation of (relatively) large ensembles of identical prepared quantum systems. MUB allow us to minimize the number of measurements needed to estimate a quantum state. The density matrix of an $N$-dimensional quantum state is determined by $N^2 - 1$ real parameters. Hence, at least $N + 1$ measurements are needed to re-construct such a density matrix. One can show that $N + 1$ measurements are sufficient if these measurements are MUB ([7]). The reason is that if two measurement bases $B_1$ and $B_2$ are mutually unbiased, then the information revealed by the outcomes of these measurements are independent. Other optimality properties of MUB with respect to state estimation are described in [11].

**Quantum cryptography.** The protocol BB84 of Bennett and Brassard ([3]) for quantum key distribution, used with the one-time pad encryption, is the first cryptographic protocol whose security does not depend on the assumption that an eavesdropper has a limited computational power. Its security is guaranteed by Heisenberg’s uncertainty principle.

MUB are the basic algebraic structure underlying $d$-dimensional analogues of the BB84 protocol ([3]). It is precisely the use of such bases which allows these protocols to make the intervention of a potential eavesdropper detectable. Alice and Bob agree on a set of $t$ orthonormal bases and Alice sends to Bob a state $a_i$ prepared in a basis $A = \{a_i\}$ taken among the $t$ bases. Bob chooses a basis in one of the $t$ bases to measure this state. If Bob chooses the right basis, he finds the good value. Now suppose an eavesdropper Eve has chosen $E = \{b_j\}$ to measure the state. Eve obtains $b_j$ as result with probability $|\langle a_i | b_j \rangle|^2$. Since for security purpose, one wants that when $E$ is the wrong base, Eve gets no information on the original state, we should require that all probabilities $|\langle a_i | b_j \rangle|^2$ are equal and hence equal to $1/N$ (i.e. $A$ and $E$ are mutually unbiased). Moreover, as the probability to choose a wrong basis is $1/t$, one wants to take the largest possible number of mutually unbiased bases. It is also known that a protocol using a larger number of mutually unbiased bases can tolerate a higher error level in the channel (see [3]).

**Wigner functions** Pure or mixed quantum states are usually represented by the density matrix. However, there is an alternative description in terms of the Wigner function. Several authors have proposed to define a Wigner function for discrete systems having $N$ degrees of freedom. It appears that the discrete Wigner function defined in [12] requires the existence of $N + 1$ mutually unbiased bases.

Finally, MUB have also been shown to be relevant to the mean king’s
problem, see [1] and references therein. An interesting source for recent results and for references is the problem page in Quantum Information at TU Braunschweig, located at http://www.imaph.tu-bs.de/qi/problems.

2 Formulas for mutually unbiased bases

A unitary transformation maps a set of MUB to a set of MUB. Hence, it is not restrictive to consider only sets \( X \) of MUB containing the standard basis \( \{ e_k \}_{1 \leq k \leq N} \) since it is always possible to choose a unitary transformation \( U \) that maps a given orthonormal basis in \( X \) to \( \{ e_k \} \) so that \( U(X) \) is a set of MUB containing \( \{ e_k \} \). If a basis \( \{ v_k \} \) is unbiased with respect to the standard basis \( \{ e_k \} \) (i.e. \( |\langle e_i | v_j \rangle| = 1/\sqrt{N} \) then \( |\langle v_k | e_l \rangle| = 1/\sqrt{N} \). Hence the coordinates of its vectors must be expressed as \( (v_k)_l = (e^{i\Theta(k,l)})/\sqrt{N} \) where \( \Theta(k,l) \) belongs to \([0, 2\pi]\).

For \( N = p^n \) where \( p \) is a prime number and \( n \) a positive integer, there always exist \( N + 1 \) mutually unbiased bases. We describe here the constructions from [1] and [8] for these dimensions.

2.1 Odd prime powers dimensions

Let the superscript \( r \) denotes the basis, \( k \) the vector in the basis and \( l \) the component. The standard basis is \( (v_k^{(0)})_l = \delta_{kl} \) for \( k, l = 0, 1, \ldots, N - 1 \). If \( N = p^n \) for a prime number \( p \neq 2 \), the other \( N = p^n \) such bases given in [1] are,

\[
(v_k^{(r)})_l = \frac{1}{\sqrt{N}} e^{(2\pi i/p)\text{Tr}(rl^2+kl)} \quad r, k, l \in \mathbb{F}_{p^n}
\]  

(1)

where \( \mathbb{F}_{p^n} \) is the finite field with \( p^n \) elements and where \( Tr \) denotes the trace map from \( \mathbb{F}_{p^n} \) into the prime field \( \mathbb{F}_p \). For \( p \geq 5 \) odd, a new formula has been proposed in [8] where the polynomial \( rl^2 + kl \) is replaced by \( (l+r)^3 + k(l+r) \). The trace map is a linear map from \( \mathbb{F}_{p^n} \), regarded as a vector space, into \( \mathbb{F}_p \). In the language of group theory, linear maps are group homomorphisms (i.e. maps that preserve sums). The trace map induces a homomorphism from the additive group of \( \mathbb{F}_{p^n} \) into the multiplicative group \( \mathbb{C}^* \) of complex numbers, defined by \( x \rightarrow e^{(2\pi i/p)\text{Tr}(x)} \).
2.2 Even prime powers dimensions

For \( N = 2^n \), W.K.Wootters and B.D.Fields (III) have used an ad hoc construction that may be reformulated in a finite ring \( R \) whose \( 4^n \) elements are sequences \((x_1, \ldots, x_n)\) with \( x_i \in \mathbb{Z}_4 \). A much easier construction has been found recently by A. Klappenecker and M. Roetteler (S) using the Galois ring \( R = GR(4, n) \). Let \( Tr \) denotes the trace map from \( GR(4, n) \) into \( \mathbb{Z}_4 \). Once again \( T : x \rightarrow e^{(2\pi i / 4) Tr(x)} \) is a group homomorphism from \( GR(4, n), + \) into \( \mathbb{C}^* \). The \( 2^n \) indexes are the elements of \( T_n \), the Teichmüller set of \( GR(4, n) \) and the \( 2^n \) bases described by

\[
(v^{(r)}_k)_l = \frac{1}{\sqrt{2^n}} e^{(2\pi i / 4) Tr((r+2k)l)} \quad r, k, l \in T_n \subset GR(4, n) \tag{2}
\]

together with the standard basis, form a complete set of mutually unbiased bases of \( \mathbb{C}^{2^n} \) (see S).

2.3 How to generalize these formulas?

Formulas (1), (2) as well as others in S, share many common characteristics. First of all, the indexes \( l, k, r \) respectively for components, vectors and bases are taken in a finite ring \( R \). Both formulas link the indexes in \( R \) to complex coordinates by a function \( f : (r, k, l) \rightarrow T(P(r, k, l)) \) where \( P \) is a polynomial and \( T \) is a homomorphism from \( R, + \) into \( \mathbb{C}^* \). We will generalize these characteristics as follows:

1. The functions \( f : (r, k, l) \rightarrow T(P(r, k, l)) \). We consider a much larger class of functions that we call functions preserving a direct sum decomposition of \( R \) (see section 4).

2. The set \( S \) of indexes. For formula (1) the set of indexes is the whole \( R \) while for formula (2), it is a remarkable subset of \( R \). We will see in subsection 2.5 that these subsets may be defined for every ring \( R \) as sets closed under multiplication and transversal to a nilpotent ideal of \( R \).

3. Distinguish the index \( r \). In formula (1) and (2), the non standard bases are indexed by \( r \) that takes all possible values of a set of size \( N \). This can only be done for dimensions \( N \) for which there exist \( N + 1 \) mutually unbiased bases. However there is, up to now, no result showing that this is true if \( N \) is not a prime power. Therefore, we propose to give up formulas that are a uniform with respect to \( r \) and to consider that each basis \( r \) may be described by a different formula. This means that
for each $r$, we choose a different function $f_r : (k, l) \rightarrow f_r(k, l)$ into $\mathbb{C}^*$ such that the vectors in basis $r$ are described by

$$(v^{(r)}_k)_l = f_r(k, l) \quad k, l \in S$$

where each $f_r$ preserve a given decomposition $R_1 \oplus R_2$ and $S \subset$ is as described in previous paragraph. For the case of polynomials $P$ and homomorphisms $T$, it amounts to choose for each $r$, different polynomials $P_r$ and homomorphism $T_r$.

2.4 Properties of rings

In this section we recall various properties of rings that are needed for this paper.

Direct sums of rings. Let $R_+$ be a ring where addition is commutative but where multiplication is not necessarily commutative. If the additive group $R_+$ is the direct sum $R_1 \oplus R_2$ of two subgroups then every element $r$ of $R$ can be written in a unique way as $r = r_1 + r_2$ where $r_i \in R_i$ and $R_1 \cap R_2$ is reduced to the zero element. An element $r \in R$ may be represented as a couple $(r_1, r_2)$ and addition in $R$ corresponds to componentwise addition of couples. If moreover, $R_1$ and $R_2$ are two-sided ideals of $R$ (i.e. $r \cdot R_i = R_i = R_i \cdot r$ for every $r$ in $R$), then multiplication in $R$ is also reduced to componentwise multiplication of couples. Indeed, for $r_i \in R_i$, $r_1.r_2$ belongs to both $R_1$ and $R_2$ since these are two-sided ideals and thus $r_1.r_2 = 0 = R_1 \cap R_2$. For two ring elements $x$ and $y$, we obtain for their product $x.y = (x_1 + x_2) \cdot (y_1 + y_2) = x_1.y_1 + x_1.y_2 + x_2.y_1 + x_2.y_2$ and since $x_i.y_j = 0$ for $i \neq j$, we get $x.y = x_1.y_1 + x_2.y_2$. Thus $(x.y)_i = x_i.y_i$. Observe also that $a.r_1 = (a_1 + a_2).x_1 = a_1.x_1$ for every $a \in R$. We say that the ring $R$ is the direct sums of its ideals $R_1$ and $R_2$. These properties also hold when $R$ is the direct sum of more than two ideals.

Polynomial functions in a ring. If a ring is a direct sum, then let us show that polynomial functions in $R$ may be evaluated componentwise. A monomial on the set of variables $\{x, y, \ldots\}$ is a finite product of elements of this set. In a commutative ring $R$, a polynomial $P(x, y, \ldots) + r_0$ is defined as a linear combination $P$ of monomials (on $\{x, y, \ldots\}$) with coefficients in $R$ and of $r_0 \in R$. A polynomial $P + r_0$ defines a polynomial function $(x, y, \ldots) \rightarrow P(x, y, \ldots) + r_0$ that maps $n$-uples of $R^n$ to elements of $R$. In the non commutative case, this definition is not very convenient since generally $x._r.y \neq r._x.y$ so that products of polynomials would not
in general be polynomials. Hence we prefer to define a non commutative polynomial function as \( P : (x, y, \ldots) \rightarrow r_0 + \sum_k m_k(x, y, \ldots) \) where \( m_k(x, y, \ldots) = w_k(x, y, \ldots; a, b, \ldots) \) is a finite product of non commuting variables in \( \{x, y, \ldots\} \) and of coefficients from a set \( \{a, b, \ldots\} \subset R \).

Assume now that the ring \( R \) is a direct sum \( R = R_1 \oplus R_2 \) and for \( r \in R \), let \( r = r_1 + r_2 \) be the corresponding decomposition. We have shown in previous paragraph that products in \( R \) may be performed componentwise. Thus, for each term \( w_k \) of a polynomial function \( w_k(x, y, \ldots; a, b, \ldots) \) is equal to \( w_k(x_1, y_1, \ldots; a_1, b_1, \ldots) + w_k(x_2, y_2, \ldots; a_2, b_2, \ldots) \). As \( x_1.a_1 = x_1.a \) and \( a_1.x_1 = a.x_1 \) for \( a, x \in R \) we may conclude that \( w_k(x_i, y_i, \ldots; a_i, b_i, \ldots) \) is equal to \( w_k(x_i, y_i, \ldots; a, b, \ldots) \), for \( i = 1, 2 \), so that \( m_k(x, y, \ldots) = m_k(x_1, y_1, \ldots) + m_k(x_2, y_2, \ldots) \). Hence, for a polynomial function \( \bar{P} = P + r_0 \) on \( R_1 \oplus R_2 \), we have \( \bar{P}(x, y, \ldots) = P(x_1, y_1, \ldots) + P(x_2, y_2, \ldots) \) and thus for \( \lambda = -r_0 = \bar{P}(0, 0, \ldots) \),

\[
\bar{P}(x, y, \ldots) = \lambda + \bar{P}(x_1, y_1, \ldots) + \bar{P}(x_2, y_2, \ldots).
\]

The Sylow decomposition of a finite ring. Let \( R \) be a finite ring and let \( |R| = \prod_i p_i^{e_i} \) be the factorization of its order into powers of distinct prime numbers. The additive group \( R_+ \) is a finite commutative group. Hence, it is equal to the direct product \( \oplus_i Syl(p_i) \) of its Sylow subgroups and thus every element \( r \) of \( R_+ \) can be written in a unique way as \( r = \sum r_i \) where \( r_i \in S_{p_i} := Syl(p_i) \). We call the element \( r_i \), the \( p_i \)-component of \( r \) and it is the unique element contained in the intersection \( S_{p_i} \cap \{r + (\oplus j \neq i S_{p_j})\} \) (see [6], chapter 3). These subgroups may be defined as \( S_{p_i} := \{x : p_i^{e_i} x = 0\} \) where \( p_i^{e_i} x \) is the repeated sum of \( p_i^{e_i} \) terms \( x \). The subgroups \( S_{p_i} \) are two-sided ideals of the ring \( R \), i.e. \( r.S_{p_i} = S_{p_i}.r = S_{p_i} \) for every \( r \in R \). This is due to the right and left distributive property of a ring since \( p_i^{e_i} (r.x) = r.x + \ldots + r.x = r(x + \ldots + x) = r.(p_i^{e_i} x) = r.0 = 0 \) if \( x \in S_{p_i} \) so that \( r.x \in S_{p_i} \) and similarly \( p_i^{e_i} \)-terms \( x.r \in S_{p_i} \). Hence every finite ring is the direct sum of its Sylow ideals and a finite ring that is not decomposable as a non trivial direct sum, must be of prime power order. Moreover, if \(|R| = d_1.d_2\) is the product of two coprime numbers (\( \geq 2 \)) then \( R = R_1 \oplus R_2 \) where \( R_i := \oplus_{p_i} Syl(p) \).

**Ring with unity.** From now on, we mainly consider rings \( R \) containing a multiplicative unity \( 1 \) such that \( x.1 = x = 1.x \) for every \( x \in R \). If \( R = \oplus_{i \in I} R_i \) has a unity \( 1 \) and \( R_i \neq \{0\} \) then \( 1_i \) is the unity of \( R_i \). An element \( x \) has a left inverse \( x_L \) (resp. right inverse \( x_R \)) if \( x_L.x = 1 \) (resp. \( x.x_R = 1 \)). An element that has both a left and a right inverse is called a **unit**. If \( x \) is a unit, then
the inverse $x^{-1}$ is unique since $x_L = x_L.(x.x_R) = (x_L.x).x_R = x_R$. The set $U(R)$ of all units of $R$ is a multiplicative group and by the componentwise multiplication $U(R_1 \oplus R_2) = U(R_1) \oplus U(R_2)$. A field is a ring where every non zero element is a unit.

**Nilpotency.** An element $n$ of a ring $R$ is called nilpotent if $n^t = 0$ for some positive integer $t$. The set $\text{Nil}(R)$ of all nilpotent element of $R$ is called the *Nilpotent radical* of $R$. Once again, by the componentwise multiplication $\text{Nil}(R_1 \oplus R_2) = \text{Nil}(R_1) \oplus \text{Nil}(R_2)$. We say that an ideal $N$ is nilpotent provided that every $n \in N$ is nilpotent. For every $r \in R$, if we have $(r.n)^t = 0$ then $(n.r)^{t+1} = n.(r.n)^t.r = 0$ and thus every nilpotent ideal is two-sided. In every ring with unity, a nilpotent element cannot be a unit but if $n$ is nilpotent ($n^t = 0$) then $1+n$ is a unit. To show this, consider $u_t(n) = 1+n+\ldots+n^{t-1}$; then since $1 = 1-n^t = (1-n)u_t(n) = u_t(n)(1-n)$, the element $u_t(-n)$ is the inverse of $1+n = 1-(n)$.

Let us show that in a commutative ring, $\text{Nil}(R)$ is an ideal. If $x^n = 0$ then $(x.x)^n = x^n.x^n = 0$ and if moreover $y^m = 0$ then $(x+y)^{n+m} = 0$ since $(x+y)^{n+m}$ is a sum of terms $x^{n+m-k}.y^k$ which are zero for $k \leq m$ and for $k \geq m$. In a non commutative ring $R$, $\text{Nil}(R)$ is not necessarily an ideal. For instance, a sum of nilpotent matrices may be invertible (and thus not nilpotent) as shown by

$$
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} + \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}.
$$

Thus for the ring $M_2(R)$ of $2 \times 2$ matrices over a ring $R$ with unity, the nilpotent radical is not an ideal. However the subring of upper triangular matrices is also non commutative but it contains the nilpotent ideal $\{(\begin{pmatrix}
0 & r \\
0 & 0
\end{pmatrix} : r \in R}\}$.

### 2.5 Generalizing Teichmüller sets

For every ring $R$ we would like to define a subset $S_R \subset R$ for the indexes $k, l$ of vectors and components, in such a way that for a Galois ring $R = GR(4, n)$, the set $S_R$ is the *Teichmüller set* $T_n$ as in formula (2), while for a finite field $R = \mathbb{F}_p^n$ we have $S_R = R$ as in formula (1).

The set $T_n \subset GR(4, n)$ has remarkable properties that are used over and over to compute easily in $GR(4, n)$ (see (9)).

1. The ideal $N := \{2t_1 : t_1 \in T_n\}$ is the nilpotent radical of $GR(4, n)$ ($N^2 = 0$).

2. Every $r \in GR(4, n)$ can be written in a unique way as $r = t_o + 2t_1$ for some $t_o, t_1$ in $T_n$. Thus $T_n$ contains exactly one representative of each coset $\{r + N\}_{r \in R}$ of $N$ in $R$; it is a transversal to the ideal $N$. 

8
3. $T_n$ is closed under multiplication. Therefore a product of elements written as $t + n$ for $t \in T_n$ and $n \in N$ is still written in this way since 
\[(t_1 + n_1)(t_2 + n_2) = \sum_{t \in T_n} t_1 t_2 + \sum_{t \in N} t_1 n_2 + n_1 t_2 + n_1 n_2 \text{.} \]
This may be generalized to every ring $R$ as follows. We require that the set of indexes $S_R$ is closed under multiplication and that it is a transversal to a nilpotent ideal $N$. Trivially, if $R$ is a field (as in formula (11)) or even a division ring, then $N = \{0\}$ is the only nilpotent ideal, $S_R = R$ is the only transversal to $N$ and it is closed under multiplication.

A commutative local ring is a ring that has a unique maximal ideal $M$ and the Galois ring $R = GR(4, n)$ is local. In a finite commutative local ring $R$, the unique maximal ideal is $Nil(R)$ and the units of $R$ are exactly the non nilpotent elements (see [2]). Hence in a local ring every ideal ($\neq R$) is nilpotent. Every finite commutative ring with unity is a direct sum of local rings (see [2], Proposition 8.7).

### 2.6 Functions preserving a direct sum decomposition

Let $R = R_1 \oplus R_2$ be a direct sum decomposition of a ring $R$ and let $r = r_1 + r_2$ be the corresponding decomposition for $r \in R$. Let $G_\ast$ be a commutative group with an operation $\ast$ (either $+$ or $\cdot$ in this paper). For a finite set of variables $\{x, y, \ldots\}$ belonging to $R$, we say that a function $f : (x, y, \ldots) \rightarrow f(x, y, \ldots) \in G_\ast$ preserves the decomposition $R_1 \oplus R_2$ if for a constant $\lambda \in R$

\[f(x, y, \ldots) = \lambda \ast f(x_1, y_1, \ldots) \ast f(x_2, y_2, \ldots) \text{ for every } x, y, \ldots \in R. \tag{4}\]

Observe that $\lambda = (f(0, 0, \ldots))^{-1}$ because $(x_1)_2 = (y_1)_2 = \ldots = 0$ implies 
\[f(x_1, y_1, \ldots) = f(x_1 + 0, y_1 + 0, \ldots) = \lambda \ast f(x_1, y_1, \ldots) \ast f(0, 0, \ldots).\]
If $f_i$ is the restriction of $f$ to $R_i$ then $f(x, y, \ldots) = \lambda \ast f_1(x_1, y_1, \ldots) \ast f_2(x_2, y_2, \ldots)$. Conversely, for arbitrary functions $f_i$ from $R_i$ into $G$ and $\lambda \in G$, this last equation defines a function that preserves $R_1 \oplus R_2$. It may happen that $f$ preserves $R_1 \oplus R_2$ but does not preserve another decomposition of $R$.

We have seen in subsection 2.4 that polynomial functions $P$ on a ring $R$ preserves every direct sum decomposition of $R$ (and in this case $\lambda = -P(0, 0, \ldots)$). Thus, since group homomorphisms preserve sums, if $T$ is a group homomorphism from $R_\ast$ into a commutative group $G_\ast$ and if $P(x, y, \ldots)$ is a polynomial function on $R$, then $(x, y, \ldots) \rightarrow T(P(x, y, \ldots))$ preserves
every direct sum decomposition of $R$. Hence, these functions generalize formula (1) and (2) for mutually unbiased bases since those rely on expressions of type $\frac{1}{\sqrt{N}}T(P(k, l))$ for $k, l \in R$ where $G_*$ is the multiplicative group of unitary complex number. This is also true for the other formulas proposed in [8].

More sophisticated such functions may be constructed by products. If $f \cdot g$ is a product of functions into a commutative group $G$ that both preserve a direct sum decomposition $R = R_1 \oplus R_2$ then it is easy to show that $f \cdot g$ also preserves $R_1 \oplus R_2$. We have $(f \cdot g)(x, y, \ldots) = \lambda_f \cdot \prod_{i=1,2} f(x_i, y_i, \ldots) \cdot \lambda_g \prod_{i=1,2} g(x_i, y_i, \ldots)$ and since the elements of $G$ commute, we may rearrange the factors as $(f \cdot g)(x, y, \ldots) = \lambda_f \cdot \lambda_g \prod_{i=1,2} (f \cdot g)(x_i, y_i, \ldots)$.

3 Such sets of MUB cannot be complete

In this section we prove that even with all these generalizations, it is not possible to construct complete sets of $N + 1$ mutually unbiased bases for $N \neq p^n$.

3.1 Preliminary results

Proposition 3.1 In a ring $R$ with 1, let $S \subset R$ be a set closed under multiplication that is a transversal to a nilpotent ideal $N$ of $R$.

1. If $R = R_1 \oplus R_2$ is a sum of rings with 1 then

$$S = (S \cap R_1) \oplus (S \cap R_2)$$

and each $S \cap R_i$ contains at least two elements ($i = 1, 2$). Moreover in each $R_i$, $S \cap R_i$ is closed under multiplication and is a transversal to $N \cap R_i$.

2. If $|S|$ is a product $d_1 \cdot d_2$ of two coprime numbers $\geq 2$ and if $R$ is finite, then $R$ is a sum $R_1 \oplus R_2$ of rings with 1 such that $|S \cap R_1| = d_1$ and $|S \cap R_2| = d_2$.

Proof. 1) $R = R_1 \oplus R_2$ has unity $(1_1, 1_2)$. First, we show that every ideal $I$ of $R$ is equal to $I_1 \oplus I_2$ where $I_j$ is an ideal of $R_j$. Since $I$ is an ideal of $R$, the sets $I_1 := I \cdot (1_1, 0)$ and $I_2 := I \cdot (0, 1_2)$ belong to $I$, are ideals of $R$ and thus in particular $I_j$ is an ideal of $R_j$. But since every $i = (i_1, i_2) \in I$ is equal to $i \cdot (1_1, 0) + i \cdot (0, 1_2)$, we have $I = I_1 \oplus I_2$. By componentwise multiplication, an element $(n_1, n_2)$ is nilpotent if and only if each $n_j$ is nilpotent in $R_j$. 

10
and the ideal \( N \) is the sum \( N_1 \oplus N_2 \) of two nilpotent ideals. Since \( S \) is a transversal to \( N \) in \( R \), it contains a unique element of each coset of \( N \). Let \( x \) be the unique element \( S \cap \{(1,0) + N\} \) then \( x = (1 + n_1, n_2) \) for some \((n_1, n_2) \in N_1 \oplus N_2 \). Since \( N \) is a nilpotent ideal, it is two-sided and we may consider the quotient ring \( R/N \) where multiplication of cosets is defined as \((x + N)(y + N) = x.y + N\). In \( R/N \), \(((1,0) + N)^2 = (1,0) + N\) so that \( x^2 \in (1,0) + N \) and also \( x^2 \in S \) because \( S \) is closed under multiplication. Therefore we have \( x^2 = S \cap \{(1,0) + N\} = x \) and we have

\[
(1 + n_1)^2 = (1 + n_1) \quad (1), \quad n_2^2 = n_2 \quad (2).
\]

By nilpotency \( n_t^t = 0 \) for some positive integer \( t \) and by \((2)\), \( n_2^t = n_2 \) whence \( n_2 = 0 \). By nilpotency of \( n_1 \), \((1 + n_1)\) has an inverse \((1 + n_1)^{-1}\) in \( R_1 \). Multiplying both sides of \((1)\) by \((1 + n_1)^{-1}\) gives \((1 + n_1) = 1\) whence \( n_1 = 0 \). Finally \( x = (1,0) \in S \) and the symmetric argument for \( S \cap \{(0,1_2) + N\} \) shows that \((0,1_2)\) also belongs to \( S \). Thus \((0,0) = (1,0)\) \((0,1_2) \in S \) and \( S \cap R_1 \) (resp. \( S \cap R_2 \)) contains at least the two elements \((0,0)\) and \((1,0)\) (resp. \((0,0)\) and \((0,1_2)\)).

As \(\{(1,0), (0,1_2)\} \in S \), for every \((s_1, s_2) \in S \), \((s_1, 0) = s.(1,0) \in S \cap R_1\) and \((0, s_2) = s.(0,1_2) \in S \cap R_2\). Conversely, it remains to show that for every \((s_1, 0) \in S \cap R_1\) and \((0, s_2) \in S \cap R_2\) we also have \((s_1, s_2) \in S \). Since \( S \) is a transversal to \( N \), it contains a unique element \( y = (s_1 + n_1, s_2 + n_2) \in S \cap \{(s_1, s_2) + N\} \) and \( y.(1,0) = (s_1 + n_1, 0) \in S \). As \((s_1 + n_1, 0)\) and \((s_1, 0)\) are in \( S \) and belong the same coset of \( N \), these must be equal and \( n_1 = 0 \). Similarly \( n_2 = 0 \) so that \( y = (s_1, s_2) \in S \) whence \( S = (S \cap R_1) \oplus (S \cap R_2) \).

Finally, let us show that \((S \cap R_i)\) is a transversal to \( N_i = N \cap R_i \) in \( R_i \). Every coset \((r_1, 0) + N_1 \) is embedded in \((r_1, 0) + N\) which contains a unique element \( s = (r_1 + n_1, n_2) \) of \( S \). Then \( s.(1,0) = (r_1 + n_1, 0) \) in \( S \cap R_1 \) and in \((r_1, 0) + N_1\) and so, \( S \cap R_1 \) contains at least one representative of each coset. If two elements of \( S \cap R_1 \) belong to the same coset of \( N_1 \) then these belong to the same coset of \( N \supset N_1 \) and thus are equal. The proof is similar for \((S \cap R_2)\).

2) Let \( \pi(d) \) denote the set of prime divisors of a positive integer \( d \). For a finite ring \( R \) and a divisor \( d \) of \(|R|\), let us define \( Syl(\pi(d)) := \oplus_{\pi \cap \pi(d)} Syl(p) \).
Since \( N \) is a transversal to the ideal \( N \) in \( R \) then \( d_1, d_2 = |S| = \frac{|R|}{|N|} \) is a divisor of \(|R|\). If \( \pi_1 = \pi(d_1) \) and \( \pi_2 = \pi(|R|) \setminus \pi(d_1) \), we know from subsection 2.4 that \( R = Syl(\pi_1) \oplus Syl(\pi_2) \). For \( i = 1, 2 \), since \( d_i \geq 2 \), the subsets \( \pi(d_i) \) are non empty so that \( \pi_1 = \pi(d_1) \) and \( \pi_2 \supset \pi(d_2) \) are non empty; whence the rings \( Syl(\pi_i) \) are not zero rings and since \( R \) has unity 1, \( Syl(\pi_i) \) has unity 1. Thus part (1) of the present proposition applies, \( S = (S \cap Syl(\pi_1)) \oplus (S \cap Syl(\pi_2)) \).
and
\[ d_1, d_2 = |S| = |S \cap Syl(\pi_1)| \cdot |S \cap Syl(\pi_2)|. \]

As \( S \cap Syl(\pi_i) \) is a transversal to the ideal \( N \cap Syl(\pi_i) \) then \( |S \cap Syl(\pi_i)| \) divides \( Syl(\pi_i) \) and so is coprime to \( d_j \) (\( j \neq i \)). Thus, by equality (5), \( |S \cap Syl(\pi_i)| \) must divides \( d_i \) and symmetrically \( d_i \) divides \( |S \cap Syl(\pi_i)| \) so that \( d_i = |S \cap Syl(\pi_i)| \).

\[ \square \]

In what follows, \( \langle \rangle \) denotes the classical hermitian product \( \langle a, b \rangle = \sum_i (a_i)^* b_i \). The tensor product \( v \otimes w \) is defined by \( (v \otimes w)_{(i,j)} = a_i b_j \) and thus \( (v_1 \otimes w_1)_{(i,j)} (v_2 \otimes w_2) = \langle v_1 | v_2 \rangle \langle w_1 | w_2 \rangle \).

**Proposition 3.2** Let \( N = N_1N_2 \) be a product of positive integers and let \( \{v_{i,j}^{(1)}\} \ldots \{v_{i,j}^{(r)}\} \) be \( r \) mutually unbiased bases of \( \mathbb{C}^N = \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2} \) (for \( (i, j) \) \( \in \tilde{N} := \{1 \ldots N_1\} \times \{1 \ldots N_2\} \)). Assume that for each \( 1 \leq t \leq r \) there are \( N_1 \) vectors \( \{a_i^{(t)}\} \) and \( N_2 \) vectors \( \{b_j^{(t)}\} \) such that
\[ v_{i,j}^{(t)} = a_i^{(t)} \otimes b_j^{(t)} \]
for every \( (i, j) \) \( \in \tilde{N} \), then for \( 1 \leq t \leq r \), \( \{a_i^{(t)} \|a_i^{(t)}\|^{-1}\} \) and \( \{b_j^{(t)} \|b_j^{(t)}\|^{-1}\} \) are \( r \) mutually unbiased bases respectively in \( \mathbb{C}^{N_1} \) and \( \mathbb{C}^{N_2} \).

**Proof.** First we show that \( \{a_i^{(t)}\} \) and \( \{b_j^{(t)}\} \) are orthogonal bases. Since othornormality of \( \{v_{i,j}^{(t)}\} \) implies \( \langle v_{i,j}^{(t)} | v_{k,l}^{(t)} \rangle = \delta_{(i,j),(k,l)} = \delta_{ik} \delta_{jl} \), we have \( \delta_{ik} \delta_{jl} = \langle a_i^{(t)} \otimes b_j^{(t)} | a_k^{(t)} \otimes b_l^{(t)} \rangle = \langle a_i^{(t)} | a_k^{(t)} \rangle \langle b_j^{(t)} | b_l^{(t)} \rangle \). For every \( j \), \( b_j^{(t)} \neq 0 \) (otherwise \( v_{i,j}^{(t)} = 0 \)) so that \( \langle a_i^{(t)} | a_k^{(t)} \rangle \langle b_j^{(t)} | b_l^{(t)} \rangle = \delta_{ik} \delta_{jl} = \delta_{ik} \) implies that \( \langle a_i^{(t)} | a_k^{(t)} \rangle = 0 \) for \( i \neq k \). Hence \( \{a_i^{(t)}\} \) is a set of \( N_1 \) mutually orthogonal vectors, thus an orthogonal basis (not necessarily orthonormal) of \( \mathbb{C}^{N_1} \).

Permuting the role of \( a \) and \( b \) gives the same result for \( \{b_j^{(t)}\} \) in \( \mathbb{C}^{N_2} \).

Furthermore if we fix \( 1 \leq j \leq N_2 \), the equalities \( 1 = \langle v_{i,j}^{(t)} | v_{i,j}^{(t)} \rangle = \langle a_i^{(t)} | a_i^{(t)} \rangle \langle b_j^{(t)} | b_j^{(t)} \rangle \) may be divided by the constant \( \langle b_j^{(t)} | b_j^{(t)} \rangle \) so that for every \( 1 \leq i \leq N_1 \), \( L_{a(i)} := \langle a_i^{(t)} | a_i^{(t)} \rangle \) is constant with respect to \( i \) and equal to \( 1/\langle b_j^{(t)} | b_j^{(t)} \rangle \). Symmetrically, \( L_{b(j)} := \langle b_j^{(t)} | b_j^{(t)} \rangle \) is also constant for every \( 1 \leq j \leq N_2 \) and
\[ L_{a(i)} = L_{b(j)} = \frac{1}{\langle b_j^{(t)} | b_j^{(t)} \rangle} = \frac{1}{L_{b(j)}} \]
for every \( (i, j) \) \( \in \tilde{N} \).

Now, it is sufficient to prove the MUB property for each couple of bases among the \( r \), in \( \mathbb{C}^{N_1} \) and \( \mathbb{C}^{N_2} \). For instance let us consider \( \{v_{i,j}^{(1)}\} \) and \( \{v_{k,i}^{(2)}\} \).
We define $A_{i,k} := |\langle a_i^{(1)} | a_k^{(2)} \rangle|$ and $B_{j,l} := |\langle b_j^{(1)} | b_l^{(2)} \rangle|$. The equality

$$1/\sqrt{N} = |\langle v_{i,j}^{(1)} | v_{k,l}^{(2)} \rangle| = A_{i,k} B_{j,l} \quad \text{for every } (i,j), (k,l) \in \mathcal{N}$$  \hspace{1cm} (7)

implies that $A_{i,k}$ and $B_{j,l}$ are non zero. Therefore if $(i,k)$ is fixed and $(j,l)$ varies, $A_{i,k}$ can be simplified and all the $B_{j,l}$ are equal to a common value $K_B$. Symmetrically, the $A_{i,k}$ are equal to a common value $K_A$.

In basis $\{a_i^{(1)}\}$ we have $a_k^{(2)} = \sum_i \lambda_i a_i^{(1)}$ for $\lambda_i = \frac{|\langle a_i^{(1)} | a_k^{(2)} \rangle|}{|\langle a_i^{(1)} | a_i^{(1)} \rangle|}$. Now, equality (6) and (7) prove that $|\lambda_i| = \frac{K_A}{L_{a^{(1)}}}$. Whence it is constant for every $(i,k)$. Therefore

$$L_{a^{(2)}} = |\langle a_k^{(2)} | a_k^{(2)} \rangle| = |\langle \sum_i \lambda_i a_i^{(1)} | \sum_i \lambda_i a_i^{(1)} \rangle|$$

$$= \sum_i^N |\lambda_i|^2 |\langle a_i^{(1)} | a_i^{(1)} \rangle| = N_1 |\lambda_i|^2 L_{a^{(1)}} = N_1 \left( \frac{K_A}{L_{a^{(1)}}} \right)^2 L_{a^{(1)}} = \frac{N_1 (K_A)^2}{L_{a^{(1)}}}.$$

that is $L_{a^{(1)}} L_{a^{(2)}} = N_1 (K_A)^2$ \hspace{1cm} (8)

Finally, we show that $\{ \frac{a_i^{(1)}}{|\langle a_i^{(1)} | a_i^{(1)} \rangle|} \}$ and $\{ \frac{a_k^{(2)}}{|\langle a_k^{(2)} | a_k^{(2)} \rangle|} \}$ are mutually unbiased bases in $\mathbb{C}^{N_1}$. Indeed

$$|\langle \frac{a_i^{(1)}}{|\langle a_i^{(1)} | a_i^{(1)} \rangle|} | \frac{a_k^{(2)}}{|\langle a_k^{(2)} | a_k^{(2)} \rangle|} \rangle|^2 = \frac{|\langle a_i^{(1)} | a_k^{(2)} \rangle|^2}{|\langle a_i^{(1)} | a_i^{(1)} \rangle||\langle a_k^{(2)} | a_k^{(2)} \rangle|^2} = \frac{(K_A)^2}{L_{a^{(1)}} L_{a^{(2)}}} = \frac{1}{N_1} \quad \text{by (8)}.$$

The result for $\mathbb{C}^{N_2}$ is obtained in the same way, using $b_j^{(2)} = \sum_j \mu_j b_j^{(1)}$ for $\mu_j = \frac{|\langle b_j^{(1)} | b_j^{(1)} \rangle|^2}{|\langle b_j^{(1)} | b_j^{(1)} \rangle|^2}$, to give $L_{b^{(1)}} L_{b^{(2)}} = N_2 (K_B)^2$. \hfill \Box

This proposition can immediately be extended as follows to $\mathbb{C}^N \cong \mathbb{C}^{N_1} \otimes \ldots \otimes \mathbb{C}^{N_k}$ for dimension $N = N_1 \ldots N_k$. Under assumption that each of the $k$ bases is a tensor product, we may use induction to conclude to the existence of $k$ mutually unbiased bases in each $\mathbb{C}^{N_i}$.

### 3.2 Main results

**Theorem 3.1** Let $R = R_1 \oplus R_2$ be a decomposition of a ring $R$. For $i = 1, 2$ let $S_i$ be a non empty subset of $R_i$ and let $N = |S_1||S_2|$. For each $1 \leq c \leq m$, let $f_c : R_+ \to \mathbb{C}^{*}$ be a two variables function that preserves the decomposition $R_1 \oplus R_2$ and let us define $N$ vectors $\{v_k^{(c)}\}$ of $\mathbb{C}^N$ as

$$v_k^{(c)} = f_c(k,l) \quad k, l \in S_1 \oplus S_2.$$

Assume that, together with the standard basis, the sets of vectors $\{v_k^{(c)}\}_{1 \leq c \leq m}$ form a set $X$ of $m + 1$ mutually unbiased bases. If $|S_i|_{i=1,2} \neq 1$, then
\[ m \leq \min_{i} |S_i| < N \text{ and the set } X \text{ is not complete.} \]

**Proof.** The function \( f_c \) preserves the decomposition \( R_1 \oplus R_2 \) so there is a constant \( \lambda_c \) such that \( f_c(k, l) = \lambda_c f_c(k_1, l_1) f_c(k_2, l_2) \). For each \( c \), let us define \( |S_1| \) vectors \( \{ a_{k_1}^{(c)} \}_{k_1 \in S_1} \) of \( \mathbb{C}^{|S_1|} \) and \( |S_2| \) vectors \( \{ b_{k_2}^{(c)} \}_{k_2 \in S_2} \) of \( \mathbb{C}^{|S_2|} \) as

\[
(a_{k_1}^{(c)})_{l_1} = \lambda_c f_c(k_1, l_1), \quad (b_{k_2}^{(c)})_{l_2} = f_c(k_2, l_2) \quad \text{for } k_i, l_i \in S_i \ (i = 1, 2).
\]

Hence \( (v_k^{(c)})_l = f_c(k, l) = \lambda_c (a_{k_1}^{(c)})_{l_1} (b_{k_2}^{(c)})_{l_2} \) and since \( l \) takes all value in \( S_1 \oplus S_2 \), the vector \( v_k^{(c)} = v_{(k_1, k_2)}^{(c)} \) is equal to the tensor product \( a_{k_1}^{(c)} \otimes b_{k_2}^{(c)} \in \mathbb{C}^{|S_1||S_2|} \). If we denote by \( \{ v_k^{(0)} \}_{k \in S_1 \oplus S_2}, \{ a_{k_1}^{(0)} \}_{k_1 \in S_1}, \{ b_{k_2}^{(0)} \}_{k_2 \in S_2} \) the standard bases respectively in \( \mathbb{C}^N, \mathbb{C}^{|S_1|} \) and \( \mathbb{C}^{|S_2|} \) then also \( v_k^{(0)} = v_{(k_1, k_2)}^{(0)} = a_{k_1}^{(0)} \otimes b_{k_2}^{(0)} \).

Therefore if the sets \( \{ v_k^{(c)} \}_{0 \leq c \leq m} \) form a set \( X \) of \( m + 1 \) mutually unbiased bases in \( \mathbb{C}^N = \mathbb{C}^{|S_1||S_2|} \), then by Proposition 3.2 there exist \( m + 1 \) mutually unbiased bases in both \( \mathbb{C}^{|S_1|} \) and \( \mathbb{C}^{|S_2|} \). By Theorem \( \square \) if each \( |S_i|_{i=1,2} \) is at least \( 2 \) then \( m + 1 \leq |S_i|_{i=1,2} \) and \( m \leq \min_{i} |S_i| < |S_1||S_2| = N \) thus \( |X| = m + 1 < N + 1 \) and \( X \) is not complete. \( \square \)

Finally, we obtain our main result: complete sets of MUB described by generalizations of known formulas only exist for prime power dimensions. Moreover, we provide an upper bound for the number of MUB described by such formulas.

**Theorem 3.2** Let \( R \) be a finite ring with unity. Let \( S \subset R \) be a subset of \( N \) elements that is closed under multiplication and transversal to a nilpotent ideal. For \( 1 \leq c \leq N \), let \( T_c : R_+ \to \mathbb{C}^*, \cdot \) be a group homomorphism and let \( P_c : R^2 \to R \) be a two variables polynomial function. Let us define \( N \) sets of vectors \( \{ v_k^{(c)} \}_{1 \leq c \leq N} \) in \( \mathbb{C}^N \) by

\[
(v_k^{(c)})_l = \frac{1}{\sqrt{N}} T_c(P_c(k, l)) \quad k, l \in S,
\]

and let the set \( X_{\{T_c\}} \) be the union of the standard basis with \( \{ v_k^{(c)} \}_{1 \leq c \leq N} \).

1. A set \( X_{\{T_c\}} \) contains at most \( 1 + \min_{i} \{ P_i^{e_i} \} \) mutually unbiased bases where \( N = \prod_i P_i^{e_i} \) is the factorization of \( N \) into powers of distinct prime numbers.

2. There exists a complete set \( X_{\{T_c\}} \) of \( N + 1 \) mutually unbiased bases if and only if \( N \) is a power of a prime number.
Proof.

1. We prove that the conditions of Theorem 3.1 applies here. First we show that every vector $v_k^{(c)}$ is unbiased with the standard basis $\{e_k\}$. Every $r \in \mathbb{R}$ has finite additive order $n_r$ ($n_r \cdot r = 0$). By the homomorphism property $T_c(n.r) = T_c(r + \ldots + r) = (T_c(r))^n$ and as $T_c(0) = 1$ we must have $|T_c(r)|^n = 1$ whence $|T_c(r)| = 1$ for every $r$ in $R$. Thus for every vector $v_k^{(c)}$ we obtain $|\langle e_l | v_k^{(c)} \rangle| = |\langle v_k^{(c)} | e_l \rangle| = |T_c(P_c(k,l))|/\sqrt{N} = 1/\sqrt{N}$ as announced. Let $m + 1$ be the maximal number of mutually unbiased bases contained in $X_{\{T_c\}}{P_c}$ and let $Y \subset X_{\{T_c\}}{P_c}$ be a set of $m + 1$ mutually unbiased bases. Since we have showed that $Y \cup \{e_k\}$ is also a set of MUB, the standard basis $\{e_k\}$ must be in $Y$.

As $|S| = N = \prod_ip_i^{e_i}$ we may use Proposition 3.1 (2) to show that there is a ring decomposition $R = \oplus_i R_i$ such that $S = \oplus_i S \cap R_i$ and $|S \cap R_i| = p_i^{e_i}$. Finally, since the functions $(k, l) \rightarrow T_c(P_c(k,l))$ preserve every direct sum decomposition of $R$ (see subsection 4), we may apply Theorem 3.1 to $Y$ to show that for every $i$ we must have $m \leq p_i^{e_i}$. Hence $1 + m \leq 1 + \min_i p_i^{e_i}$.

2. If we have $N+1$ such MUB, then by (1), $\prod_ip_i^{e_i} = N \leq \min_i p_i^{e_i}$, which implies that $N = \min_i p_i^{e_i}$ and thus $N$ is a prime power. Conversely if $N = p_i^{e_i}$, we have shown in section 2 that the sets of $N+1$ MUB given by formulas (1) and (2) may be described as sets $X_{\{T_c\}}{P_c}$.

\[\square\]

The bound $1 + \min_i p_i^{e_i}$ can be easily reached for dimension $N = \prod_ip_i^{e_i}$. It suffices to view $\mathbb{C}^N$ as $\bigotimes_i \mathbb{C}^{p_i^{e_i}}$. As $\langle b_i \otimes c_k | b_j \otimes c_l \rangle = \langle b_i | b_j \rangle \cdot \langle c_k | c_l \rangle$ we may conclude that a tensor product of two sets with $t$ MUB is a set of $t$ MUB in the product space. Since there exist at least $1 + \min_i p_i^{e_i}$ MUB in each $\mathbb{C}^{p_i^{e_i}}$ we may construct by tensor product of these, a set of $1 + \min_i p_i^{e_i}$ MUB in the product space $\mathbb{C}^N$. 

15
3.3 Discussion on larger generalizations and conclusion

In order to further generalize formula (3), it could be tempting to allow the index set \( S \) to be any subset of a finite ring. Unfortunately, this leads to a situation where any set of vectors could be described by such a formula. To see this, let us recall that functions \( f \) that preserve a decomposition of a ring \( R \) are arbitrary functions on each component \( R_i \). If we choose a ring \( R \) that has no decomposition (a field for instance) then such a \( f \) is arbitrary on \( R \). If \( S \) is a subset \( \{s_1, \ldots, s_N\} \) of \( N \) elements, then we may associate an arbitrary set of vectors \( \{v_k\}_{1 \leq k \leq N} \) in \( \mathbb{C}^N \) to the couples in \( S \times S \) by \((s_k, s_l) \rightarrow (v_k)_l\).

This may be extended (in many ways) to a two variables function from \( R \times R \) into \( \mathbb{C} \) that preserves every decomposition of \( R \) (since \( R \) cannot be decomposed). One cannot expect to reach algebraic conclusions that are valid for all \( N \times N \) arrays with arbitrary complex entries \( (v_k)_l \).

For these reasons, it is difficult to generalize formula (3) much more. It indicates that for dimensions that are not prime powers, algebraic formulas providing complete sets of MUB should have a radically new structure. However, do these complete sets exist for any dimension? Mathematicians are used to properties that behave differently for some particular dimensions but such an answer is unsatisfactory from a physical point of view. W.K. Wootters has showed that the absence of \( N + 1 \) MUB for a dimension \( N \) would be problematic for defining a discrete Wigner function in systems having \( N \) degrees of freedom (see [12]). A negative answer to the MUB problem might have other physical consequences and these could be used to guide mathematical investigations.

Acknowledgement

I gratefully acknowledge Sofyan Iblisdir for introducing me to this problem and for all our fruitful discussions.

References


